

Dyson–Schwinger Approach to Hamiltonian QCD

Davide Campagnari^{1,a}, Hugo Reinhardt¹, Markus Q. Huber², Peter Vastag¹, and Ehsan Ebadati¹

¹*Institut of Theoretical Physics, University of Tübingen, Auf der Morgenstelle 14, 72076 Tübingen, Germany*

²*Institute of Physics, University of Graz, NAWI Graz, Universitätsplatz 5, 8010 Graz, Austria*

Abstract. Dyson–Schwinger equations are an established, powerful non-perturbative tool for QCD. In the Hamiltonian formulation of a quantum field theory they can be used to perform variational calculations with non-Gaussian wave functionals. By means of the DSEs the various n -point functions, needed in expectation values of observables like the Hamilton operator, can be thus expressed in terms of the variational kernels of our trial ansatz. Equations of motion for these variational kernels are derived by minimizing the energy density and solved numerically.

1 Introduction

Coulomb gauge Yang–Mills theory has been investigated in the continuum both in the Hamiltonian [1–4] and the Lagrangian [5–8] approach. In the Hamiltonian approach, the use of variational methods with Gaussian wave functionals [3, 4] has led to an equal-time gluon propagator which behaves like the photon in the UV but is strongly suppressed in the IR, signalling confinement. The obtained propagator also compares favourably with the available lattice data [9]; some deviations in the mid-momentum regime can be accounted for by using non-Gaussian wave functionals [10]. Such functionals can be handled by means of DSEs, exploiting the formal similarity between vacuum expectation values in the Hamiltonian formalism and correlation functions in Euclidean quantum field theory. The emerging DS-type of equations were dubbed canonical recursive Dyson–Schwinger equations [10, 11]. In this talk we report about the derivation of the CRDSEs for full QCD and their application to the study of the spontaneous breaking of chiral symmetry.

Chiral symmetry breaking in Coulomb gauge has been studied for example in Refs. [12–14]. While it has been shown that an infrared enhanced potential can account for chiral symmetry breaking, the calculated physical quantities, such as the dynamical mass and chiral condensate, turn out to be far too small. A wave functional including the coupling of the quarks to the transverse gluons improves the results towards the phenomenological findings [15]. We apply here the techniques developed in Refs. [10, 11] to the wave functional proposed in Ref. [15] and extended in Refs. [16, 17].

2 Hamiltonian Approach to Quantum Chromodynamics

Canonical quantization of QCD relies usually on the temporal gauge $A_0^a = 0$, and results in a functional Schrödinger equation. Gauge invariance of the wave functional is enforced by the Gauss law, which

^aPresenter; e-mail: d.campagnari@uni-tuebingen.de.

can be explicitly solved in Coulomb gauge, yielding the gauge-fixed Hamiltonian

$$\begin{aligned}
H = & \frac{1}{2} \int d^3x \left(J_A^{-1} \frac{\delta}{\delta A_i^a(\mathbf{x})} J_A \frac{\delta}{\delta A_i^a(\mathbf{x})} + B_i^a(\mathbf{x}) B_i^a(\mathbf{x}) \right) \\
& + \frac{g^2}{2} \int d^3x d^3y J_A^{-1} \rho^a(\mathbf{x}) J_A F_A^{ab}(\mathbf{x}, \mathbf{y}) \rho^b(\mathbf{x}) \\
& + \int d^3x \hat{\psi}^\dagger(\mathbf{x}) [-i\alpha_i \partial_i + \beta m - g\alpha_i t_a A_i^a(\mathbf{x})] \hat{\psi}(\mathbf{x}). \quad (1)
\end{aligned}$$

Here, β and α_i are the usual Dirac matrices; t_a are the generators of the gauge group in the fundamental representation (or, in general, in the representation to which the fermion fields belong); $J_A = \text{Det}(G_A^{-1})$ is the Faddeev–Popov determinant; and

$$[G_A^{ab}(\mathbf{x}, \mathbf{y})]^{-1} = (-\delta^{ab} \partial^2 - g f^{abc} A_i^c(\mathbf{x}) \partial_i) \delta(\mathbf{x} - \mathbf{y}) \quad (2)$$

is the Faddeev–Popov operator, with g being the coupling constant and f^{abc} being the structure constants of the $\text{su}(N_c)$ algebra. The second term in Eq. (1) represents the Coulomb-like interactions of the total colour charge

$$\rho^a = \hat{\psi}^\dagger t_a \hat{\psi} - i f^{abc} A_i^b \frac{\delta}{\delta A_i^c}$$

through the Coulomb kernel

$$F_A \equiv G_A(-\nabla^2) G_A. \quad (3)$$

3 Dyson–Schwinger Equations in the Hamiltonian Approach

The derivation of the CRDSEs in pure Yang–Mills theory has been presented in Ref. [10], and the generalization to full QCD has been given in Ref. [11]. Here we summarize briefly the steps leading from the choice of the wave functional to the corresponding CRDSEs.

A state $|\Phi\rangle$ in the fermionic Fock space is conveniently represented by coherent-states defined in terms of Grassmann fields ξ and ξ^\dagger [11]

$$\langle \xi | \Phi \rangle \equiv \Phi(\xi^\dagger, \xi). \quad (4)$$

The Dirac field operators $\hat{\psi}$ and $\hat{\psi}^\dagger$ act onto this state as

$$\hat{\psi} \Phi(\xi^\dagger, \xi) = \left(\xi_- + \frac{\delta}{\delta \xi_+^\dagger} \right) \Phi(\xi^\dagger, \xi), \quad \hat{\psi}^\dagger \Phi(\xi^\dagger, \xi) = \left(\xi_+^\dagger + \frac{\delta}{\delta \xi_-} \right) \Phi(\xi^\dagger, \xi), \quad (5)$$

where $\xi_\pm = \Lambda_\pm \xi$, and Λ_\pm are the projectors onto states of positive and negative energy of the free Dirac operator. We have put explicitly a hat over the fermion operators $\hat{\psi}$, $\hat{\psi}^\dagger$ in Eq. (5) to distinguish them from the classical Grassmann fields ξ , ξ^\dagger used in the coherent-state representation.

The expectation value in the vacuum state Ψ of an operator K depending on bosonic and fermionic fields is given by

$$\langle K[A, \Pi, \hat{\psi}, \hat{\psi}^\dagger] \rangle = \int \mathcal{D}\xi^\dagger \mathcal{D}\xi \mathcal{D}A J_A e^{-\xi^\dagger (\Lambda_+ - \Lambda_-) \xi} \Psi^*[A, \xi^\dagger, \xi] K[A, \frac{\delta}{i\delta A}, \hat{\psi}, \hat{\psi}^\dagger] \Psi[A, \xi^\dagger, \xi]. \quad (6)$$

In Eq. (6) the functional integration runs over transverse gauge field configurations satisfying the Coulomb gauge condition, $\partial_i A_i^a = 0$, and is restricted to the first Gribov region; the exponential factor

occurring in the fermionic functional integration in Eq. (6) arises from the completeness relation of the coherent fermion states [11, 18]. Once the fermion field operators and the gluon canonical momentum act onto the vacuum wave functional, Eq. (6) boils down to an expectation value of some functional of the fields to be integrated over.

The vacuum state of QCD can be assumed to be of the form

$$\Psi[A, \xi, \xi^\dagger] =: \exp \left\{ -\frac{1}{2} S_A[A] - S_f[\xi, \xi^\dagger, A] \right\}, \quad (7)$$

where S_A is a functional of the gauge field only, while S_f contains both the fermion and the gluon fields. The expectation value of a function $f(A, \xi, \xi^\dagger)$ of the fields is a path integral of the form

$$\langle F \rangle = \int \mathcal{D}A \mathcal{D}\xi^\dagger \mathcal{D}\xi J_A e^{-\xi^\dagger(\Lambda_+ - \Lambda_-)\xi} e^{-S_A - S_f - S_f^*} f(A, \xi, \xi^\dagger)$$

which looks like a correlation function of a Euclidean field theory with ‘‘action’’

$$S = S_A + S_f + S_f^* + \xi^\dagger(\Lambda_+ - \Lambda_-)\xi - \text{Tr} \ln G_A^{-1}$$

Correspondingly, we can derive Dyson–Schwinger equations for the gluon and quark correlators by starting from the identity

$$0 = \int \mathcal{D}A \mathcal{D}\xi^\dagger \mathcal{D}\xi \frac{\delta}{\delta\phi} \left\{ J_A e^{-\xi^\dagger(\Lambda_+ - \Lambda_-)\xi} \Psi^*[A, \xi, \xi^\dagger] K[A, \xi, \xi^\dagger] \Psi[A, \xi, \xi^\dagger] \right\} \quad (8)$$

with $\phi \in \{A, \xi, \xi^\dagger\}$, while ghost DSEs can be derived by starting from the operator identity Eq. (2). (It may be useful to introduce ghost fields in the Hamiltonian approach but this is not strictly necessary.)

The ‘‘Dyson–Schwinger’’ equations derived from Eq. (8) are not equations of motion in the usual sense, but rather relate the Green functions of the theory to the (so far undetermined) vacuum wave functional. This is why we dubbed them canonical recursive Dyson–Schwinger equations (CRDSEs).

4 The Vacuum Wave Functional

The explicit form of the vacuum wave functional is unknown. We will therefore solve the Schrödinger equation in an approximate fashion by means of the *variational principle*: we take an ansatz for the wave functional, depending on some variational kernels, we evaluate the expectation value of the Hamilton operator Eq. (1), thereby using the CRDSEs, and minimize the resulting vacuum energy density with respect to the variational kernels.

To proceed further, we need an explicit ansatz for the vacuum wave functional Eq. (7). For the Yang–Mills part we choose an ‘‘action’’ S_A involving up to quartic terms in the gluon field

$$S_A = \omega(1, 2) A(1) A(2) + \frac{1}{3!} \gamma_3(1, 2, 3) A(1) A(2) A(3) + \frac{1}{4!} \gamma_4(1, 2, 3, 4) A(1) A(2) A(3) A(4), \quad (9)$$

where ω , γ_3 , and γ_4 are variational kernels. We are using here a compact notation where a repeated numerical label denotes integration over the spatial coordinates as well as summation over discrete indices (colour, Lorentz. . .). For the quark wave functional we choose the ansatz [15–17]

$$S_f = \xi_+^\dagger(1) K_A(1, 2) \xi_-(2), \quad \xi_\pm = \Lambda_\pm \xi.$$

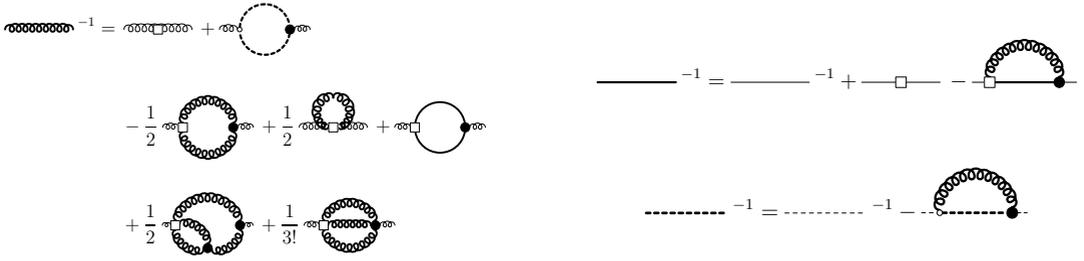


Figure 1. CRDSEs for the gluon propagator (left), quark propagator (top right), and ghost propagator (bottom right). Empty boxes stand for the variational kernels, fat dots for the full (one-particle irreducible) vertices, and fat lines for fully dressed propagators.

The kernel K_A contains both a purely fermionic part and the coupling of the quarks to the transverse gluons

$$K_A^{mn}(\mathbf{x}, \mathbf{y}) = \delta^{mn} \beta s(\mathbf{x}, \mathbf{y}) + g t_a^{mn} \int d^3z [v(\mathbf{x}, \mathbf{y}; \mathbf{z}) \alpha_i + w(\mathbf{x}, \mathbf{y}; \mathbf{z}) \beta \alpha_i] A_i^a(\mathbf{z}). \quad (10)$$

Here, the functions s , v and w are the variational kernels. Note that the quark-gluon coupling in Eq. (10) contains besides the leading-order term $\sim \alpha_i$ known from perturbation theory [19] also a second Dirac structure $\sim \beta \alpha_i$, which turns out to be of fully non-perturbative nature.

With the ansatz given by Eqs. (9)–(10) we find e.g. the propagator equations represented diagrammatically in Fig. 1.

5 Yang–Mills Correlation Functions

The first investigations in the Hamiltonian approach [3, 4, 20] used a Gaussian-type wave functional, which allows to evaluate the expectation value of the Hamilton operator in terms of products of propagators by means of Wick’s theorem. While the obtained equal-time gluon propagator agrees with lattice simulations in the IR and in the UV, there is some mismatch in the mid-momentum regime. Figure 2a shows the results for the gluon propagator with a non-Gaussian functional [10], showing that the presence of a three-gluon coupling (and hence of a gluon loop in the CRDSE) improves the agreement with the lattice data. Figure 2b illustrates the effect of the quark loop onto the gluon propagator.

The truncated ghost-gluon vertex CRDSE is shown diagrammatically in Fig. 3a. The first loop integral (the “abelian diagram”) contains only ghost-gluon vertices, while the second loop integral (the “non-abelian diagram”) involves also a three-gluon vertex (see later). The results [21] for the two diagrams are separately shown in Fig. 3c. Figure 3b shows an alternative CRDSE, which differs from the one given in Fig. 3a by the leg attached to the bare vertex (remember that in general a n -point functions has n possible DSEs). Because of truncation artefacts the two equations do not yield exactly the same result. The differences depend on the specific momentum configuration: in general they are small and they even practically vanish for some momentum configurations, see Fig. 3d.

Figure 4 shows the truncated CRDSE for the three-gluon vertex. The full four-gluon vertex is replaced by the bare one in the actual numerical calculation. It turns out that the gluon triangle diagram is sub-leading in comparison to the ghost triangle diagram; neglecting the swordfish diagrams altogether, on the other hand, turns out to be too crude an approximation [21].

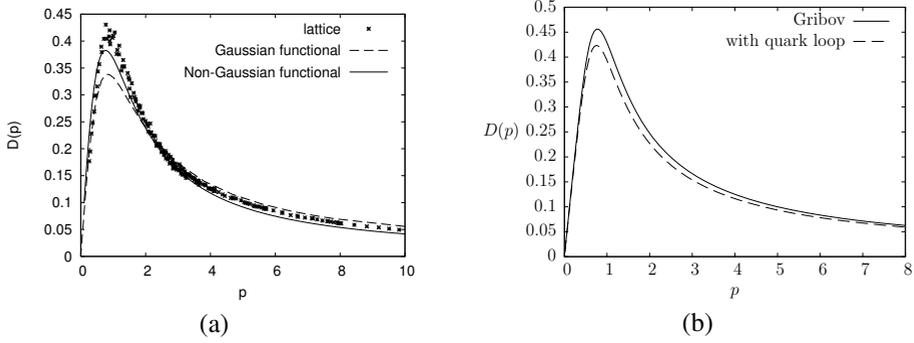


Figure 2. (a) Quenched gluon propagator with Gaussian and non-Gaussian wave functional. (b) Gluon propagator with one dynamical quark family.

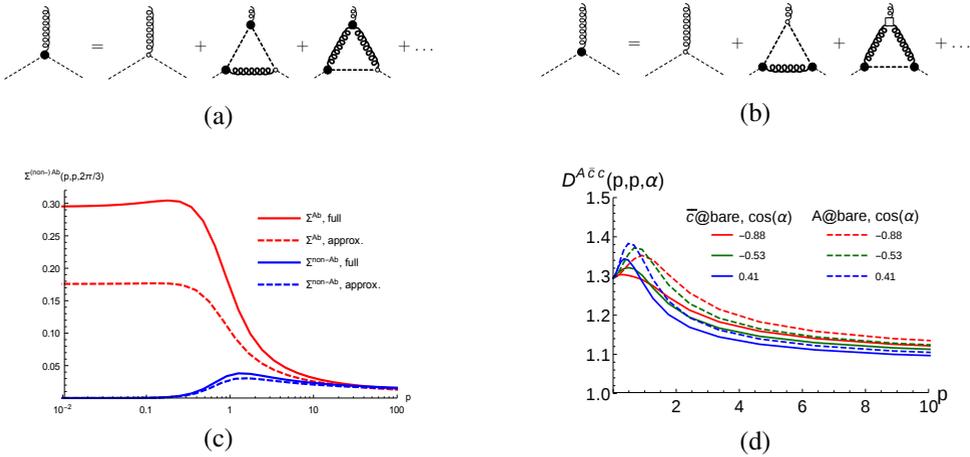


Figure 3. (a) and (b): possible forms of the CRDSE for the ghost-gluon vertex. (c) Result for the loop integrals in the ghost-gluon vertex DSE: the dashed lines represent the lowest-order approximation, where all vertices are taken to be bare, while the continuous lines are the result of the full calculation. (d) Dressing function of the ghost-gluon vertex for equal ghost and antighost momentum and various angles as calculated from the two different DSEs.

6 Quark Mass Function and Chiral Condensate

The first calculations [15, 22] in the quark sector used an ansatz similar to Eq. (10) where, however, only the Dirac structure α_i was considered. The resulting equations were plagued by linear divergences whose presence suggested that this simple ansatz needed to be extended. While a full variational equation for the quark-gluon kernel exists [11] it is easier to motivate the presence of the second Dirac structure in the following way: Consider first-order perturbation theory, where the relevant part of the Hamilton operator has the form

$$H = H_D + g\hat{\psi}^\dagger \boldsymbol{\alpha} \cdot \mathbf{A}\hat{\psi},$$

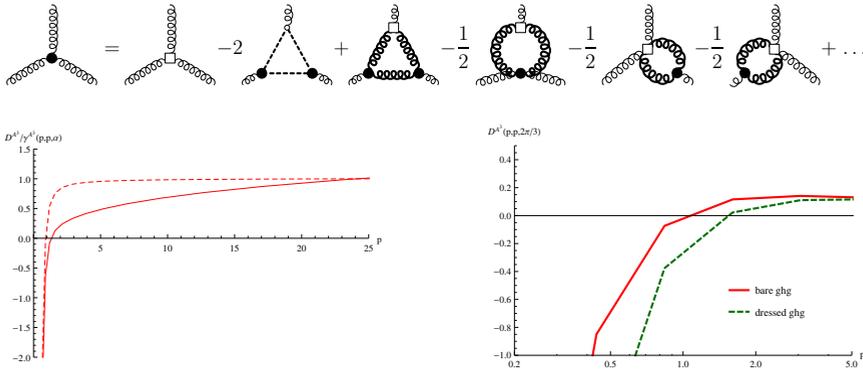


Figure 4. (top) Truncated CRDSE for the three-gluon vertex. (bottom left) Dressing function of the perturbative tensor structure of the three gluon vertex at the symmetric point: the dashed line is the result from the ghost triangle only, while the continuous line is the result of the full calculation. (bottom right) Zoom around the zero crossing showing the difference between a bare (continuous line) and dressed (dashed line) ghost-gluon vertex.

with H_D being the Hamiltonian of the free Dirac theory. The first-order correction $|0\rangle^{(1)}$ to the vacuum state $|0\rangle^{(0)}$ is given as usual by

$$|0\rangle^{(1)} = - \sum \frac{\langle n | g \hat{\psi}^\dagger \boldsymbol{\alpha} \cdot \mathbf{A} \hat{\psi} | 0 \rangle^{(0)}}{E_n} |n\rangle^{(0)} \tag{11}$$

where $|n\rangle^{(0)}$ are the excited eigenstates of the free theory. Working out the matrix element [19] one finds (apart from numerical factors)

$$|0\rangle^{(1)} \propto \hat{\psi}^\dagger \boldsymbol{\alpha} \cdot \mathbf{A} \hat{\psi} |0\rangle^{(0)}.$$

Once the coherent-state representation is used one arrives at a functional of the form of Eq. (10) with $s = 0$ and $w = 0$. The first term in Eq. (10) is a BCS-type of wave functional, already used in the Coulomb-gauge pairing model [12–14]. If we use this “vacuum” (which, admittedly, is not an eigenstate of H_D) as bare state and evaluate Eq. (11), we find a first-order correction coming with a Dirac structure $\beta_i \alpha_i$. The inclusion of such a term in the ansatz is straightforward [16, 17], and the resulting gap equation for the quark mass function shows no spurious linear divergences any more. The resulting equation for the quark mass function has the structure

$$M(\mathbf{p}) \sim \text{Coulomb-like interaction} + \text{gluon exchange}$$

where the first diagram involves a Coulomb-like interaction and the second one a gluon exchange. Using for the Coulomb interaction [see second term in Eq. (1) and Eq. (3)] the sum of a confining ($\propto 1/p^4$, i.e. linearly rising in coordinate space) and a perturbative ($\sim 1/p^2$) term

$$F(\mathbf{p}) = \frac{8\pi\sigma_C}{p^4} + \frac{\alpha}{p^2} \tag{12}$$

and taking the Coulomb string tension σ_C to be 2.5 times the Wilson string tension we recover a chiral condensate of $\sim (-235 \text{ MeV})^3$. Spontaneous breaking of chiral symmetry breaking does not occur if the confining part of the non-abelian Coulomb potential (12) is discarded.

7 Conclusions

We have presented some results of the CRDSE approach to treat non-Gaussian wave functionals in the Hamiltonian formulation of a quantum field theory, presented first in Ref. [10] in the framework of pure Yang–Mills theory and generalized to full QCD in Ref. [11]. In particular, we have shown that a non-trivial Dirac structure is needed in order to avoid spurious linear divergences. With the inclusion of the coupling of the quarks to the gluons in our vacuum wave functional our approach is capable to reproduce the phenomenological value of the quark condensate.

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