

Methods for Simulating the Heavy Core Instability

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Abstract. Vortices have been proposed as the sites of planet formation, where dust collects and grows into planetesimals, the building blocks of planets. However, for very small dust particles that can be treated as a pressure-less fluid, we have recently discovered the “heavy core” instability, driven by the density gradient in the vortex. In order to understand the eventual outcome of this instability, we need to study its non-linear development. Here, we describe our ongoing work to develop highly accurate numerical models of a vortex with a density gradient embedded within a protoplanetary disk.

1 Introduction

The *Kepler* Space Telescope has dramatically increased the number of observed planetary systems and has highlighted their diversity. In order to understand this diversity, we need a robust theory for the formation of planets. This formation process begins in protoplanetary accretion disks around forming stars. Understanding how to collect μm size interstellar dust into km sized planetesimals in these disks remains a significant challenge. Here, we consider the vortex hypothesis, in which long-lived vortices within protoplanetary disks collect dust, greatly increasing the dust to gas ratio and setting up conditions favorable for gravitational collapse of the dust layer [1, 2].

Recently, we discovered a heavy core instability that acts in these vortices as mass loading proceeds [3]. Here, we outline our recent developments toward understanding the non-linear stage of this instability, with the eventual goal of understanding its saturation. We first give an overview of the background flow, then we discuss the complications arising from the fact that the density within the vortex is not constant (despite the flow being incompressible), and we then describe a numerical algorithm to treat this situation.

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2 Background Flow

Here, as in our previous linear treatment, we considered a shearing-sheet geometry [e.g. 4] in which Keplerian rotation is described as a linear shear flow around a guiding center at some radius R from the central star,

$$u_x = S y, \quad (1)$$

with the shear rate $S = 3/2$ for Keplerian rotation.

For our first numerical experiments, we consider a Kida vortex, which is a steady solution of the shearing sheet describing an elliptical vortex atop the background rotating shear flow. A stationary Kida vortex with aspect ratio $\chi = a/b$ (with semi-major axis a and semi-minor axis b) has vorticity

$$\omega_v = -\frac{S \chi + 1}{\chi \chi - 1}. \quad (2)$$

We also consider a density field which we consider to be a power law in b , the semi-minor axis which labels the ellipse. This gradient is fully specified by $\partial \ln \rho / \partial \ln b$.

Recently, one of us has developed *Dedalus*¹, a new Python-based toolkit for pseudospectral simulation. *Dedalus* includes both incompressible and Boussinesq approximations for shearing box hydrodynamics and magnetohydrodynamics, making it ideal for studying the heavy core instability. We first test the code to make sure it correctly reproduces the steady nature of the Kida solution in the absence of a density gradient. Figure 1 shows the vorticity in a 256×64 -mode *Dedalus* simulation after several vortex turnover times. The vortex remains steady and coherent, indicating that the shearing box hydrodynamics are working correctly. Next, we describe the necessary extensions to the code in order to allow it to solve variable-density incompressible flow.

3 Equations and Numerical Solution Methods

The heavy core instability is incompressible, but requires the presence of a background density gradient. We thus consider the equations of variable-density incompressible flow,

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = 0, \quad (3)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla p}{\rho} + \mathbf{f}_{shear}, \quad (4)$$

and

$$\nabla \cdot \mathbf{u} = 0. \quad (5)$$

Here, density is advected like a passive scalar, but it affects the dynamics via the $-\nabla p/\rho$ term. Treating such flows numerically is challenging, because one needs to maintain the incompressibility constraint (equation 5) while at the same time allowing density to advect and add buoyant forces [5].

Because the system is incompressible and the delicate instability mechanisms involved in parametric instability require very accurate solvers [e.g. 6], we use a pseudospectral algorithm to solve equations 3-5.

¹<https://bitbucket.org/jsoishi/dedalus>

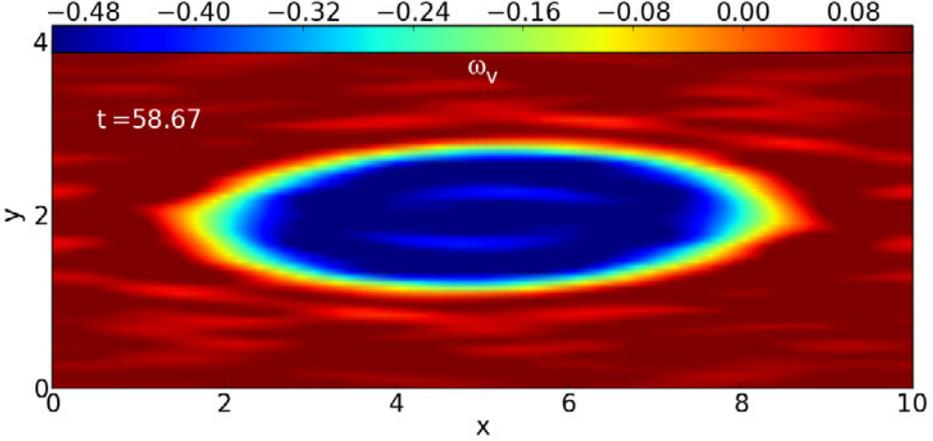


Figure 1. Vorticity in a 2D *Dedalus* simulation of a Kida vortex after several vortex turnover times. The vortex remains steady and coherent. This vortex does *not* include a background density gradient and is thus stable to the heavy core instability.

3.1 Maintaining Incompressibility

There are a number of approaches to maintaining incompressibility, that is, ensuring $\nabla \cdot \mathbf{u} = 0$ during a simulation. One of the most accurate methods for doing so is the *projection method*, in which the pressure in the Navier-Stokes equation (equation 4) is determined by inverting a Poisson equation. This ensures that the divergence is kept to machine precision, but its cost can be very high. However, when using a pseudospectral technique on a fully periodic domain, the FFT allows a Poisson solve at trivial cost. Given that our simulation meet these restrictions, it is natural to use the projection method.

The projection method for a constant density incompressible flow is well known and is described in numerous textbooks, [e.g. 7]. By taking the gradient of equation 4, one constructs a Poisson equation for the pressure. However, in the variable density case,

$$\nabla \cdot \left(\frac{\nabla P}{\rho} \right) = \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{f}_{shear}, \quad (6)$$

and one must instead consider a generalized Poisson equation for the pressure, because ρ^{-1} is no longer a constant. Taking $a = \rho^{-1}$, $u = P$, and $f = \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{f}_{shear}$, we write this schematically as

$$\nabla \cdot (a \nabla u) = f. \quad (7)$$

We must find a suitable method to solve this within our pseudospectral framework.

3.2 A Generalized Poisson Solver

Here, we outline the method derived in reference [8] for computing electrostatic fields in molecular solvents. We then describe how we adapt it to enforcing incompressibility in variable density flows.

The first step is to convert equation 6 into a form that eliminates the first derivative term,

$$-\nabla^2 w + pw = q, \quad (8)$$

with $w = a^{1/2}u$, $p = a^{-1/2}\nabla^2 a^{1/2}$, and $q = a^{-1/2}f$ [9]. Because our domain is periodic, we can apply Fourier transforms to this, leading to a matrix equation that can be solved with standard methods. Because *Dedalus* is written in the Python language, we have easy access to a large number of high quality linear algebra libraries, including the highly optimized, parallel PETSc² system. Combined with the simplicity of our codebase, we expect a very short implementation time.

4 Conclusions

Previously, we have described the heavy core instability, which could disrupt or slow the collection of dust in protoplanetary disk vortices. Here, we have shown a preliminary test of our pseudospectral toolkit *Dedalus*. This test shows that the code can correctly evolve the steady background state for the instability without destabilizing density gradient. We have described the ongoing algorithmic developments that will allow us to study the non-linear phase of this instability using high-resolution spectral methods. We are currently implementing this algorithm in *Dedalus*, and we will soon have a complete, non-linear description of the heavy core instability. This description will help us to determine if the instability disrupts dust gathering vortices, increases the velocity dispersion of dust within them, or simply settles on a new equilibrium.

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²<http://www.mcs.anl.gov/petsc/>