

# Poincaré-invariant constituent quark model for light mesons: capabilities and constraints

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**Abstract.** We present a brief survey of some results on electroweak properties of composite systems that are obtained in the frameworks of our version of the instant form of relativistic quantum mechanics (RQM). Our approach describes well the  $\pi$ - and the  $\rho$ -mesons in wide range of momentum transfers  $Q^2$ . At large  $Q^2$  the obtained pion form factor asymptotics coincides with that of QCD predictions. The method permits to perform analytic continuation of pion form factor to complex plane of momentum transfers that is in accordance with predictions of quantum field theory.

## 1 Introduction

The purpose of this paper is to present a version of relativistic composite quark model developed by the authors (see, e.g., [1]) for the study of composite systems of light quarks. The investigation of electroweak properties of light mesons is an important part of the study of the transition region where the perturbative QCD behavior starts to be valid. In this connection such particles are in the focus of experiments on up-to-day accelerators. The most important results here are the following: measurement of the pion form factor at large momentum transfers in JLab (see, e.g., [2, 3]), the obtaining of the  $\rho$ -meson decay constant from the reaction  $\tau \rightarrow \rho \nu_\tau$  [4, 5], the magnetic moment measurement of the radiation transition  $\rho \rightarrow \pi \gamma^*$ . The interesting result obtained by BABAR collaboration concerning the deviation of the behavior of the transition form factor  $F_{\pi \rightarrow \gamma \gamma^*}(Q^2)$  for large momentum transfer from that predicted by perturbative QCD [6] remains unexplained.

The theoretical description of systems containing light quarks, requires the account of relativistic effects even at low energies. The method in the relativistic theory of composite systems which will be used here is based on the direct realization of the Poincaré algebra on the set of dynamic observable systems and dates back to the P.Dirac's paper [7]. This approach is called the relativistic quantum mechanics with fixed number of particles (RQM) (see, e.g., review [1] and references therein). From the point of view of the principles underlying it, the RQM occupies the intermediate position between the local quantum field theory and nonrelativistic quantum mechanical models.

It should be noted that the field theory and the RQM are formulated as fundamentally different structures and the establishment of a connection between them is a complicated problem that has

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not been solved yet. Unlike the field theory, in the Poincaré-invariant quantum mechanics, the finite number of degrees of freedom are separated initially, that represents some model. The covariance of description in the RQM is provided by the construction on the Hilbert state space of the composite system with the finite number of degrees of freedom of the unique unitary transformation of the inhomogeneous group  $SL(2, C)$ , which is the universal covering of the Poincaré group. The RQM can be realized using different methods and in different forms of dynamics (instant form, point form, light-front form). In this survey basic attention was paid to the description of the electroweak structure of composite systems in the framework of the instant form (IF) of RQM.

One of the main problem in construction of the relativistic constituents models is known to be the problem of construction of operators of transition currents. It should be noted that the approach to describing the electroweak structure of two-particle composite systems presented in this survey has the following characteristic features: the matrix element of the electroweak current of the composite system automatically satisfies the relativistic covariance conditions; the matrix element of the electromagnetic current satisfies the conservation law; the impulse approximation (IA) is formulated in a relativistically invariant way and in the case of the electromagnetic current and with account of the conservation law, the so called modified impulse approximation (MIA) is formulated.

## 2 Instant form of RQM

Relativistic invariance means that on the Hilbert state space of the system the unitary representation of the Poincaré group (or, more precisely, of the inhomogeneous group  $SL(2, C)$  which is the universal covering of the Poincaré group) is realized [8].

Considering infinitesimal transformations and introducing the generators of translations  $\hat{P}^\mu$  and space-time rotations  $\hat{M}^{\mu\nu}$ , we arrive at the Poincaré algebra in the common way. The construction of the representation of the Poincaré group in the Hilbert space is reduced to finding the generators  $\hat{P}^\mu$ ,  $\hat{M}^{\mu\nu}$  in terms of dynamic variables of the system. In the case of the system of noninteracting particles generators in the Poincaré algebra have the clear physical meaning,  $\hat{P}^0 \equiv \hat{H}$  is the operator of total energy,  $\hat{\vec{P}} = (\hat{P}^1, \hat{P}^2, \hat{P}^3)$  is the operator of total 3- momentum,  $\hat{\vec{J}} = (\hat{M}^{23}, \hat{M}^{31}, \hat{M}^{12})$  is the operator of total angular momentum, and  $\hat{\vec{N}} = (\hat{M}^{01}, \hat{M}^{02}, \hat{M}^{03})$  are the Lorentz boost generators. However, inclusion of the interaction between particles in this approach involves some problems; the essence of these problems can be illustrated by considering first the quantum nonrelativistic theory and its invariance group, the Galilean group. After the known transition to the central extension of the Galilean group to the covering group  $SU(2)$  [9] we obtain the 11-parametric group with the set of generators:  $\hat{H}$ ,  $\hat{\vec{P}}$ ,  $\hat{\vec{J}}$ ,  $\hat{\vec{K}}$ ,  $\hat{M}$ , where  $\hat{\vec{K}}$  are the generators of the Galilean boosts and  $\hat{M}$  is the mass operator. The other generators coincide with the corresponding operators of the Poincaré group. The following nonzero commutation relations are contained in the Galilean algebra:

$$\begin{aligned} [\hat{J}_i, \hat{J}_j] &= i\epsilon_{ijk} \hat{J}_k, & [\hat{J}_i, \hat{K}_j] &= i\epsilon_{ijk} \hat{K}_k, & [\hat{J}_i, \hat{P}_j] &= i\epsilon_{ijk} \hat{P}_k, \\ [\hat{K}_i, \hat{H}] &= -i \hat{P}_i, & [\hat{K}_i, \hat{P}_k] &= -i\delta_{ik} \hat{M}. \end{aligned} \quad (1)$$

In nonrelativistic quantum mechanics the operator of interaction is added to the operator of total energy,  $\hat{H} \rightarrow \hat{H} + \hat{V}$ . In order to preserve the Galilean invariance of the theory, i.e., to preserve algebra (1) under such re-definition of the operator of total energy, the following conditions should be imposed on the operator of interaction:

$$[\hat{\vec{P}}, \hat{V}] = [\hat{\vec{J}}, \hat{V}] = [\hat{\vec{\nabla}}_p, \hat{V}] = [\hat{M}, \hat{V}] = 0. \quad (2)$$

Since the generator  $\hat{H}$  is absent in the right-hand sides of relations (1), it is not necessary to include the interaction to other generators of the group in order to preserve the Galilean algebra.

The matter is different in the case of the Poincaré group. Let us consider one of the generators of the Poincaré algebra (see, e.g., [1]):

$$[\hat{P}^j \hat{N}^k] = i \delta^{jk} \hat{H} . \quad (3)$$

At inclusion of the interaction to the operator of total energy described above, the right-hand side of (3) depends on the interaction; therefore, either both generators in the left-hand side of (3), or one of them should depend on the interaction. Thus, in order to preserve commutation relations of the Poincaré algebra it is necessary to make other generators in this algebra dependent on the interaction. The generators of the algebra are separated into two types in this case: generators independent of interaction which form the so called kinematical subgroup and generators depending on interaction, Hamiltonians. The separation of the generators into kinematical generators and Hamiltonians is not unambiguous. Different methods for the separation of the kinematical subgroup result in different forms of dynamics. Further, the instant form of dynamics will be used in which the kinematical subgroup is comprised of the generators of the group of rotations and shifts of the Euclidean space,  $\hat{J}$ ,  $\hat{P}$ , the other generators are the Hamiltonians, i.e., depend on the interaction,  $\hat{P}^0$ ,  $\hat{N}$ .

One of the technical methods for inclusion of interaction to the Poincaré algebra allowing to preserve commutation relations is the additive inclusion of interaction to the mass operator, the so called Bakamajian-Thomas procedure [10] (see also [1]):

$$\hat{M}_0 \rightarrow \hat{M}_I = \hat{M}_0 + \hat{V} . \quad (4)$$

Here,  $\hat{M}_0$  is the operator of the invariant mass of the system without interaction,  $\hat{M}_I$  is the mass operator of the system with interaction. In the instant form of dynamics the operator of interaction should satisfy the first three conditions in Eqs. (2). The first three equalities (2) provide the satisfaction of algebraic relations in the Poincaré algebra in the system with an interaction.

The wave function of the system of interacting particles in the RQM is determined as the eigenfunction of the total commuting set of operators. In the instant form of dynamics this set consists of the operators

$$\hat{M}_I^2 \text{ (or } \hat{M}_I), \quad \hat{J}^2, \quad \hat{J}_3, \quad \hat{P} . \quad (5)$$

$\hat{J}^2$  is the operator of the total angular momentum squared. In the instant form of dynamics the operators  $\hat{J}^2$ ,  $\hat{J}_3$ ,  $\hat{P}$  coincide with the corresponding operators of the system without interaction. Thus, only the operator  $\hat{M}_I^2$  ( $\hat{M}_I$ ) depends on the interaction in (5).

It is possible to introduce a basis in which the motion of the center of mass of two particles is separated and three of four operators of set (5) are diagonal:

$$|\vec{P}, \sqrt{s}, J, l, S, m_J\rangle , \quad (6)$$

where  $P_\mu = (p_1 + p_2)_\mu$ ,  $P_\mu^2 = s$ ,  $\sqrt{s}$  is the invariant mass of the system of two particles,  $l$  is the orbital momentum in the center-of-mass system,  $\vec{S}^2 = (\vec{S}_1 + \vec{S}_2)^2 = S(S + 1)$ ,  $S$  is the total spin in the center-of-mass system,  $J$  is the total angular momentum,  $m_J$  is the projection of the total angular momentum.

Basis (6) diagonalizes the operators  $\hat{J}^2$ ,  $\hat{J}_3$ ,  $\hat{P}$  in (5). Thus, the problem of calculation of the wave function of the system is reduced to the diagonalization of the operator  $\hat{M}_I^2$  (or  $\hat{M}_I$ ).

### 3 Construction of the electroweak current matrix elements

Let us describe now a general method of construction (parameterization) of local operator matrix elements briefly (see e.g., [1, 11] for details).

The main idea of the parametrization can be formulated as follows. Using the variables entering the state vectors which define the matrix elements one has to construct two types of objects.

1. A set of linearly independent matrices which are Lorentz scalars (scalars or pseudoscalars). This set describes transition matrix elements non-diagonal in spin projections in the initial and finite states, as well as the properties defined by the discrete space–time transformations.

2. A set of linearly independent objects with the same tensor dimension as the operator under consideration (for example, four–vector, or four–tensor of some rank). This set describes the matrix element behavior under the action of Lorentz group transformations.

The operator matrix element is written as a sum of all possible objects of the first type multiplied by all possible objects of the second type. The coefficients in this representation as a sum are just the reduced matrix elements – form factors.

To demonstrate this let us consider the parameterization of the matrix elements taken between the states of a free particle of mass  $M$  in different simple cases. Let us normalize the state vectors as follows:

$$\langle \vec{p}, m | \vec{p}', m' \rangle = 2p_0 \delta(\vec{p} - \vec{p}') \delta_{mm'}, \quad p_0 = \sqrt{M^2 + \vec{p}^2}, \quad (7)$$

here  $\vec{p}, \vec{p}'$  are three–momenta of particle,  $m, m'$  are spin projections.

Let us consider now the 4-vector operator of electromagnetic current  $j_\mu(0)$ . To parameterize the matrix element one needs a set of quantities of the appropriate tensor dimension. Using the variables entering the particle state vectors one can construct one pseudovector of the covariant spin operator  $\Gamma^\mu(p')$  (see, e.g., [1]):

$$\Gamma_0(p) = (\vec{p} \cdot \vec{j}), \quad \vec{\Gamma}(p) = M \vec{j} + \frac{\vec{p}(\vec{p} \cdot \vec{j})}{p_0 + M} \quad (8)$$

and three independent vectors:

$$K_\mu = (p - p')_\mu = q_\mu, \quad K'_\mu = (p + p')_\mu, \quad R_\mu = \epsilon_{\mu\nu\lambda\rho} p^\nu p'^\lambda \Gamma^\rho(p'). \quad (9)$$

Here  $\epsilon_{\mu\nu\lambda\rho}$  is a completely anti-symmetric pseudo-tensor in four dimensional space-time with  $\epsilon_{0123} = -1$ .

The set in question of linearly independent matrices in spin projections of the initial and the final states giving the set of independent Lorentz scalars is presented by  $2j + 1$  quantities:

$$D^j(p, p') (p_\mu \Gamma^\mu(p'))^n, \quad n = 0, 1, \dots, 2j. \quad (10)$$

The operator matrix element contains the matrix elements of the listed quantities multiplied by Wigner's rotation matrix  $D^j(p, p')$  from the left. Each of such products is to be multiplied by the sum of linearly independent scalars (10):

$$\langle \vec{p}, M, j, m | j_\mu(0) | \vec{p}', M, j, m' \rangle = \sum_{m''} \langle m | D^j(p, p') | m'' \rangle \langle m'' | F_1 K'_\mu + F_2 \Gamma_\mu(p') + F_3 R_\mu + F_4 K_\mu | m' \rangle, \quad (11)$$

where

$$F_i = \sum_{n=0}^{2j} f_{in}(Q^2) (ip_\mu \Gamma^\mu(p'))^n. \quad (12)$$

Let us impose some additional conditions on the operator: self-adjointness, orthogonality of the vectors in the parametrization (11), parity conservation and the conservation condition:  $q_\mu j_\mu(0) = 0$ .

So, finally we have the parametrization of the matrix element which has following form for particle with spin 1/2:

$$\begin{aligned} & \langle \vec{p}, M, 1/2, m | j_\mu(0) | \vec{p}', M, 1/2, m' \rangle = \\ & = \sum_{m''} \langle m | D^{1/2}(p, p') | m'' \rangle \langle m'' | f_{10}(Q^2) K'_\mu + i f_{30}(Q^2) R_\mu | m' \rangle, \end{aligned} \quad (13)$$

The form factors  $f_{10}(Q^2)$  and  $f_{30}(Q^2)$  are the electric and the magnetic form factors of the particle, respectively. These form factors are connected with Sachs form factors  $G_E(Q^2)$  and  $G_M(Q^2)$ .

The developed procedure can be applied to the construction of the electromagnetic current two-quark system. The following integral representation for the pion form factor in the MIA (see, e.g., [1, 12, 13]) holds:

$$F_\pi(Q^2) = \int d\sqrt{s} d\sqrt{s'} \varphi(k) g_0(s, Q^2, s') \varphi(k'). \quad (14)$$

Here  $\varphi(k)$  is pion wave function in the sense of RQM,  $g_0(s, Q^2, s')$  is the free two-particle form factor describing the electromagnetic properties of the two noninteracting quarks without interaction with the quantum numbers of the pion.

The wave function in (14) has the following structure:

$$\varphi(k) = \sqrt[4]{s} u(k) k, \quad s = 4(k^2 + M^2). \quad (15)$$

$M$  is mass of constituent quarks. Below for the function  $u(k)$  we use some phenomenological wave functions.

The function  $g_0(s, Q^2, s')$  is written in terms of the quark electromagnetic form factors in the form

$$g_0(s, Q^2, s') = A_1(s, Q^2, s') [G_E^q(Q^2) + G_E^{\bar{q}}(Q^2)] + A_2(s, Q^2, s') [G_M^q(Q^2) + G_M^{\bar{q}}(Q^2)]. \quad (16)$$

Functions  $A_1$ ,  $A_2$  may be obtained explicitly by the methods of relativistic kinematics (see, e.g., [12]).

Expressions similar to (14) take a place for charge  $G_C(Q^2)$ , quadrupole  $G_Q(Q^2)$  and magnetic  $G_M(Q^2)$  form factors of the  $\rho$ -meson. The explicit form of corresponding free two-particle form factors with the quantum numbers of the  $\rho$ -meson is cumbersome; it can be found in Ref. [14].

The method of the parameterization of the current matrix elements has been generalized in the paper [15] to the case of the nondiagonal in the total angular momentum matrix elements. In particular the expression for the lepton decay constant of the  $\rho$ -meson was obtained in the 4-fermion approximation:

$$f_\rho = \frac{\sqrt{3}}{\sqrt{2\pi}} \int_0^\infty dk k^2 u(k) \frac{(\sqrt{k^2 + M^2} + M)}{(k^2 + M^2)^{3/4}} \left( 1 + \frac{k^2}{3(\sqrt{k^2 + M^2} + M)^2} \right). \quad (17)$$

## 4 Parameters of the model

For the calculation of the  $\pi$ - and  $\rho$ -meson characteristics basing on the relations (14)–(17) we use the following model wave functions (see, e.g., [16]).

1. The Gaussian or harmonic oscillator wave function

$$u(k) = N_{HO} \exp\left(-k^2/2 b_{\pi,\rho}^2\right). \quad (18)$$

2. The power-law wave function:

$$u(k) = N_{PL} (k^2/b_{\pi,\rho}^2 + 1)^{-n}, \quad n = 2, 3. \quad (19)$$

In eqs. (18), (19)  $b_{\pi,\rho}$  are parameters of wave functions for pion and  $\rho$ -meson, respectively.

The electromagnetic form factors of constituent quarks in (16) are taken in the form [1, 17]:

$$G_E^q(Q^2) = e_q f_q(Q^2), \quad G_M^q(Q^2) = (e_q + \kappa_q) f_q(Q^2), \quad (20)$$

where  $e_q$  is the quark charge and  $\kappa_q$  is the quark anomalous magnetic moment,

$$f_q(Q^2) = \frac{1}{1 + \ln(1 + \langle r_q^2 \rangle Q^2/6)}, \quad (21)$$

$\langle r_q^2 \rangle$  is the MSR of the constituent quark. Values of all parameters used in these expressions are taken from the  $\pi$ -meson calculation, see, e.g., Ref. [18].

So, the following parameters enter our calculations:

1) the parameters that describe the constituent quarks *per se* (the quark mass  $M$ , the anomalous magnetic moments of the quarks  $\kappa_q$ , that enter our formulae through the sum  $s_q = \kappa_u + \kappa_{\bar{d}}$ , and the quark mean square radius (MSR)  $\langle r_q^2 \rangle$ );

2) the parameters  $b_{\pi,\rho}$  that enter the quark wave functions (18), (19) and is determined by the quark interaction potential.

In the paper [12] on pion we have shown that in our approach all the parameters of the first group are the functions of the quark mass  $M$  and are defined by its value. In particular, for the quark MSR we can use the relation (see, also [19]):

$$\langle r_q^2 \rangle \simeq 0.3/M^2. \quad (22)$$

To calculate electroweak properties of the  $\rho$  meson, we use the same values of quark parameters from the first group as that we have used for the pion [12]. So, the wave function parameters  $b_{\pi,\rho}$  are the only free parameters in our calculations.

## 5 Asymptotics of the pion form factor at high momentum transfers

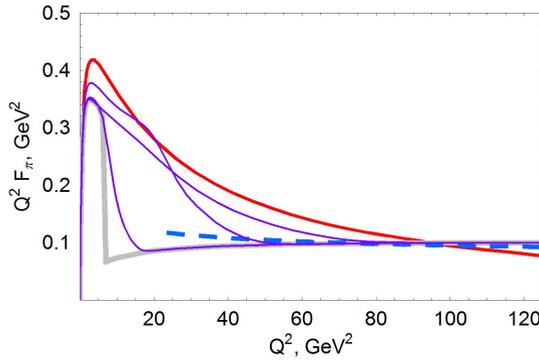
It is worth to consider the pion form factor asymptotic behavior (14) at  $Q^2 \rightarrow \infty$  especially. In our paper [17] it was shown that in our approach, the pion form-factor asymptotics at

$$Q^2 \rightarrow \infty, \quad M(Q^2) \rightarrow 0 \quad (23)$$

does not depend on the choice of a wave function but is defined by the relativistic kinematics only. We consider the fact that the asymptotics obtained in our nonperturbative approach does coincide with that predicted by QCD as a very significant one. Our approach occurs to be consistent with the asymptotic freedom, and this feature surely distinguishes it from other nonperturbative approaches.

Let us note that it is obvious that at very high momentum transfers the quark mass decreases as it goes to zero at the infinity. Our approach permits to take into account the dependence  $M(Q^2)$  beginning from the range where this becomes necessary to correspond to experimental data. It is possible that this will take place at the values of  $Q^2$  lower than  $6 \text{ GeV}^2$ . So, it is obtained in Ref. [17] that

$$F_\pi(Q^2) \sim Q^{-2}. \quad (24)$$



**Figure 1.** Examples of the allowed solutions for  $F_\pi(Q^2)$  (thin lines) demonstrating how the QCD asymptotics, Eq.(25) (dashed line) settles down. The thick gray line bounds from below the range of all solutions allowed by the experimental constraints. The thick full (red) line represents the solution with  $M = \text{const}$ , Refs. [12, 26].

The asymptotics of the pion electromagnetic form factor  $F_\pi$  at momenta transfer  $Q^2 \rightarrow \infty$  has been determined [20–22], in the QCD frameworks, as

$$Q^2 F_\pi(Q^2) \rightarrow 8\pi\alpha_s^{1\text{-loop}}(Q^2)f_\pi^2, \quad (25)$$

where  $\alpha_s^{1\text{-loop}}(Q^2) = 4\pi/(\beta_0 \log(Q^2/\Lambda_{\text{QCD}}^2))$  is the one-loop running strong coupling constant,  $\beta_0 = 11 - 2N_f/3$  is the first beta-function coefficient,  $N_f$  is the number of active quark flavours and  $f_\pi \approx 130$  MeV is the pion decay constant. It is important to note that this asymptotical behaviour, consistent with the quark counting rules [23, 24], includes the one-loop coupling only and is to be modified whenever the one-loop approximation fails, but not by means of a simple replacing of  $\alpha_s$  with its more precise value. Involved QCD calculations have been performed to obtain corrections to Eq. (25), see e.g. [25]. The QCD does not predict the value of  $Q^2$  at which this asymptotics should be reached.

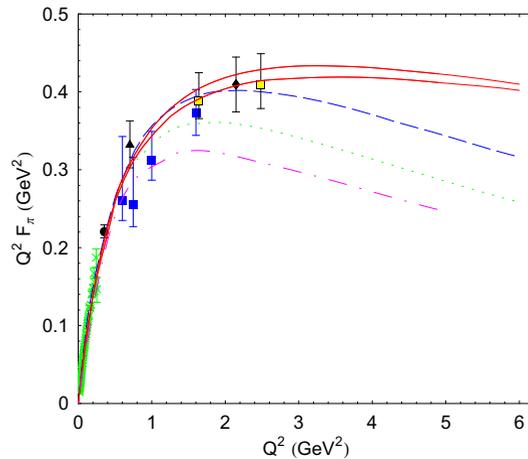
The results of the calculation of  $F_\pi$  from Ref. [18] are presented in Fig. 1. As can be seen the our asymptotic coincides with QCD prediction at different scenarios for the zero limit of constituent mass (23).

## 6 The pion form factor in the region of the JLab experiments

The results of the calculation of the charge pion form factor using the wave functions (18), (19) and the value of constituent-quark mass  $M = 0.22$  GeV (this parameter has been fixed as early as in 1998 [12] from the data at  $Q^2 \leq 0.26$  (GeV)<sup>2</sup> [27]) are shown in Fig. 2. Let us note that our RQM describes well the experimental data for the pion form factor including the recent points [3]. The upper of our curves corresponds to the model (18), the lower - to the models (19) with  $n = 3$ .

Let us emphasize that the parameters used in our calculations were obtained from the fitting to the experimental data up to  $Q^2 \approx 0.26$  GeV<sup>2</sup> [27]. At that time the data for higher  $Q^2$  was not correlated in different experiments and had significant uncertainties. The later data for pion form factor in JLab experiments up to  $Q^2 = 2.45$  GeV<sup>2</sup> were obtained with rather good accuracy. All experimental points obtained in JLab up to now agree very well with our prediction of 1998.

So, our way of fixing the model parameters constrains effectively the behavior of wave functions both at small and at large relative momenta. The structure of our relativistic integral representation



**Figure 2.** Our predictions for the pion form factor given in 1998 [12] (full red lines; upper line: wave functions (18), lower line: wave functions (19) with  $n = 3$ ) compared with data and with some other models. Experimental data are from Refs. [2, 3, 27–29]. Other theoretical curves are: those by C.-W. Hwang [30] (blue dashed); Cardarelli *et al.* [19] and (precisely coinciding with it) instant-form predictions [31] (magenta dash-dotted); Ref. [32] (green dotted).

(14) is so, that the form-factor behavior in the region of small momentum transfers is determined by the wave function at small relative momenta, and the behavior of the form factor in the region of high momentum transfer — by the wave function at large relative momenta. The constraints for the wave functions provide the limitations for the form factor, and this is seen in the results of the calculation.

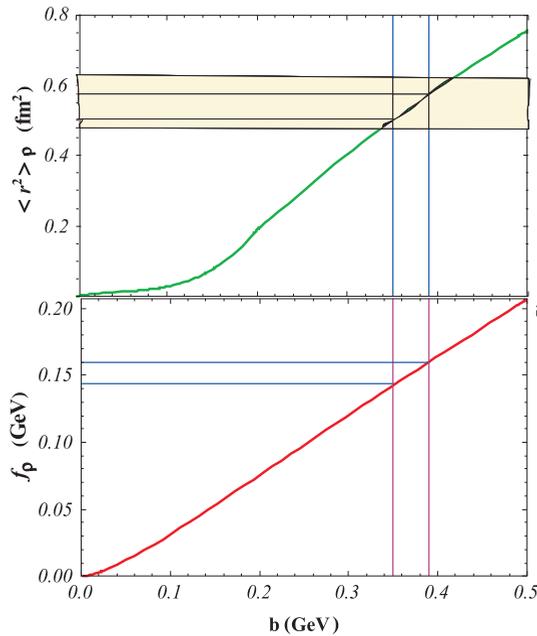
## 7 Lepton decay constant and MSR of the $\rho$ -meson

Let us describe the procedure of calculation of the  $\rho$ -meson MSR in detail, starting from the quark parameter,  $M = 0.22$  GeV, used in a successful calculation of the pion parameters [12]. As it has been demonstrated in Ref. [12], the actual choice of the wave-function form does not affect the result provided the quark parameters are fixed. In what follows, we illustrate the procedure with the wave function (19) with  $n = 3$ .

In the lower panel of Fig. 3, the  $\rho$ -meson lepton decay constant  $f_\rho$  as a function of the only free parameter of the model,  $b_\rho$ , is presented. The interval on the vertical axis representing the experimental values of  $f_\rho$ , that is  $f_\rho^{\text{exp}} = (152 \pm 8)$  MeV [4, 5], is shown. It corresponds to the interval of the values of  $b_\rho$  which give, through our calculation, the correct experimental values of the decay constant. This interval,  $b_\rho = (0.385 \pm 0.019)$  GeV, is shown on the horizontal axis of Fig. 3.

The calculated MSR of the  $\rho$  meson is presented in the upper panel of Fig. 3. The interval of admissible values of  $b_\rho$  gives now the corresponding interval of MSR predicted in the present study,  $\langle r_\rho^2 \rangle = (0.56 \pm 0.04)$  fm<sup>2</sup>.

Table 1 presents a comparison of our results with the results of calculations of electroweak properties of the  $\rho$  meson in other approaches. The values of  $\langle r_\rho^2 \rangle$ , while not measured directly, are important for testing various conjectures about strongly interacting systems. One of the interesting related prediction was introduced as a consequence of the so-called Wu–Yang hypothesis [38] (see also



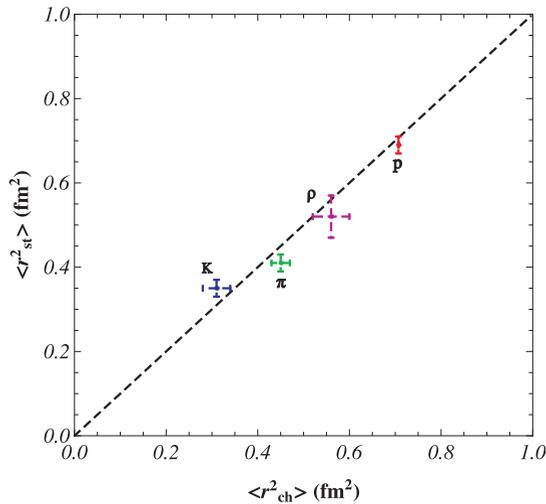
**Figure 3.** The decay constant  $f_\rho$  and the  $\rho$ -meson MSR,  $\langle r_\rho^2 \rangle$ , as functions of the only free parameter of the model,  $b_\rho$ . The experimental data fix the value of  $f_\rho$ , as shown in the lower panel. This fixes the value of  $b_\rho$ , which determines in turn the value of  $\langle r_\rho^2 \rangle$ , as shown in the upper panel. The values of quark parameters are taken from Ref. [12]; the wave function (19) with  $n = 3$  is used.

**Table 1.** The lepton decay constant  $f_\rho$  and MSR of the  $\rho$  meson calculated within different approaches.

Model	$f_\rho$ , MeV	$\langle r_\rho^2 \rangle$ , fm <sup>2</sup>
<b>This work</b>	$152 \pm 8$ (fixed)	$0.56 \pm 0.04$
[4]	154	0.268
[33]	146	0.54
[34]	130	0.312
[35]	—	0.67
[36]	—	0.67
[37]	134	0.296

Ref. [39]), though it is remarkable by itself. Namely, one may define the radius of a hadron either in terms of the electroweak interaction (the mean square charge radius,  $\langle r_{\text{ch}}^2 \rangle$ , calculated for the  $\rho$  meson in this paper) or in terms of the strong interaction (this radius,  $\langle r_{\text{st}}^2 \rangle$ , is defined by the slope of the cross section of hadron–proton scattering). The conjecture [39], which may be derived from, though not necessary implies, the hypothesis of Ref. [38], is the equality of the two radii,

$$\langle r_{\text{st}}^2 \rangle = \langle r_{\text{ch}}^2 \rangle. \quad (26)$$



**Figure 4.** Relation between the strong-interaction hadronic radius  $\langle r_{st}^2 \rangle$  and the charge radius  $\langle r_{ch}^2 \rangle$  for light hadrons.

This remarkable equality has been verified experimentally with a great degree of accuracy for the proton,  $\pi$ - and  $K$ - mesons.

Even more demonstrative is Fig. 4, analogous to a figure from the paper [39], but presenting more recent data. We can see that the value of the  $\rho$ -meson charge radius obtained in this paper fits perfectly the conjecture (26).

## 8 Analytic continuation of the pion form factor to the complex plain of momentum transfers

As mentioned above, one unsolved theoretical problem of RQM is its relations to the fundamental QFT principles. This paper partially addresses this issue. We compare the corollaries of the QFT principles with the model-independent corollaries of the RQM axioms. In Ref. [40] we compared the predictions for analytic properties of the pion form factor in the complex plane of transferred momenta  $t = -Q^2$  that follow from QFT general principles with those obtained by the analytic continuation of the form factor integral representation (14) in the spacelike domain derived in the framework of the instant form of RQM [13]. It follows from the QFT microcausality condition that the pion form factor is an analytic function in the complex plane of the parameter  $t$  with a cut running from  $4m_\pi^2$  to  $\infty$ , where  $m_\pi$  is the pion mass (see, e.g., [41] and the references therein). The obtained in our approach analytic properties of the form factor differ from those obtained in the QFT approach only by the position of the branch point on the real axis: it is  $4M^2$  in our case. The main experimentally observed qualitative feature of the form factor behavior in the complex plane is the presence of two resonances, which correspond to  $\rho$  and  $\omega$  mesons and are located on the positive part of the real axis (in the timelike domain). We can demonstrate that wave functions in the momentum representation without poles in the momentum complex plane do not yield the resonance behavior of the pion form factor for timelike transferred momenta. In particular, this is the case for a wave function of the Gaussian type (18) widely used in composite models, and this result holds independently of the model parameter

values. We therefore encounter the problem of finding a wave function that produces resonances and establishing the relation between the locus of the resonance in the pion form factor and the locus of the wave function poles. In solving this problem, the main idea is to introduce wave functions with poles such that poles of form factor (14) appear on the nonphysical sheet.

So, the wave functions  $u(k)$  (15) resulting in the resonance behavior of the pion form factor for timelike transferred momenta must have a pole at the point  $k_s$ :

$$k_s^2 = \frac{M}{4}(z_s - 2M), \quad z_s = z_s'' + i\sqrt{t_r - 4M^2}, \quad (27)$$

where  $t_r$  is resonance location,  $z_s'' < 0$ .

Let the function  $u(k)$  (15) be even and satisfies condition of reality ( $\text{Im } u(k) = 0$ ) at real  $k$  also. The simplest rational function satisfying all the above conditions is

$$u_M(k) = N \left[ (k^2 - k_s^2)(k^2 - (k_s^*)^2) \right]^{-n}. \quad (28)$$

The obtained analytic continuation with wave functions of form (28) therefore contains the following parameters:  $t_r$  and  $z_s''$  from (27), the constituent mass  $M$ , and the exponent  $n$  of the wave function. We fix these data by fitting experimental data for the parameters of the  $\rho$  resonance in the pion form factor.

To describe the experimental data pertaining to the pion form factor, we use the fitting by the celebrated Breit-Wigner formula (see [42]):

$$F_{\pi BW}(t) = \frac{4\kappa^2}{4\kappa^2 - t - 2i\kappa\Gamma}, \quad (29)$$

where  $\kappa = 0.375$  GeV,  $\Gamma = 0.1$  GeV. (we take the parameters from [42]).

We numerically tuned the parameters of our model such that the absolute value of the form factor best agrees with the Breit-Wigner approximation of the experimental data, which gives

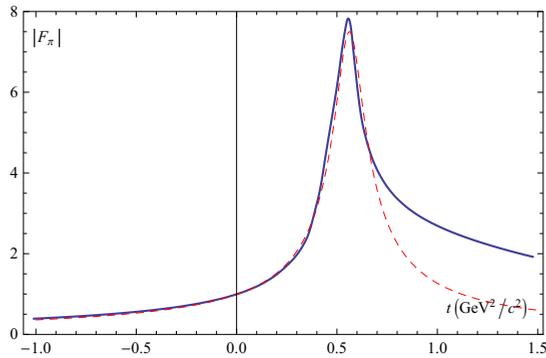
$$t_r = 0.57 \text{ GeV}^2, \quad z_s'' = -0.01, \quad M = 0.32 \text{ GeV}, \quad n = 0.86. \quad (30)$$

We note that the parameter  $t_r$  is the resonance mass, and its value is very close to the value  $0.5625 \text{ GeV}^2$  provided by formula (29); the value of the constituent quark mass is typical for different formulations of the composite quark model in which constituent masses are in the interval  $(0.2 - 0.33) \text{ GeV}$  (see, e.g., [12, 19]). The parameter  $z_s''$  determining the position of the form factor singularity on the nonphysical sheet and the wave function parameter  $n$  are purely model parameters determined by the tuning procedure.

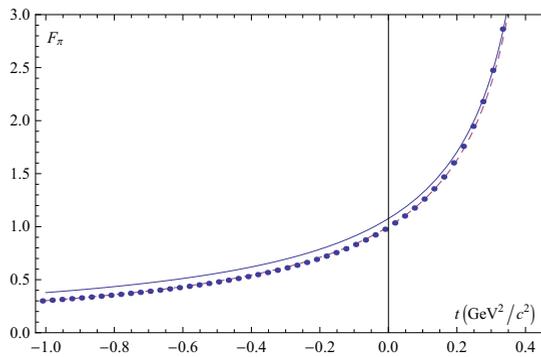
The results of numerically integrating expression (14) with wave function (28) are shown in Fig. 5. We can see that our analytic continuation of pion form factor (14) provides a good qualitative description of the form factor behavior in the timelike domain already for the simplest wave function choice given by (28).

We can demonstrate the correctness of our constructed analytic continuation as follows. We verify how well the constructed form factor satisfies the known dispersion relation in the spacelike domain

$$F_\pi(t) = \frac{1}{\pi} \int_{4M^2}^{\infty} \frac{\text{Im}[F_\pi(t')]}{t' - t} dt', \quad (31)$$



**Figure 5.** The function  $|F_\pi(t)|$  from: the solid curve is expression (14), and the dashed curve is the Breit-Wigner approximation.



**Figure 6.** Comparing the calculated pion form factor with the known dispersion relations: points correspond to a straightforward calculation with formula (14), the solid line is obtained from dispersion relation (31), and the dashed line is obtained from dispersion relation (32). The results of calculating by formula (14) practically coincide with those obtained using dispersion relation (32).

and also the dispersion relation with one subtraction (with the weakened condition of decreasing at infinity)

$$F_\pi(t) = 1 + \frac{t}{\pi} \int_{4M^2}^{\infty} \frac{\text{Im}[F_\pi(t')]}{t'(t'-t)} dt'. \quad (32)$$

In Fig. 6, we compare our result with formulas (31) and (32). We use parameter values (30). It can be seen that the constructed pion form factor in the complex plane satisfies the dispersion relations with good accuracy.

The problem of a more detailed description of the character of the resonance behavior of the pion form factor in the framework of our approach is related to the problem of choosing a wave function of interacting quarks that is more realistic than (28).

## 9 Conclusions

The approach developed in IF RQM has following main features: predictivity: parameters of model are fixed from experimental data at small momentum transfers. The experimental data for the pion form factor at large momentum transfers obtained later are described without tuning of parameters; robustness: the behavior of the pion and  $\rho$ -meson electromagnetic form factors does not depend on the choice of wave functions and are determined by the mass of the constituent quarks; the approach gives the asymptotics that agrees with QCD asymptotic behavior at large momentum transfers; the approach gives the self-consistent description of the electroweak properties of the pion and  $\rho$ -meson; the approach gives the right analytical properties of the pion form factor in the complex plane of momentum transfers.

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