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Janus-Facedness of the Pion: Analytic Instantaneous Bethe–Salpeter Models

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Abstract. Inversion enables the construction of interaction potentials underlying — under fortunate circumstances even analytic — instantaneous Bethe–Salpeter descriptions of all lightest pseudoscalar mesons as quark–antiquark bound states of Goldstone-boson nature.

1 Introduction: quark–antiquark bound states of Goldstone-boson identity

Within quantum chromodynamics, the pions or, as a matter of fact, all *light pseudoscalar mesons* must be interpretable as both *quark–antiquark bound states* and almost massless (pseudo) *Goldstone bosons* related to the spontaneously (and, to a minor extent, also explicitly) broken chiral symmetries of QCD.

Relativistic quantum field theory describes bound states by their Bethe–Salpeter amplitudes, $\Phi(p)$, controlled by the *homogeneous Bethe–Salpeter equation* defined (for two bound particles of individual and relative momenta $p_{1,2}$ and p) by their full propagators $S_{1,2}(p_{1,2})$ and the integral kernel $K(p,q)$ that encompasses their interactions (notationally suppressing dependences on the total momentum p_1+p_2):

$$\Phi(p) = \frac{\mathrm{i}}{(2\pi)^4}\, S_1(p_1) \int \mathrm{d}^4q\, K(p,q)\, \Phi(q)\, S_2(-p_2) .$$

The application of suitably adapted *inversion* techniques [1] allows us to retrieve all the underlying interactions — rooted, of course, in QCD — analytically in the form of a (configuration-space) *central potential* $V(r)$, $r \equiv |\boldsymbol{x}|$, from presumed solutions to the Bethe–Salpeter equation [2]. By that, we are put in a position to construct *exact analytic* Bethe–Salpeter *solutions* for all massless pseudoscalar mesons [3] in the sense of establishing in a rigorous manner the analytic relationships between interactions and resulting solutions: all analytic findings [4] can be confronted with associated numerical outcomes [5].

2 Sequence of simplifying assumptions crucial for the inversion formalism

By a few steps, we cast the Bethe–Salpeter equation into a shape that allows us to talk about potentials.

1. Assuming, for each involved quark, both *instantaneous* interactions and *free* propagation, with a mass dubbed as *constituent*, simplifies the Bethe–Salpeter equation to a bound-state equation for the *Salpeter amplitude* $\phi(\boldsymbol{p})$, obtained from the Bethe–Salpeter amplitude by integration over p_0:

$$\phi(\boldsymbol{p}) \propto \int \mathrm{d}p_0\, \Phi(p) .$$

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Generically, for a spin-$\frac{1}{2}$ fermion and a spin-$\frac{1}{2}$ antifermion of equal constituent masses m, bound to a spin-singlet state (which, for instance, clearly is the case for any such pseudoscalar state), its three-dimensional wave function involves just two independent components, here called $\varphi_{1,2}(\boldsymbol{p})$:

$$\phi(\boldsymbol{p}) = \left[\varphi_1(\boldsymbol{p})\,\frac{\gamma_0\,(\boldsymbol{\gamma}\cdot\boldsymbol{p}+m)}{E(p)} + \varphi_2(\boldsymbol{p})\right]\gamma_5\,, \qquad E(p) \equiv \sqrt{\boldsymbol{p}^2+m^2}\,, \qquad p \equiv |\boldsymbol{p}|\,.$$

2. Upon supposing that the quark interactions in the kernel respect spherical and Fierz symmetries, our bound-state equation for $\phi(\boldsymbol{p})$ collapses to the system of *coupled radial eigenvalue equations*

$$2\,E(p)\,\varphi_2(p) + 2\int\limits_0^\infty \frac{\mathrm{d}q\,q^2}{(2\pi)^2}\,V(p,q)\,\varphi_2(q) = \widehat{M}\,\varphi_1(p)\,, \qquad 2\,E(p)\,\varphi_1(p) = \widehat{M}\,\varphi_2(p)\,, \qquad q \equiv |\boldsymbol{q}|\,,$$

   for the bound-state mass eigenvalue $\widehat{M}$ [6]. Therein, $V(r)$ enters via its Fourier–Bessel transform

$$V(p,q) \equiv \frac{8\pi}{p\,q}\int\limits_0^\infty \mathrm{d}r\,\sin(p\,r)\,\sin(q\,r)\,V(r)\,.$$

3. In the strictly massless (Goldstone) case $\widehat{M}=0$, the system decouples: one Salpeter component, $\varphi_1(p)$, is doomed to vanish, $\varphi_1(p)\equiv 0$, whereas the surviving Salpeter component $\varphi_2(p)$ satisfies

$$E(p)\,\varphi_2(p) + \int\limits_0^\infty \frac{\mathrm{d}q\,q^2}{(2\pi)^2}\,V(p,q)\,\varphi_2(q) = 0\,.$$

   Denoting the Fourier–Bessel transform of the kinetic term $E(p)\,\varphi_2(p)$ by $T(r)$, the potential $V(r)$ may be simply read off from the configuration-space representation of this bound-state equation:

$$T(r) + V(r)\,\varphi_2(r) = 0 \qquad \Longrightarrow \qquad V(r) = -\frac{T(r)}{\varphi_2(r)}\,.$$

3 Constraints on lightest-pseudoscalar-meson Bethe–Salpeter amplitudes

Information on the input Salpeter component $\varphi_2(p)$ can be gained from the *full quark propagator* $S(p)$, which is determined by its mass function $M(p^2)$ and its wave-function renormalization function $Z(p^2)$:

$$S(p) = \frac{\mathrm{i}\,Z(p^2)}{\not{p} - M(p^2) + \mathrm{i}\,\varepsilon}\,, \qquad \not{p} \equiv p^\mu\,\gamma_\mu\,, \qquad \varepsilon \downarrow 0\,.$$

Studies of $S(p)$ within the Dyson–Schwinger framework, preferably done in Euclidean space signalled by underlined quantities, allow for pivotal insights. *In the chiral limit*, a Ward–Takahashi identity links [7] this quark propagator to the flavour-nonsinglet *pseudoscalar-meson Bethe–Salpeter amplitude* [3]:

$$\Phi(\underline{k}) \approx \frac{M(\underline{k}^2)}{\underline{k}^2 + M^2(\underline{k}^2)}\,\underline{\gamma}_5 + \text{subleading contributions}\,.$$

First, in order to devise *analytically* accessible scenarios, we exploit two crucial pieces of information:

1. *In the chiral limit*, phenomenologically sound Dyson–Schwinger studies [8] imply, for the quark mass function $M(\underline{k}^2)$, at large Euclidean momenta $\underline{k}^2$ a decrease essentially proportional to $1/\underline{k}^2$.

2. From axiomatic quantum field theory, we may deduce [9] that the presence of an *inflection point at finite space-like momenta* $\underline{k}^2 > 0$ in the quark mass function $M(\underline{k}^2)$ entails *colour confinement*.

Of course, any imposition of such kind of requirements on $M(\underline{k}^2)$ has to be *reflected by* $\Phi(\underline{k})$. An *ansatz* for $\Phi(\underline{k})$ compatible with both constraints, involving a mass parameter, μ, and a mixing parameter, η, is

$$\Phi(\underline{k}) = \left[\frac{1}{(\underline{k}^2+\mu^2)^2} + \frac{\eta\,\underline{k}^2}{(\underline{k}^2+\mu^2)^3}\right]\underline{\gamma}_5 , \qquad \mu > 0 , \qquad \eta \in \mathbb{R} .$$

An integration of this $\Phi(\underline{k})$ with respect to the time component of the Euclidean momentum $\underline{k}$ results in

$$\varphi_2(p) \propto \frac{1}{(p^2+\mu^2)^{3/2}} + \eta\,\frac{p^2+\mu^2/4}{(p^2+\mu^2)^{5/2}} , \qquad p \equiv |\boldsymbol{p}| ,$$

in configuration space expressible in terms of modified Bessel functions of the second kind $K_\sigma(z)$ [10]:

$$\varphi_2(r) \propto 4\,(1+\eta)\,K_0(\mu\,r) - \eta\,\mu\,r\,K_1(\mu\,r) . \tag{1}$$

For η values satisfying $\eta < -1$ or $\eta > 0$, $\varphi_2(r)$ has one zero, which clearly induces a *singularity* in $V(r)$.

4 Analytic outcomes [3, 4] for interquark potentials exhibiting confinement

For a few particular values of the dimensionless ratio m/μ, the *analytic* expression of $V(r)$ can be found [3, 4]. (Throughout this section, any quantity has to be understood in units of the adequate power of μ.) As a consequence of our ansatz for $\Phi(\underline{k})$, giving rise to the particular form (1) of $\varphi_2(r)$, for $\eta \neq -1$ each extracted $V(r)$ will develop, at the spatial origin $r = 0$, a logarithmically softened Coulomb singularity:

$$V(r) \xrightarrow[r\to 0]{} \frac{\text{const}}{r\ln r} \xrightarrow[r\to 0]{} -\infty \qquad (\text{const} > 0) \qquad \text{for } \eta \neq -1 .$$

4.1 Analytically manageable scenario of massless quarks, i.e., of constituent mass $m = 0$

For our choice of $\varphi_2(r)$, $V(r)$ involves both modified Bessel (I_n) and Struve ($\mathbf{L}_n$) functions [10] ($n \in \mathbb{N}$), and rises in a confinement-betraying manner to infinity either at the zero of $\varphi_2(r)$ or for $r \to \infty$ (Fig. 1):

$$V(r) = \frac{\pi\,[4+\eta\,(4+r^2)]\,[\mathbf{L}_0(r)-I_0(r)] + \pi\,(4+5\,\eta)\,r\,[\mathbf{L}_1(r)-I_1(r)] + 4\,(2+3\,\eta)\,r}{2\,r\,[4\,(1+\eta)\,K_0(r)-\eta\,r\,K_1(r)]} .$$

4.2 Analytically expressible observation for quarks with common constituent mass $m = \mu$

For $m = \mu$, the kinetic term $T(r)$ is a mixture of Yukawa and exponential behaviour, whence (cf. Fig. 2)

$$V(r) = -\frac{\pi\,[8+\eta\,(8-3\,r)]\exp(-r)}{4\,r\,[4\,(1+\eta)\,K_0(r)-\eta\,r\,K_1(r)]} \xrightarrow[r\to\infty]{} -\frac{\text{const}}{\sqrt{r}} \xrightarrow[r\to\infty]{} 0 \qquad (\text{const} > 0) .$$

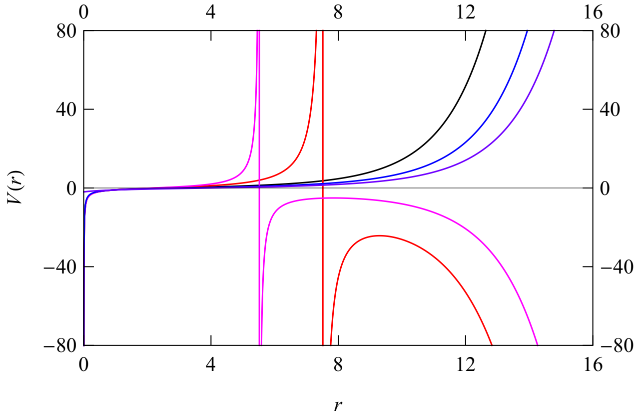

Figure 1. Configuration-space interquark potential $V(r)$ of the Fierz-symmetric kernel $K(p,q)$, for the constituent quark mass $m = 0$ and mixture $\eta = 0$ [3] (black), $\eta = 1$ (red), $\eta = 2$ (magenta), $\eta = -0.5$ (blue), or $\eta = -1$ (violet).

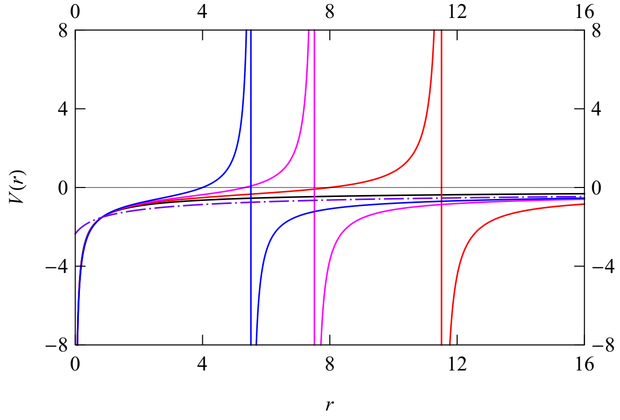

Figure 2. Configuration-space interquark potential $V(r)$ of the Fierz-symmetric kernel $K(p,q)$, for the constituent quark mass $m = 1$ and mixture $\eta = 0$ [3] (black), $\eta = 0.5$ (red), $\eta = 1$ (magenta), $\eta = 2$ (blue), and $\eta = -1$ (violet).

5 Reliability check of findings: numerical determination of the potential [5]

Our findings may be scrutinized by use of the chiral-limit quark mass function's pointwise form $M(\underline{k}^2)$, provided graphically in Ref. [8] and shown in Fig. 3 as $M(\underline{k})$ with $\underline{k} \equiv (\underline{k}^2)^{1/2}$, which we parametrize by

$$M(\underline{k}) = 0.708\ \text{GeV} \exp\left(-\frac{\underline{k}^2}{0.655\ \text{GeV}^2}\right) + \frac{0.0706\ \text{GeV}}{\left[1 + \left(\frac{\underline{k}^2}{0.487\ \text{GeV}^2}\right)^{1.48}\right]^{0.752}} .$$

Note that the *product* of the two exponents in the second term above yields $1.48 \times 0.752 \approx 1.1$, which is pretty close to unity, as demanded by the large-$\underline{k}$ constraint. Feeding this $M(\underline{k})$ parametrization into our inversion procedure, we obtain potentials that are finite at $r = 0$ and, for sufficiently small m, rise with r to infinity but, for large m, remain negative, as illustrated in Fig. 3 for selected constituent mass values.

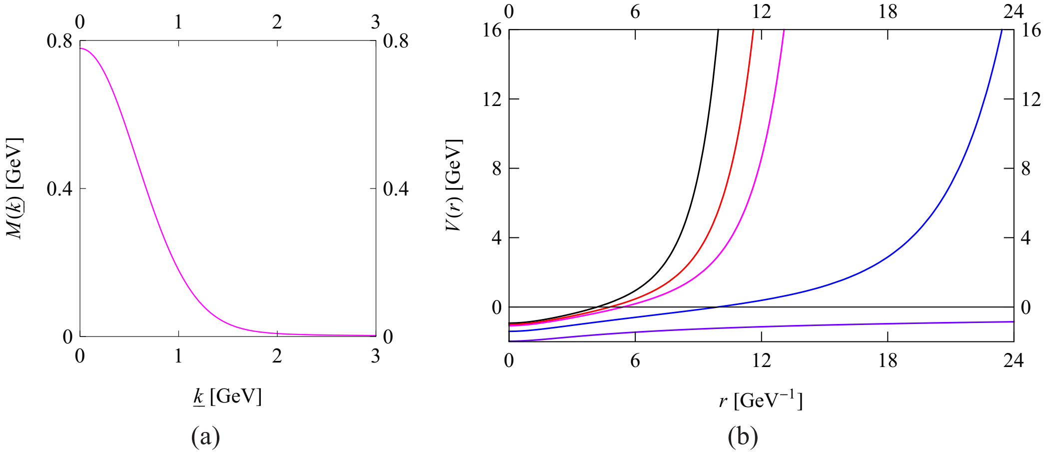

Figure 3. (a) Mass function $M(\underline{k})$ deduced from the Dyson–Schwinger model of Ref. [8] for the quark propagator. (b) Configuration-space interquark potential $V(r)$ numerically determined from $M(\underline{k}^2)$, for constituent quark mass $m = 0$ (black), $m = 0.35$ GeV (red), $m = 0.5$ GeV (magenta), $m = 1.0$ GeV (blue), and $m = 1.69$ GeV (violet) [5].

6 Summary of results, observations, discussion, conclusion, perspectives

We constructed confining potentials $V(r)$ that in cooperation with a Fierz-symmetric interaction kernel describe massless pseudoscalar quark–antiquark bound-state solutions of the Bethe–Salpeter equation. This is possible even analytically if focusing to specific aspects of the quark mass function's behaviour. Two obstacles call for a particularly careful treatment: Numerically, $M(p^2)$ is known for only a limited range of p^2. For large r, both $T(r)$ and $\varphi_2(r)$ approach zero; thus, pinning down $V(r)$ in the limit $r \to \infty$ boils down to a division of zero by zero. Dropping the free quark propagation constraint [11] allows us to thoroughly take into account the effects of $M(p^2)$ *and* the quark wave-function renormalization [12].

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