

The mass-gap in Quantum Chromodynamics

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Abstract. It is shown that the fundamental Lagrangian of Quantum Chromodynamics should be modified by the adding gluon masses to produce the mass-gap in accordance with the Källén-Lehmann spectral representation. On mass-shell renormalizability and unitarity of the resulting theory is demonstrated.

1 Introduction

The obtaining the mass-gap in Quantum Chromodynamics (QCD) is the well known long standing task. QCD was quite quickly established as the theory of strong interactions after the discovery [1] of asymptotic freedom since physicists got the small parameter, the strong coupling constant at high energies, and it became possible to perform perturbative calculations. The gauge bosons of the theory, the gluons, are considered to be massless to have gauge invariance and correspondingly renormalizability. The introduction of the non-zero Lagrangian gluon masses into the fundamental QCD Lagrangian was considered to be forbidden because of the violating either renormalizability or unitarity of the corresponding theory. Giving masses to gauge bosons via the Higgs mechanism [2] is also not allowed since colored Higgs particles are rejected by experiments.

Quite recently it was found that the non-abelian Yang-Mills theory [3] with masses of the Proca type is in fact on mass-shell renormalizable [4]. Unitarity of the theory of the Proca type is obvious since it contains only physical degrees of freedom (no ghosts).

In the present paper it is shown that it is impossible to obtain the necessary mass-gap in QCD with zero Lagrangian gluon masses. The fundamental QCD Lagrangian should be modified by the adding non-zero Lagrangian gluon masses to get the theory of the Proca type to produce the mass-gap in accordance with the Källén-Lehmann spectral representation [5]. On mass-shell renormalizability of the resulting theory is discussed.

2 The main part

2.1 Definitions and notations

The QCD Lagrangian is

$$L_{QCD} = -\frac{1}{4}F_{\mu\nu}^a F^{a\ \mu\nu} + i\bar{\psi}_f \gamma^\mu (\partial_\mu - igA_\mu^a T^a) \psi_f - m_f \bar{\psi}_f \psi_f \quad (1)$$

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$$-\frac{1}{\xi}(\partial^\mu A_\mu^a)^2 + \partial^\mu \bar{c}^a (\partial_\mu c^a - g f^{abc} c^b A_\mu^c)$$

plus counterterms. The notations are standard. The summation over the flavour index $f = u, \dots, t$ is assumed. ξ is the gauge parameter of the usually used general covariant gauge chosen because of its convenience for perturbative calculations. $m_f = m_f(\mu)$ is the renormalized (current) quark mass, $g = g(\mu)$ is the renormalized strong coupling constant, $g^2/(16\pi^2) \equiv a_s$, where μ is the renormalization point.

It is known [6] that the covariant gauge chosen in the Lagrangian (1) does not fix the gauge ambiguity uniquely and is valid strictly speaking only within perturbation theory. Hence the theory described by the Lagrangian (1) with the covariant gauge condition (chosen because of its convenience for perturbative calculations) should not be seen as the complete theory of strong interaction, but it can be considered as the theory which reproduces the perturbative part of the complete theory [7]. The theories, see e.g. [8], which do not reproduce conventional perturbative QCD expansions are not considered here.

2.2 Obtaining the mass-gap

To demonstrate the necessity of the introducing the Lagrangian gluon masses for the obtaining the mass-gap let us consider the vacuum polarization function $\Pi(q^2)$

$$(-q^2 g_{\mu\nu} + q_\mu q_\nu) \Pi(q^2) = i \int dx e^{iqx} \langle 0 | T (j_\mu(x) j_\nu(0)) | 0 \rangle. \quad (2)$$

where $j_\mu = \sum_f e_f \bar{\psi}_f \gamma_\mu \psi_f$ is the quark electromagnetic current, e_f being the electromagnetic quark charge.

According to the general principles of local quantum field theory the function $\Pi(q^2)$ satisfies the fundamental Källén-Lehmann [5] spectral representation

$$\Pi(q^2) = \frac{1}{12\pi^2} \int_{4m_\pi^2}^{\infty} ds \frac{R(s)}{s - q^2 - i0} + C, \quad (3)$$

where the ratio $R(s) = \sigma_{tot}(e^+e^- \rightarrow hadrons)/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$ is the normalized total cross-section of electron-positron annihilation into hadrons, m_π is a pion mass. The representation is written with one subtraction and C is the subtraction constant (the number of necessary subtractions is not important for our considerations).

We would like to stress here that this spectral representation is more fundamental than just a dispersion relation. It can not be improved or modified. To write a dispersion relation for a function one should know analytic properties of a function. The Källén-Lehmann representation itself determines the analytic properties of $\Pi(q^2)$ which should be an analytic function in the complex q^2 -plane with the cut starting from the first physical threshold, i.e. as it is dictated by experiments from the two-pion threshold $q^2 = 4m_\pi^2$. One gets for the discontinuity of $\Pi(q^2)$ on the cut

$$\Delta\Pi(q^2) \equiv \Pi(q^2 + i0) - \Pi(q^2 - i0) = \begin{cases} i R(q^2)/(6\pi) & \text{at } s > 4m_\pi^2 \\ 0 & \text{at } s < 4m_\pi^2. \end{cases} \quad (4)$$

On the other hand one obtains within Perturbative QCD for the discontinuity

$$\Delta\Pi(q^2)_{pQCD} = \theta(q^2) \rho_{gl}(q^2) + \theta(q^2 - 4M_u^2) \rho_q(q^2). \quad (5)$$

The gluon spectral density $\rho_{gl}(q^2)$ contributes for $q^2 > 0$ as it is shown by the theta-function $\theta(q^2)$. This is the famous zero threshold. It arises from those absorptive parts of Feynman diagrams of $\Pi(q^2)$ which are produced by purely gluonic cuts of the diagrams (i.e. by the Cutkosky cuts which cross only gluon propagators of diagrams). As it is well known such diagrams appear for the first time at the four-loop level in the order a_s^3 (corresponding cuts cross 3 gluon propagators).

The quark spectral density

$$\rho_q(q^2) = 3 \sum_f \theta(q^2 - 4M_f^2) e_f^2 (1 + 2M_f^2/q^2) \sqrt{1 - 4M_f^2/q^2} + O(a_s) \quad (6)$$

arises from the quark cuts of the diagrams (i.e. from the cuts which cross two or more quark propagators of the diagrams). It contributes for $q^2 > 4M_u^2$ where M_u is the perturbative pole mass of the lightest u -quark, defined as the perturbative pole of the quark propagator. The perturbative quark pole mass $M_f = m_f(\mu) + O(a_s)$ naturally appears as the pole of the quark propagator after summation of perturbative loop corrections to the propagator. It is a renormalization group invariant quantity, i.e. independent on the renormalization point μ and on the subtraction scheme. In this sense it behaves as a physical quantity and that is why it is natural to use a perturbative pole quark mass to parametrize the theory.

We do not discuss here the important by themselves questions of convergence or divergence of corresponding perturbative QCD series at low or high energies. We will just accept the constructive approach that our conventional perturbation theory is adequate to the exact solution of the theory, i.e. it correctly reproduces the perturbative expansion of the exact solution. We assume here that the exact solution is in principle obtainable if we know enough mathematics.

Thus one gets within QCD that $\Delta\Pi(q^2)$ is non-zero in the energy region $0 < q^2 < 4m_\pi^2$ since the perturbative contribution $\Delta\Pi(q^2)_{pQCD}$ is non-zero in this interval. Here we would like to stress that one should get in QCD an exact zero below the two-pion threshold as it is dictated by experiments. There are also non-perturbative contributions to $\Delta\Pi(q^2)$, i.e. contributions of the type e^{-1/a_s} which are invisible in the perturbative expansion at the point $a_s = 0_+$ (that is $e^{-1/a_s} = 0 \cdot a_s + 0 \cdot a_s^2 + \dots$).

But we note that the non-perturbative contributions can not exactly cancel the perturbative ones in the continuous interval $0 < q^2 < 4m_\pi^2$ because of the different analytical dependence on the coupling constant a_s . To get the condition $\Delta\Pi(q^2) = 0$ at $0 < q^2 < 4m_\pi^2$ in agreement with experiments we should shift perturbative gluon and quark thresholds above the two-pion threshold $q^2 = 4m_\pi^2$. That is why one should introduce the non-zero gluon masses into the fundamental QCD Lagrangian.

One could object that perturbative contributions in the exact solution for $\Delta\Pi(q^2)$ could be moved above the two-pion threshold with the help of the suitably arranged θ -function. E.g. the gluon perturbative contribution could be moved above the two-pion threshold in the following way

$$\theta(q^2 - M^2 f(a_s)) \rho_{gl}(q^2), \quad (7)$$

where $f(a_s)$ is some function which is zero at $a_s = 0$ and M^2 is a dimensional parameter such that $M^2 f(a_s)$ is a renormalization group invariant quantity which produces the necessary mass-gap below the two-pion threshold.

But perturbative expansion of the contribution of eq.(7) would contain terms with $\delta(q^2)$ and its derivatives arising from the differentiating the θ -function and we do not see such terms in real perturbative expansion. Thus the contributions of the type of eq.(7) are excluded in the exact solution of the theory.

The standard naive objection here is that nobody trusts perturbation theory at such low energies, since the corresponding perturbative series is heavily divergent below the two-pion threshold. But

only the principal existence of the perturbative series with well defined coefficients below the two-pion threshold is of importance here independently on the question of its convergence: the non-zero perturbative expansion means the non-zero exact function generating this expansion.

Thus one gets the restrictions on the perturbative pole masses of gluons and quarks

$$M_{gl} > \frac{2}{\sqrt{3}}m_\pi, \quad (8)$$

$$M_u > m_\pi.$$

The restriction on M_u seems to be quite strong for the lightest u -quark but it is not excluded from the first principles.

2.3 Constructing QCD with gluon masses

To construct QCD with massive gluons we will follow the approach of [4]. Presently this is the only known way to get (on mass-shell) renormalizable theory of massive gluons without color scalars (color scalars are rejected by experiments). Within this approach one starts from a renormalizable theory with scalar fields using the Englert-Brout-Higgs mechanism of spontaneous symmetry breaking [2] and after transition to the unitary gauge removes remaining massive scalar fields. Thus we add to the massless QCD Lagrangian (1) the scalar part to begin with the following general Lagrangian

$$L_{QCD+SCALARS} = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + i\bar{\psi}_f \gamma^\mu D_\mu \psi_f - m_f \bar{\psi}_f \psi_f + \quad (9)$$

$$(D_\mu \Phi)^+ D^\mu \Phi + (D_\mu \Sigma)^+ D^\mu \Sigma - \lambda_1 (\Phi^+ \Phi - v_1^2)^2 - \lambda_2 (\Sigma^+ \Sigma - v_2^2)^2$$

$$- \lambda_3 (\Phi^+ \Phi + \Sigma^+ \Sigma - v_1^2 - v_2^2)^2 - \lambda_4 (\Phi^+ \Sigma) (\Sigma^+ \Phi) + L_{gf} + L_{gc}$$

plus counterterms, where we introduced two triplets $\Phi(x)$ and $\Sigma(x)$ of complex scalar fields in the fundamental representation of the $SU(3)$ color group to get all gluon massive. L_{gf} is the gauge fixing part of the Lagrangian in some chosen gauge and L_{gc} is the corresponding gauge compensating part with the Faddeev-Popov ghost fields.

We can choose the following shifts of scalar fields by the quantities v_1 and v_2 to generate masses of all eight gluons

$$\Phi(x) = \begin{pmatrix} \phi_1(x) + i\phi_2(x) + v_1 \\ \phi_3(x) + i\phi_4(x) \\ \phi_5(x) + i\phi_6(x) \end{pmatrix}, \quad \Sigma(x) = \begin{pmatrix} \sigma_1(x) + i\sigma_2(x) \\ \sigma_3(x) + i\sigma_4(x) + v_2 \\ \sigma_5(x) + i\sigma_6(x) \end{pmatrix}. \quad (10)$$

Choosing for simplicity $v_1 = v_2 \equiv v$ one obtains the following massive terms for gluons in the Lagrangian

$$L_M = m_{gl}^2 \left[(A^1)^2 + (A^2)^2 + (A^3)^2 + \frac{1}{2}(A^4)^2 + \right. \quad (11)$$

$$\left. \frac{1}{2}(A^5)^2 + \frac{1}{2}(A^6)^2 + \frac{1}{2}(A^7)^2 + \frac{1}{3}(A^8)^2 \right],$$

where $m_{gl}^2 \equiv g^2 v^2$ is the gluon mass parameter of the theory.

Now one can make transition to the unitary gauge. All ghost fields as usual disappear from the Lagrangian. Following the approach of [4] one can remove in the unitary gauge all Higgs fields from the Lagrangian preserving on mass-shell renormalizability of the theory.

To give the derivation of this statement let us consider as an example the simplified case, the generalization to the above QCD+SCALARS case will be straightforward. Let us consider the known model given by the initial $SU(2)$ -invariant Lagrangian of interaction of vector bosons and scalar fields possessing the spontaneously broken symmetry

$$L = -\frac{1}{4}F_{\mu\nu}^a F_{\mu\nu}^a + (D_\mu\Phi)^\dagger D_\mu\Phi - \lambda(\Phi^\dagger\Phi - v^2)^2 \quad (12)$$

with the doublet of scalar fields $\Phi(x)$ in the fundamental representation of the group.

Here $D_\mu\Phi = (\partial_\mu - ig\frac{\tau^a}{2}W_\mu^a)\Phi$ is the covariant derivative, τ^a are the Pauli matrices, $\lambda > 0$, $v^2 > 0$. (This model can be considered as the Standard Model of electroweak interactions without $U(1)$ -interaction and fermions. The derivation given below can be applied also to the complete Standard Model, the γ_5 -matrix being treated within dimensional regularization according to the technique of [9].)

To obtain the complete Lagrangian one makes the standard shift of the scalar field fixes the gauge and adds ultraviolet counterterms. Let us consider two gauges: the widely used R_ξ -gauge [10], [11] with an arbitrary parameter ξ and the unitary gauge.

The theory in the R_ξ -gauge describes three physical massive vector bosons with the mass $m = gv/\sqrt{2}$, and the physical Higgs field χ with the mass $M = 2\lambda v$. Here are also Goldstone ghosts ϕ^a and Faddeev-Popov ghosts c^a with masses ξm^2 . The structure of the counterterms (consistent with gauge invariance and Slavnov-Taylor identities [12, 13] to ensure unitarity) is well known, see e.g. [7].

The propagators in the unitary gauge defined by the gauge condition $\phi^a = 0$ are obtained from those of the R_ξ -gauge by taking the limit $\xi \rightarrow \infty$. The theory in the unitary gauge is renormalizable only on mass-shell, i.e. Green functions are divergent at $\epsilon \rightarrow 0$ (ϵ being the parameter of dimensional regularization) but the S-matrix elements are finite.

To consider renormalization for our purpose it is convenient to use the Bogoliubov-Parasiuk-Hepp subtraction scheme [14]. As it is well known in this scheme a counterterm of e.g. a primitively divergent Feynman diagram is the truncated Taylor expansion of the diagram itself at some fixed values of external momenta. Hence counterterms of mass dependent diagrams are also mass dependent. Needless to say that subtractions should respect Slavnov-Taylor identities.

Let us consider S-matrix elements in the R_ξ -gauge without external Higgs bosons (i.e. with external W-bosons only in this simplified model). We will analyze the dependence of diagrams on the Higgs mass M by using for convenience the expansion in large M (after renormalization but before the removing regularization). The algorithm for the large mass expansion of Feynman diagrams is given e.g. in [15] (where it is quite reasonably checked in calculations of the 4-loop diagrams for the Z-boson decay into hadrons). It can be rigorously derived with the technique of [16].

We separate all diagrams into physical ones which are not nullified in the limit $\xi \rightarrow \infty$ and unphysical ones which are nullified. In this limit the propagator of the W-boson reduces to the known unitary form

$$\begin{aligned} \lim_{\xi \rightarrow \infty} \langle T(W_\mu^a W_\nu^b) \rangle &= -i\delta^{ab} \lim_{\xi \rightarrow \infty} \left(\frac{g_{\mu\nu} - k_\mu k_\nu / m^2}{k^2 - m^2} + \frac{k_\mu k_\nu / m^2}{k^2 - \xi m^2} \right) = \\ &= -i\delta^{ab} \frac{g_{\mu\nu} - k_\mu k_\nu / m^2}{k^2 - m^2} \end{aligned}$$

The propagators of the Goldstone bosons ϕ^a and ghosts c^a vanish in this limit and correspondingly all diagrams which contain these propagators are also nullified.

Thus in our notations the physical diagrams are the diagrams which do not contain Goldstone bosons propagators or ghosts propagators and the unphysical diagrams are the diagrams which contain such propagators.

Within the large- M expansion the physical diagrams with χ -propagators contain either terms with integer negative powers of M^2 : $\frac{1}{M^{2n}}$, $n = 1, 2, 3, \dots$ or terms with non-integer powers of M^2 (non-integer powers contain ϵ): $\frac{1}{M^{2(k+l\epsilon)}}$, $k - \text{integer}$, $l - \text{positive integer}$. This is because each vertex with the large factor M^2 has three or four attached χ -propagators due to the structure of the Higgs boson self-coupling.

In contrast, unphysical diagrams can have polynomial in M terms due to the four- ϕ vertex with the large factor M^2 . But they are ξ -dependent (they are nullified in the limit $\xi \rightarrow \infty$) and this polynomial terms cancel in S-matrix elements.

In the renormalizable R_ξ -gauge one can present ultraviolet renormalization in a standard form of the Bogoliubov-Parasiuk R-operation for individual diagrams. This ensures that after renormalization the M -dependent terms are finite at $\epsilon \rightarrow 0$ separately from M -independent terms. Thus if one removes all M -dependent terms one is left with a finite expression.

On the Lagrangian level it means in the unitary gauge that one removes from L_U all terms containing the field χ and also all M -dependent terms in the counterterms. This should be done in the unitary gauge because in the R_ξ -gauge some diagrams containing propagators of Higgs particle can give contributions not depending on the Higgs mass M . In contrast in the unitary gauge all diagrams containing Higgs propagators give only contributions depending on M so there is one to one correspondence between M -dependent diagrams and terms in the Lagrangian containing the Higgs field χ .

The resulting theory is on mass-shell finite. This is the massive Yang-Mills theory of the Proca type

$$L_{YM} = -\frac{1}{4}(\partial_\mu W_\nu^a - \partial_\nu W_\mu^a + \frac{z_1}{z_2} g f^{abc} W_\mu^b W_\nu^c)^2 + m^2 W_\mu^a W_\mu^a$$

plus counterterms.

Thus the Higgs mechanism can be considered as an efficient mathematical tool to observe on mass-shell renormalizability of the massive Yang-Mills theory of the Proca type which is far from to be obvious directly.

Let us now return to our Lagrangian (9) with two scalar triplets which after spontaneous symmetry breaking has four Higgs particles. Following the above approach we can remove in the unitary gauge all four Higgs fields from the Lagrangian preserving on mass-shell renormalizability of the theory. The Lagrangian of the resulting QCD with massive gluons is

$$L_{\text{massive QCD}} = L_M - \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + i \bar{\psi}_f \gamma_\mu D_\mu \psi_f - m_f \bar{\psi}_f \psi_f \quad (13)$$

plus counterterms, where the part L_M with gluon mass terms is given in eq.(11).

We would like to note that on mass-shell renormalizability does not mean that one should consider quarks and gluons as free external particles which would contradict to confinement. It means only that in the $SU(3) \times SU(2) \times U(1)$ theory the S-matrix elements with the physical external particles are finite.

One can calculate the one-loop β -function in this theory [4] to obtain that asymptotic freedom remains valid in the considered theory with massive gluons.

3 Conclusions

Thus we have demonstrated that it is impossible to produce the mass-gap in accordance with the fundamental Källén-Lehmann spectral representation within QCD with zero Lagrangian gluon masses. Correspondingly one should introduce the non-zero gluon masses into the fundamental QCD Lagrangian to generate the necessary mass-gap.

We have also demonstrated that it is possible to introduce the gluon masses into the QCD Lagrangian preserving on mass-shell renormalizability and unitarity of the theory.

Acknowledgments

The author is grateful to the collaborators of the Theory division of INR for helpful discussions.

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