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Extended granular micromechanics

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Abstract. We analyze an extended mechanical model in the conservative case in order to describe a dilatant granular material with rotating grains for which the kinetic energy, in addition to the usual translational one, consists of three terms owing to microstructural motions: in particular, it includes the rotation of the grains, the dilatational expansion and contraction of the individual granules and of the granules relative to one another. Hence, we model the body as a continuum with a peculiar microstructure; after we follow classical procedures and define a variational principle of local type for a perfect fluid with microstructure, in accordance with the fluid-like behavior of granular materials: the motion equations are in good agreement with those obtained by other authors. At the end the particular case of a suspension of rotating rigid granules puts in evidence the possibility for granular materials to support shear stresses through the generation of micro-rotational gradients.

1 Preamble

A thermodynamic theory for the study of dilatant granular materials with rotating compressible granules was presented in [10] (see, also, [1] and [20]), a theory that generalized the model of distributed bodies in [16], which was widely used to study the slow flows of granular materials.

The material was assumed to consist of a suspension of dry cohesionless compressible spheres of uniform size in a compressible gas of negligible mass; the flow behavior has required a combination of suggestions from both fluid and solid mechanics owing to the fact that the material has an essentially fluid–like behavior, but it can also be heaped and, moreover, its bulk compressibility depends on the initial voids distribution in the reference placement (see the experimental results in [2]); we also assume a volume concentration close to that of packed particles, so that the mean free path of the particles is very short in comparison to the size of the particles themselves (as it is the case of cohesionless soil or sand with rough surface grains).

Additional balance equations for the microinertia was needed for new independent kinematical variables introduced in [10]: the volume fraction of the grains which describes the local arrangements of grains themselves and the micro-rotations of granules; so the granular material is a special case of perfect fluid with microstructure [5, 19].

In this work we analyze the conservative case for the model of granular materials proposed in [10] following the procedure used in [12]: an approach which was already successfully applied to a model of compressible immiscible mixtures of two perfect fluids [11] and of liquids with diffusing gas bubbles [13]. Also, an interesting application of the granular theory [12] was investigated in [14] for the study of seismic waves propagating through a sediment filled basin in the case of rigid grains; one advantage of our model, with respect to purely propagative models, was the reproduction of a nonlinear effect experimentally observed for real seismic waves: site amplification decreases as the amplitude of the incident wave increases.

The dynamic equations of motion are so obtained from a Hamiltonian variational principle of local type for a perfect fluid with microstructure, in accordance with the fluid–like behavior of granular materials, and then compared with those previously obtained by other authors.

At last, we consider the constrained case of a suspension of rotating rigid granules in a fluid matrix and notice that the microstructure behaves as that of a microrigid Cosserat's continua. By considering possible rotations of grains during the motion, we also show that, even when the volume fraction of the grain distribution is constant, the model predicts the possibility of supporting shear stress through the generation of microrotational gradients.

2 The model

The continuum model for a dry cohesionless dilatant granular materials with rotating grains here proposed is directly referred to the model proposed in [4], as specialized in [1, 7]. A material element of the body $\mathcal{B}$ is a sort of quasi–particle which consists of a grain and its immediate neighbours and that is called a 'chunk' of material; the chunks are thought of as envelopes which fill the body without voids between them and without diffusion of the grains through the envelope itself. The quasi-static motion of the body admits only dilatations, or contractions, of the individual (compressible) grain and of the grains relative to one another as well as rotations of the granule itself; there-

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fore the continuum model depict the granular material as a medium with constrained affine microstructure [5].

The total kinetic energy per unit mass κ of the medium is then the sum of four components: a) the usual translational kinetic energy $\kappa_t := 2^{-1}\mathbf{v}^2$ related to the velocity $\mathbf{v}$ of the centre of mass $\mathbf{x}$ of the material element; b) the 'fluctuation' kinetic energy κ_f associated with the dilatancy of the chunk, that is the homogeneous expansions of it represented by the motion of individual grains relative to the centre of mass and expressed in terms of the rate of change of the chunk mass density ρ by $2^{-1}\gamma(\rho)\dot{\rho}^2$; the dot denotes the material time derivative, *i.e.*, $\dot{(\cdot)} := \frac{\partial(\cdot)}{\partial\tau} + \mathbf{v}\cdot\operatorname{grad}(\cdot)$, with τ the time. We observe that the chunk mass density ρ is equal to ν times the proper mass density ρ_m: $\rho = \rho_m\nu$, where $\nu \in (0,1)$ is the volume fraction of the grains; c) the dilatational kinetic energy κ_d related to local expansions (or contractions) of the compressible inclusions in the chunk and written in terms of the rate of change of the proper mass density of the grains ρ_m by $2^{-1}\alpha(\rho_m)\dot{\rho}_m^2$; d) the rotational kinetic energy $\kappa_r := 2^{-1}3\rho_m^2\alpha(\rho_m)\dot{\mathbf{R}}\cdot\dot{\mathbf{R}}$ due to grain rotations, with $\mathbf{R}$ a proper orthogonal tensor.

The form of constitutive coefficients $\gamma(\rho)$ and $\alpha(\rho_m)$ can be very general as they depend on geometric configurations of the grains and of the chunks and on the admissible micro-motions (see §2 of [4]). In [10, 12] simple microstructural motions and peculiar geometrical shapes for chunks and granules were assumed in order to compute them explicitly. In a mental magnification, one can imagine the chunk consisting of a spherical grain and its immediate spherical neighbors, and the envelope of the chunk as a spherical surface of variable radius ς containing all these spherical compressible inclusions of variable radius ϑ, with interstices filled with a fluid of negligible mass; the chunks and the compressible grains are also supposed to expand or contract homogeneously with independent purely radial pulsations, while rotations of single grains are not necessarily related to macro-rotations of the own chunk. Therefore, if the envelopes and the grains have the radius ς_* and ϑ_*, respectively, in a reference placement $\mathcal{B}_*$ of the material, one can calculate analytically the following expressions for functions γ and α:

$$\gamma(\rho) = \gamma_*\rho^{-\frac{8}{3}}, \quad \alpha(\rho_m) = \alpha_*\rho_m^{-\frac{8}{3}}, \tag{1}$$

with $\gamma_* = (16/351)\rho_*^{\frac{2}{3}}\varsigma_*^2$ and $\alpha_* = 15^{-1}\rho_{m*}^{\frac{2}{3}}\vartheta_*^2$.

The density per unit mass of the total kinetic energy of the material turns out to be

$$\kappa = 2^{-1}\mathbf{v}^2 + 2^{-1}\gamma(\rho)\dot{\rho}^2 + 2^{-1}\alpha(\rho_m)\left(\dot{\rho}_m^2 + 3\rho_m^2\,\dot{\mathbf{R}}\cdot\dot{\mathbf{R}}\right). \tag{2}$$

In order to reduce the developments to the essential things, we consider the case when the suspension is contained in a fixed rigid vessel, very large with respect to grains and to a single element of the medium. If $\mathcal{B}$ is the region of space delimited by the vessel and $\partial\mathcal{B}$ its boundary, kinematic compatibility imposes the condition

$$\mathbf{v}\cdot\hat{\mathbf{n}} = 0 \quad \text{on } \partial\mathcal{B}, \tag{3}$$

where $\hat{\mathbf{n}}$ is the unit exterior vector normal to $\partial\mathcal{B}$; moreover, the conservation of mass requires

$$\dot{\rho} + \rho\operatorname{div}\mathbf{v} = 0 \quad \text{in } \mathcal{B}. \tag{4}$$

3 The Hamiltonian variational principle

In order to obtain the dynamical balance equations of motion we use a Hamiltonian variational principle of local type in the conservative case, even if, in general, the viscous effects could not be ignored under dynamic conditions. The choice of a Eulerian variational principle, rather than Lagrangian, is not in contrast with the previous appeal to a reference placement; in fact the bulk compressibility of a granular material depends on the initial porosity (see [2]) and therefore the material has a preferred reference placement with respect to volume distribution, thus the choice for the former or the latter formulation is not so peremptory for such materials (see [3]).

So we suppose that the internal actions on $\mathcal{B}$ derive from a potential energy φ per unit mass; since the material is essentially thought as a perfect fluid in its behavior, we assume φ to be a function of the state of $\mathcal{B}$ through ρ, $\operatorname{grad}\rho$, ρ_m, $\operatorname{grad}\rho_m$, $\mathbf{R}$ and $\operatorname{grad}\mathbf{R}$.

When a variation $\delta\mathbf{x}(\mathbf{x},\tau)$, with $\mathbf{x}$ in $\mathcal{B}$ and the time τ in the interval $[\tau_0,\tau_1]$ ($\tau_1 > \tau_0$) of duration of the mechanical process, is added to the motion $\mathbf{x}$, a corresponding variation of the domain $\mathcal{B}$ and of the fields $\delta\rho_m$ and $\delta\mathbf{R}$ defined on $\mathcal{B}\times[\tau_0,\tau_1]$ is implied. Moreover, these fields are subject to their arbitrary variation in addition to that induced by $\delta\mathbf{x}$: for instance, the total variation of ρ_m is defined as

$$\hat{\delta}\rho_m := \delta\rho_m + (\operatorname{grad}\rho_m)\cdot\delta\mathbf{x}. \tag{5}$$

The condition (3) and the conservation of mass (4) require restrictions on the variation $\delta\mathbf{x}$, which must be tangent to the boundary, and the total variation of $\hat{\delta}\rho$:

$$\delta\mathbf{x}\cdot\hat{\mathbf{n}} = 0 \quad \text{on } \partial\mathcal{B}, \tag{6}$$

$$\hat{\delta}\rho + \rho\operatorname{div}\delta\mathbf{x} = 0 \quad \text{in } \mathcal{B}; \tag{7}$$

besides, we consider only variations $\delta\mathbf{x}$, $\delta\rho_m$ and $\delta\mathbf{R}$ which vanish at the extremes of the interval $[\tau_0,\tau_1]$ and on $\partial\mathcal{B}$.

In view of condition (6), we can define the global virtual work of external actions consequent to the virtual displacement $\delta\mathbf{x}$, to a change $\delta\rho_m$ of the proper density of the matrix and $\delta\mathbf{R}$ of the rotation of grains, as the quantity

$$\delta\mathcal{L} := \int_{\mathcal{B}} \rho\left(\mathbf{f}\cdot\delta\mathbf{x} + \beta\,\hat{\delta}\rho_m + \mathbf{O}\cdot\hat{\delta}\mathbf{R}\right)d\mathcal{B}, \tag{8}$$

where $\mathbf{f}$, β and $\mathbf{O}$ are the densities for unit mass of the external body, dilatational and tensor moment forces, respectively. Then the Hamiltonian principle asserts that, during the natural motion of the body, the equality

$$\hat{\delta}\int_{\tau_0}^{\tau_1} d\tau \int_{\mathcal{B}} \rho(\kappa-\varphi)d\mathcal{B} + \int_{\tau_0}^{\tau_1}\delta\mathcal{L}d\tau = 0 \tag{9}$$

holds for all virtual processes, where the applied operator $\hat{\delta}$ means the variation of the next integral functional defined on a variable region (see §37, ch. 7, of [8]).

The use of the transport theorem leads to the equality

$$\hat{\delta}\int_{\tau_0}^{\tau_1} d\tau \int_{\mathcal{B}} \rho(\kappa-\varphi)d\mathcal{B} = \int_{\tau_0}^{\tau_1} d\tau \int_{\mathcal{B}} \rho\hat{\delta}(\kappa-\varphi)\,d\mathcal{B}, \tag{10}$$

so we can write the variational principle (9) as follows

$$\begin{aligned}&\int_{\tau_0}^{\tau_1} d\tau \int_{\mathcal{B}} \Big\{\rho\Big[\mathbf{v}\cdot\hat{\delta}\mathbf{v} + 2^{-1}\gamma'(\rho)\dot{\rho}^2\hat{\delta}\rho + \gamma(\rho)\dot{\rho}\,\hat{\delta}(\dot{\rho})\\&+\alpha(\rho_m)\left(\dot{\rho}_m\hat{\delta}(\dot{\rho}_m) + 3\,\rho_m\,\dot{\mathbf{R}}\cdot\dot{\mathbf{R}}\,\hat{\delta}\rho_m + 3\,\rho_m^2\,\dot{\mathbf{R}}\cdot\hat{\delta}(\dot{\mathbf{R}})\right)\\&+2^{-1}\alpha'(\rho_m)\left(\dot{\rho}_m^2 + 3\,\rho_m^2\,\dot{\mathbf{R}}\cdot\dot{\mathbf{R}}\right)\hat{\delta}\rho_m\Big] - \frac{\pi}{\rho}\hat{\delta}\rho - \zeta\hat{\delta}\rho_m\\&-\mathbf{r}\cdot\hat{\delta}(\operatorname{grad}\rho) - \mathbf{s}\cdot\hat{\delta}(\operatorname{grad}\rho_m) - \mathbf{\Sigma}\cdot\hat{\delta}(\operatorname{grad}\mathbf{R})\\&-\mathbf{N}\cdot\hat{\delta}\mathbf{R} + \rho(\mathbf{f}\cdot\delta\mathbf{x} + \beta\hat{\delta}\rho_m + \mathbf{O}\cdot\hat{\delta}\mathbf{R})\Big\}\,d\mathcal{B} = 0,\end{aligned} \tag{11}$$

where the prime $'$ denotes derivation with respect to the argument and we have introduced the notations

$$\begin{aligned}&\pi := \rho^2\frac{\partial\varphi}{\partial\rho}, \quad \mathbf{r} := \rho\frac{\partial\varphi}{\partial(\operatorname{grad}\rho)}, \quad \zeta := \rho\frac{\partial\varphi}{\partial\rho_m},\\&\mathbf{s} := \rho\frac{\partial\varphi}{\partial(\operatorname{grad}\rho_m)}, \ \mathbf{N} := \rho\frac{\partial\varphi}{\partial\mathbf{R}}, \ \mathbf{\Sigma} := \rho\frac{\partial\varphi}{\partial(\operatorname{grad}\mathbf{R})}.\end{aligned} \tag{12}$$

The velocity variation $\hat{\delta}\mathbf{v}$ corresponding to $\delta\mathbf{x}$ is

$$\hat{\delta}\mathbf{v} = (\delta\mathbf{x})^{\cdot}, \tag{13}$$

while we can obtain the relations

$$\hat{\delta}(\dot{\rho}) = (\hat{\delta}\rho)^{\cdot}, \ \hat{\delta}(\operatorname{grad}\rho) = \operatorname{grad}\hat{\delta}\rho - (\operatorname{grad}\delta\mathbf{x})^T\operatorname{grad}\rho, \tag{14}$$

for the total variations of, *e.g.*, $\dot{\rho}$ and $\operatorname{grad}\rho$, from definitions of the time derivative and (5); ^{T} means transposition.

Usual developments (as the insertion of (13) and (14) into (11), the integration by parts, as well as the application of divergence and transport theorems with conditions (3) and (6), besides the vanishing of variations on the boundary $\partial\mathcal{B}$ and in τ_0 and τ_1) lead us to write (11) as follows:

$$\begin{aligned}&\int_{\tau_0}^{\tau_1} d\tau \int_{\mathcal{B}} \{[\rho(\mathbf{f}-\dot{\mathbf{v}}) - \operatorname{div}(\operatorname{grad}\rho\otimes\mathbf{r} + \operatorname{grad}\rho_m\otimes\mathbf{s}\\&+(\operatorname{grad}\mathbf{R})^T\odot\mathbf{\Sigma}^T)\Big]\cdot\delta\mathbf{x} + \{\rho\beta - \zeta + \operatorname{div}\mathbf{s}\\&-\rho\left[\alpha(\rho_m)\left(\ddot{\rho}_m + \rho_m\dot{\mathbf{R}}\cdot\dot{\mathbf{R}}\right) + 2^{-1}\alpha'(\rho_m)\dot{\rho}_m^2\right]\Big\}\hat{\delta}\rho_m\\&-\left[\rho\left(\gamma(\rho)\ddot{\rho} + 2^{-1}\gamma'(\rho)\dot{\rho}^2\right) + \frac{\pi}{\rho} - \operatorname{div}\mathbf{r}\right]\hat{\delta}\rho\\&+\left[\rho\left(\mathbf{O} - 3\left(\rho_m^2\alpha(\rho_m)\dot{\mathbf{R}}\right)^{\cdot}\right) - \mathbf{N} + \operatorname{div}\mathbf{\Sigma}\right]\cdot\hat{\delta}\mathbf{R}\Big\}\,d\mathcal{B} = 0.\end{aligned} \tag{15}$$

where we used the expression $(1)_2$ of $\alpha(\rho_m)$; the tensor product $\odot$ between third–order tensors is so defined: $(\mathbf{\Gamma}\odot\mathbf{\Sigma})_{ij} := \mathbf{\Gamma}_{ihk}\mathbf{\Sigma}_{jhk}$.

By using relation (7) and other elementary calculations, we can express the integral in (15) as a linear functional of $\delta\mathbf{x}$, $\hat{\delta}\rho_m$ and $\hat{\delta}\mathbf{R}$; so, by requiring the integral to vanish for any values of them, we obtain the motion equations for dilatant granular materials with rotating grains

$$\begin{aligned}\rho\dot{\mathbf{v}} &= \rho\mathbf{f} - \operatorname{grad}\left[\pi - \rho\operatorname{div}\mathbf{r} + \rho^2\left(\gamma(\rho)\ddot{\rho} + 2^{-1}\gamma'(\rho)\dot{\rho}^2\right)\right]\\&-\operatorname{div}\left[\operatorname{grad}\rho\otimes\mathbf{r} + \operatorname{grad}\rho_m\otimes\mathbf{s} + (\operatorname{grad}\mathbf{R})^T\odot\mathbf{\Sigma}^T\right],\end{aligned} \tag{16}$$

$$\rho\left[\alpha(\rho_m)\left(\ddot{\rho}_m + \rho_m\dot{\mathbf{R}}\cdot\dot{\mathbf{R}}\right) + 2^{-1}\alpha'(\rho_m)\dot{\rho}_m^2\right] = \rho\beta - \zeta + \operatorname{div}\mathbf{s}, \tag{17}$$

$$\rho\,\rho_m\,\alpha(\rho_m)\left(3\,\rho_m\ddot{\mathbf{R}} - 2\dot{\rho}_m\dot{\mathbf{R}}\right) = \rho\mathbf{O} - \mathbf{N} + \operatorname{div}\mathbf{\Sigma}. \tag{18}$$

The balance equation for the moment of momentum is directly recovered when we impose the principle of material objectivity to the scalar potential φ, *i.e.*, its value is invariant if the observer changes, although its constitutive variables $\operatorname{grad}\rho$, $\operatorname{grad}\rho_m$, $\mathbf{R}$ and $\operatorname{grad}\mathbf{R}$ vary (the measures of ρ and ρ_m are not affected by changes of frame). This condition coincides with the following relation:

$$\mathcal{E}\left[\operatorname{grad}\rho\otimes\mathbf{r} + \operatorname{grad}\rho_m\otimes\mathbf{s} + (\operatorname{grad}\mathbf{R})^T\odot\mathbf{\Sigma}^T\right] = 0, \tag{19}$$

with $\mathbf{r}$, $\mathbf{s}$ and $\mathbf{\Sigma}$ given by $(12)_{2,4,6}$ and $\mathcal{E}$ being the Ricci's permutation tensor: (19) coincides with the symmetry of the tensor in the square parentheses and, consequently, of the Cauchy's stress tensor $\mathbf{T}$. Indeed, the balance (16) reduces, at least formally, to the Cauchy's equation

$$\rho\dot{\mathbf{v}} = \rho\mathbf{f} + \operatorname{div}\mathbf{T}, \tag{20}$$

if the symmetric Cauchy's tensor is defined as follows:

$$\begin{aligned}\mathbf{T} &= \left[\rho\operatorname{div}\mathbf{r} - \pi - \rho^2\left(\gamma(\rho)\ddot{\rho} + 2^{-1}\gamma'(\rho)\dot{\rho}^2\right)\right]\mathbf{I}-\\&-\operatorname{sym}\left[\operatorname{grad}\rho\otimes\mathbf{r} + \operatorname{grad}\rho_m\otimes\mathbf{s} + (\operatorname{grad}\mathbf{R})^T\odot\mathbf{\Sigma}^T\right],\end{aligned} \tag{21}$$

where $\operatorname{sym}(\cdot)$ is the symmetric part of the tensor $(\cdot)$.

Relations (17) and (18) are the balance equations for the micromomentum with ζ and $\mathbf{N}$ the densities per unit volume of the internal forces of dilatancy and of rotation, while $\mathbf{s}$ and $\mathbf{\Sigma}$ are the dilating microstress vector and the third order spinning hyperstress tensor, respectively. Here, we can recognize the dynamics equations of §8 of [5] that describe the motion of continua with microstructure, even if the constitutive relation (21) for the stress tensor $\mathbf{T}$ generalizes the analogous one for granular materials in §17 of [5] where less complex physical phenomena are studied.

The four basic fields ρ, $\mathbf{v}$, ρ_m and $\mathbf{R}$ are determined by the four equations (4), (17), (18) and (20), so the additional differential equation (3.16) for ρ which appears in [4] arouses some perplexity. However, it is possible to eliminate the Lagrange multiplier μ in the (3.15) of [4] by using (3.16) and reduce equations in agreement with us.

At the end, a comparison with equations $(69)_2$ and (85) of [10] that rule the motion of continua with spherical microstructure (see also [6]) is possible only if we insert the skew microspin tensor $\mathbf{Y} := -\dot{\mathbf{R}}\mathbf{R}^T$ in (17) and (18), so

$$\rho\left[\alpha(\rho_m)\left(\ddot{\rho}_m + \rho_m\mathbf{Y}\cdot\mathbf{Y}\right) + 2^{-1}\alpha'(\rho_m)\dot{\rho}_m^2\right] = \rho\beta - \zeta + \operatorname{div}\mathbf{s}, \tag{22}$$

$$\rho\left(\rho_m^2\,\alpha(\rho_m)\mathbf{Y}\right)^{\cdot} + \rho\rho_m^2\alpha(\rho_m)\mathbf{Y}^2 = 3^{-1}\mathbf{R}\left(\rho\mathbf{O} - \mathbf{N} + \operatorname{div}\mathbf{\Sigma}\right)^T. \tag{23}$$

After we can take the deviatoric and the skew part of (23)

$$\begin{aligned}&\operatorname{dev}\left[3\rho\rho_m^2\alpha(\rho_m)\mathbf{Y}^2 - \mathbf{R}\left(\rho\mathbf{O} - \mathbf{N} + \operatorname{div}\mathbf{\Sigma}\right)^T\right] = \mathbf{0},\\&\rho\left(\rho_m^2\,\alpha(\rho_m)\mathbf{Y}\right)^{\cdot} = 3^{-1}\operatorname{skw}\left[\mathbf{R}\left(\rho\mathbf{O} - \mathbf{N} + \operatorname{div}\mathbf{\Sigma}\right)^T\right],\end{aligned} \tag{24}$$

to have, with (22), the requested equations.

4 The case of rigid grains

In this section we consider the flow of a large number of discrete inelastic particles at relatively high concentrations and with interstices filled by a fluid or a gas of negligible mass (as it is the case of cohesionless soil, such as sand with rough surface grains, or of fluidized particulate beds). Hence, we have to assume that the granules are rigid and the proper mass density ρ_m is constant, thus, for the constitutive hypothesis b) of §2, the chunk mass density ρ comes down to be proportional to the volume fraction ν of grains ($\rho = \rho_{m*}\nu$) and the conservation of mass (4) reduces to

$$\dot{\nu} + \nu \operatorname{div} \mathbf{v} = 0. \tag{25}$$

Moreover, the expression of kinetic energy changes, owing to such constraint; in particular we have from (2) and (1)

$$\hat{\kappa} = 2^{-1}\mathbf{v}^2 + 2^{-1}\hat{\gamma}(\nu)\dot{\nu}^2 + 2^{-1}\mu_* \dot{\mathbf{R}} \cdot \dot{\mathbf{R}}, \tag{26}$$

with $\hat{\gamma}(\nu) = (16/351)\, \nu_*^{\frac{2}{3}} \varsigma_*^2 \nu^{-\frac{8}{3}}$, and $\mu_* = 5^{-1}\, \vartheta_*^2$.

The potential energy ψ is now a function of the state of $\mathcal{B}$ only through ν, $\mathbf{R}$, grad ν and grad $\mathbf{R}$; we also suppose that the external actions derive from a potential ω, which is a function of the place $\mathbf{x}$ in $\mathcal{B}$, of the volume fraction ν and of the rotation $\mathbf{R}$ (see [18] and [15]). Besides the relation (7) binds the fraction volume to vary as follows

$$\hat{\delta}\nu = -\nu \operatorname{div} \delta\mathbf{x} \quad \text{in } \mathcal{B}. \tag{27}$$

Therefore, now, the Hamiltonian principle is

$$\hat{\delta} \int_{\tau_0}^{\tau_1} d\tau \int_{\mathcal{B}} \rho(\hat{\kappa} - \psi - \omega) d\mathcal{B} = 0; \tag{28}$$

by going on as in the previous section we obtain equations

$$\begin{aligned} \nu\, \dot{\mathbf{v}} &= \nu\, \hat{\mathbf{f}} - \operatorname{div}\left[\operatorname{grad} \nu \otimes \hat{\mathbf{r}} + (\operatorname{grad} \mathbf{R})^T \odot \hat{\boldsymbol{\Sigma}}^T\right] \\ &\quad - \operatorname{grad}\left[\hat{\pi} + \nu^2\left(\hat{\gamma}(\nu)\ddot{\nu} + 2^{-1}\hat{\gamma}'(\nu)\dot{\nu}^2 - \hat{\beta}\right) - \nu \operatorname{div}\hat{\mathbf{r}}\right], \end{aligned} \tag{29}$$

$$\mu_*\, \nu\, \ddot{\mathbf{R}} = \nu\, \hat{\mathbf{O}} - \hat{\mathbf{N}} + \operatorname{div} \hat{\boldsymbol{\Sigma}}, \tag{30}$$

where we introduced the notations

$$\begin{aligned} &\hat{\pi} := \nu^2 \frac{\partial \psi}{\partial \nu},\ \hat{\mathbf{r}} := \nu\, \frac{\partial \psi}{\partial(\operatorname{grad}\nu)},\ \hat{\boldsymbol{\Sigma}} := \nu\, \frac{\partial \psi}{\partial(\operatorname{grad} \mathbf{R})}, \\ &\hat{\mathbf{N}} := \nu \frac{\partial \psi}{\partial \mathbf{R}},\ \hat{\mathbf{f}} := -\frac{\partial \omega}{\partial \mathbf{x}},\ \hat{\beta} := -\frac{\partial \omega}{\partial \nu},\ \hat{\mathbf{O}} := -\frac{\partial \omega}{\partial \mathbf{R}}. \end{aligned} \tag{31}$$

In (29) the usual pressure for fluids $\hat{\pi}$ is related to the compressibility of chunks; the spherical part of a Reynolds' stress tensor of the turbulence theory $-\nu^2\left(\hat{\gamma}(\nu)\ddot{\nu} + 2^{-1}\hat{\gamma}'(\nu)\dot{\nu}^2\right)\mathbf{I}$ evaluates the agitation within a chunk of material; finally, the stress of Ericksen's type $-\left[\operatorname{grad} \nu \otimes \hat{\mathbf{r}} + (\operatorname{grad} \mathbf{R})^T \odot \hat{\boldsymbol{\Sigma}}^T\right]$ measures the ability of rigid granular continua to support shear stresses in equilibrium, by inducing the generation of micro-rotation gradients, even when the proper mass density and the volume fraction of the grain distribution are constant.

Again, if we insert the microspin tensor $\mathbf{Y}$ into the balance (30) and apply the skew and symmetric operators, we calculate equations which govern the micromotion for the microrigid Cosserat's continua:

$$\begin{aligned} \nu\dot{\mathbf{Y}} &= \operatorname{skw}\left[\mu_*^{-1}\mathbf{R}\left(\nu\hat{\mathbf{O}} - \hat{\mathbf{N}} + \operatorname{div}\hat{\boldsymbol{\Sigma}}\right)^T\right], \\ \operatorname{sym}\left[\mathbf{R}\left(\hat{\mathbf{N}} - \operatorname{div}\hat{\boldsymbol{\Sigma}}\right)^T\right] &= \nu\left[\operatorname{sym}\left(\mathbf{R}\hat{\mathbf{O}}^T\right) - \mu_*\mathbf{Y}^2\right]. \end{aligned} \tag{32}$$

In particular, the first one represents the balance of rotational micromomentum (see equation (63) of [17])

$$\mu_*\nu\dot{\mathbf{Y}} = \nu\,\bar{\mathbf{O}} - \bar{\mathbf{N}} + \operatorname{div}\bar{\boldsymbol{\Sigma}}, \tag{33}$$

where $\bar{\mathbf{O}} := \operatorname{skw}(\mathbf{R}\hat{\mathbf{O}}^T)$ ($\bar{\mathbf{N}} := \operatorname{skw}(\mathbf{R}\hat{\mathbf{N}}^T)$) is the external (internal) rotational tensor moment and $\bar{\boldsymbol{\Sigma}}$, such that $\bar{\boldsymbol{\Sigma}}\mathbf{w} := \operatorname{skw}[\mathbf{R}(\hat{\boldsymbol{\Sigma}}\mathbf{w})^T], \forall \mathbf{w}$, the third-order spinning hyperstress tensor; the second one is the equation for the reactions to the constraint ρ_{m*}=const.: indeed, if we consider the material expression of (27): $\nu = \nu_*\, \iota^{-1}$, with ι the determinant of deformation gradient, we observe that the volume fraction ν of grains is bound to the macrodeformation.

Remark: When the effects of rotations of chunks and of grains are also negligible ($\mathbf{R}$=$\mathbf{I}$), we recover the essence of the theory in [9]: in particular, the Coulomb's model for the stress at equilibrium in a granular material with incompressible grains: $\mathbf{T}_C = (\beta_0 - \beta_1\nu^2 + \beta_2 \operatorname{grad}\nu \cdot \operatorname{grad}\nu + 2\beta_3\, \nu\, \Delta\nu)\mathbf{I} - 2\beta_4 \operatorname{grad}\nu \otimes \operatorname{grad}\nu$ (β_i material constants) is obtained as a peculiar example (see, also, equation (9.1) of [16]). Alternatively, the complementary case in which the grains are elastic, is studied in [12].

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References

[1] Ahmadi G, Int. J. Non–Linear Mech. **17**, 21-33 (1982)
[2] Arthur JR, Menzies BK, Geotech. **22**, 115-129 (1972)
[3] Bampi F, Morro A, J.Math.Phys.**25**, 2418-2421 (1984)
[4] Bedford A, Drumheller DS, Acta Mechanica **48**, 173-184 (1983)
[5] Capriz G, *Continua with Microstructure*, Springer Tracts in Nat. Phil. **35**, (Springer-Verlag, Berlin, 1989)
[6] Capriz G, Podio-Guidugli P, Arch. Rational Mech. Anal. **75**, 269-279 (1981)
[7] Chen K-C, Lan J-Y, Tai Y-C, Int. J. Solids Struct. **46**, 3882-3893 (2009)
[8] Gelfand IM, Fomin SV, *Calculus of Variations* (Prentice-Hall, Englewood Cliffs, New Jersey, 1963)
[9] Giovine P, Math. Phys. An. Geom. **2**, 179-196 (1999)
[10] Giovine P, in Capriz G, Giovine P, Mariano PM (Eds.): *Math. Mod. Gran. Matter*, Lect. Notes Appl. Math.S. **1937** (Springer, New-York, 2008) 167-192
[11] Giovine P, Oliveri F, Acta Mechan. **96** 85-96 (1993)
[12] Giovine P, Oliveri F, Meccanica **30**, 341-357 (1995)
[13] Giovine P, Archives Mechanics **58**, 363-380 (2006)
[14] Godano C, Oliveri F, Int. J. Non-linear Mech. **34**, 457-468 (1999)
[15] Goodman M, Cowin SC, ZAMM **56**, 281-286 (1976)
[16] Goodman M, Cowin SC, Arch. Rational Mech. Anal. **44**, 249-266 (1972)
[17] Grioli G, Cont. Mech. Thermod. **15**, 441-450 (2003)
[18] Jenkins JT, J. Applied Mech. **42**, 603-606 (1975)
[19] Jenkins JT, La Ragione L, in Giovine P, Goddard JD, Jenkins JT (Eds.): *IUTAM-ISIMM Symp. Math. Modeling Phys. Instances Granular Flows*, Conf.Proc.Series **1227** (AIP, Melville, New York, 2010) 31-40
[20] Kanatani KI, Int. J. Engng. Sci. **17**, 419-432 (1979)