Affinity of NJL$_2$ and NJL$_4$ model results on duality and pion condensation in chiral asymmetric dense quark matter

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Abstract.   In this paper we investigate the phase structure of a (1+1) and (3+1)-dimensional quark model with four-quark interaction and in the presence of baryon (μ_B), isospin (μ_I) and chiral isospin (μ_{I5}) chemical potentials. It is shown that the chemical potential μ_{I5} promotes the appearance of the charged PC phase with nonzero baryon density. Results of both models are qualitatively the same, this fact enhances one's confidence in the obtained predictions. It is established that in the large-N_c limit (N_c is the number of colored quarks) there exists a duality correspondence between the chiral symmetry breaking phase and the charged pion condensation one.

1 Introduction

Recently, much attention has been paid to the investigation of the QCD phase diagram in the presence of baryonic as well as isotopic (isospin) chemical potentials. The reason is that dense baryonic matter which can appear in heavy-ion collision experiments has an evident isospin asymmetry. Moreover, the dense hadronic/quark matter inside compact stars is also expected to be isotopically asymmetric.

However, theoretical investigations of QCD encounter considerable difficulties in the low-energy as well as low-temperature and density region, where perturbative methods do not work. The only possible first principle calculation in QCD at low energies is lattice QCD. Unfortunately, its main method (Monte Carlo simulations) cannot be applied at finite baryon chemical potential due to the sign problem. In order to study the phase diagram of QCD at nonzero chemical potential one usually use effective field theories. One of the most widely used effective theory is Nambu–JonaLasinio (NJL) model [1] (see for review [2–4]). The model is tractable and can be used as low energy effective theory for QCD. In this way, QCD phase diagrams including chiral symmetry restoration [5–7], color superconductivity [8–10], and charged pion condensation (PC) phenomena [11–14] were investigated under heavy-ion experimental and/or compact star conditions, i.e. in the presence of temperature, chemical potentials and possible external (chromo)magnetic fields. (3+1)-dimensional NJL models are non-renormalizale and depend on the cutoff parameter which is typically chosen to be of the

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order of 1 GeV, so that the results of their usage are valid only at *comparatively low energies, temperatures and densities (chemical potentials)*. But there exists also a class of renormalizable theories, the (1+1)-dimensional chiral Gross–Neveu (GN) or NJL type models , that can be used as a laboratory for the qualitative simulation of specific properties of QCD at *arbitrary energies*. Renormalizability, asymptotic freedom, as well as the spontaneous breaking of chiral symmetry (in vacuum) are the most fundamental inherent features both for QCD and all NJL_2 type models. In addition, the $\mu_B - T$ phase diagram is qualitatively the same for the QCD and NJL_2 models [20–22].

Among all the above mentioned phenomena, which can be observed in dense baryonic matter, the existence of the charged PC phase is predicted without sufficient certainty. If the electric charge neutrality constraint is imposed, the charged pion condensation phenomenon depends strongly on the bare (current) quark mass values. In particular, it turns out that the charged PC phase with *nonzero baryonic density* is forbidden in the framework of NJL models for the physically acceptable values of the bare quark masses (see ref. [14]). Due to these circumstances, the question arises whether there exist factors promoting the appearance of charged PC phenomenon in dense baryonic matter. A positive answer to this question was obtained in the papers [15, 16], where it was shown that a charged PC phase might be realized in dense baryonic system with finite size or in the case of a spatially inhomogeneous pion condensate.

In the present paper we will show that a chiral imbalance of dense and isotopically asymmetric baryon matter is another interesting factor, which can induce a charged PC phase. Recall that chiral imbalance, i.e. a nonzero difference between densities of left- and right-handed fermions, may arise from the chiral anomaly in the quark-gluon-plasma phase of QCD and possibly leads to the chiral magnetic effect [17] in heavy-ion collisions. It might be realized also in compact stars or condensed matter systems [18] (see also the review [19]).

2 NJL_2 model

We consider a (1+1)-dimensional NJL model in order to mimic the phase structure of real dense quark matter with two massless quark flavors (*u* and *d* quarks). Its Lagrangian, which is symmetrical under global color SU(N_c) group, has the form

$$L = \bar{q}\Big[\gamma^\nu i\partial_\nu + \frac{\mu_B}{3}\gamma^0 + \frac{\mu_I}{2}\tau_3\gamma^0 + \frac{\mu_{I5}}{2}\tau_3\gamma^0\gamma^5\Big]q + \frac{G}{N_c}\Big[(\bar{q}q)^2 + (\bar{q}i\gamma^5\vec{\tau}q)^2\Big], \tag{1}$$

where the quark field $q(x) \equiv q_{i\alpha}(x)$ is a flavor doublet ($i = 1, 2$ or $i = u, d$) and color N_c-plet ($\alpha = 1, ..., N_c$) as well as a two-component Dirac spinor (the summation in (1) over flavor, color, and spinor indices is implied); τ_k ($k = 1, 2, 3$) are Pauli matrices in two-dimensional flavor space. The Dirac γ^ν-matrices ($\nu = 0, 1$) and γ^5 in (1) are matrices in two-dimensional spinor space,

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \qquad \gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

It is evident that the model (1) is a generalization of the two-dimensional GN model with a single massless quark color N_c-plet to the case of two quark flavors and additional baryon μ_B-, isospin μ_I- and axial isospin μ_{I5} chemical potentials. These parameters are introduced in order to describe in the framework of the model (1) quark matter with nonzero baryon n_B-, isospin n_I- and axial isospin n_{I5} densities, respectively. It is evident that Lagrangian (1), both at $\mu_{I5} = 0$ and $\mu_{I5} \neq 0$, is invariant with respect to the abelian $U_B(1)$, $U_{I_3}(1)$ and $U_{AI_3}(1)$ groups, where

$$U_B(1): \ q \to \exp(i\alpha/3)q; \ U_{I_3}(1): \ q \to \exp(i\beta\tau_3/2)q; \ U_{AI_3}(1): \ q \to \exp(i\omega\gamma^5\tau_3/2)q. \tag{3}$$

So the quark bilinears $\frac{1}{3}\bar{q}\gamma^0 q$, $\frac{1}{2}\bar{q}\gamma^0\tau^3 q$ and $\frac{1}{2}\bar{q}\gamma^0\gamma^5\tau^3 q$ are the zero components of corresponding conserved currents. Their ground state expectation values are just the baryon n_B-, isospin n_I- and chiral (axial) isospin n_{I5} densities of quark matter, i.e. $n_B = \frac{1}{3}\bar{q}\gamma^0 q$, $n_I = \frac{1}{2}\bar{q}\gamma^0\tau^3 q$ and $n_{I5} = \frac{1}{2}\bar{q}\gamma^0\gamma^5\tau^3 q$. As usual, the quantities n_B, n_I and n_{I5} can be also found by differentiating the thermodynamic potential of the system with respect to the corresponding chemical potentials. We would like also to remark that, in addition to (3), Lagrangian (1) is invariant with respect to the electromagnetic $U_Q(1)$ group,

$$U_Q(1): \ q \rightarrow \exp(iQ\alpha)q, \tag{4}$$

where $Q = \text{diag}(2/3, -1/3)$.

To find the thermodynamic potential of the system, we use a semi-bosonized version of the Lagrangian (1), which contains composite bosonic fields $\sigma(x)$ and $\pi_a(x)$ ($a = 1, 2, 3$) (in what follows, we use the notations $\mu \equiv \mu_B/3$, $\nu = \mu_I/2$ and $\nu_5 = \mu_{I5}/2$):

$$\widetilde{L} \ = \ \bar{q}\Big[\gamma^\rho i\partial_\rho + \mu\gamma^0 + \nu\tau_3\gamma^0 + \nu_5\tau_3\gamma^0\gamma^5 - \sigma - i\gamma^5\pi_a\tau_a\Big]q - \frac{N_c}{4G}\Big[\sigma\sigma + \pi_a\pi_a\Big]. \tag{5}$$

In (5) the summation over repeated indices is implied. From the Lagrangian (5) one gets the Euler–Lagrange equations for the bosonic fields

$$\sigma(x) = -2\frac{G}{N_c}(\bar{q}q); \quad \pi_a(x) = -2\frac{G}{N_c}(\bar{q}i\gamma^5\tau_a q). \tag{6}$$

Note that the composite bosonic field $\pi_3(x)$ can be identified with the physical π_0 meson, whereas the physical $\pi^\pm(x)$-meson fields are the following combinations of the composite fields, $\pi^\pm(x) = (\pi_1(x) \pm i\pi_2(x))/\sqrt{2}$. Obviously, the semi-bosonized Lagrangian $\widetilde{L}$ is equivalent to the initial Lagrangian (1) when using the equations (6). In this case, the order parameters are the ground state expectation values of the composite fields, i.e. $\sigma(x)$ and $\pi_a(x)$ ($a = 1, 2, 3$), the coordinates of the global minimum point of the thermodynamic potential $\Omega(\sigma, \pi_a)$ of the system. It is clear that if $\sigma(x) \neq 0$ and/or $\pi_3(x) \neq 0$, then the axial isospin $U_{AI_3}(1)$ symmetry of the model is spontaneously broken down, whereas at $\pi_1(x) \neq 0$ and/or $\pi_2(x) \neq 0$ we have a spontaneous breaking of the isospin $U_{I_3}(1)$ symmetry. Since in the last case the ground state expectation values (condensates) of both the fields $\pi^+(x)$ and $\pi^-(x)$ are not zero, this phase is usually called charged pion condensation (PC) phase. In addition, it is easy to see from (6) that the nonzero condensates $\pi_{1,2}(x)$ (or $\pi^\pm(x)$) are not invariant with respect to the electromagnetic $U_Q(1)$ transformations (4) of the flavor quark doublet. Hence in the charged PC phase the electromagnetic $U_Q(1)$ invariance of the model (1) is also broken spontaneously, and superconductivity is an unavoidable property of this phase, one can call this phase pion superfluid one.

Starting from the theory (5), one obtains in the leading order of the large N_c-expansion (i.e. in the one-fermion loop approximation) the following path integral expression for the effective action

$$\exp(iS_{\text{eff}}(\sigma, \pi_a)) = N' \int [d\bar{q}][dq] \exp\Big(i\int \widetilde{L}d^2 x\Big),$$

Using the definition of thermodynamic potential (TDP)

$$\Omega(\sigma, \pi_a) \ \equiv -\frac{S_{\text{eff}}(\sigma, \pi_a)}{N_c \int d^2 x}\bigg|_{\sigma,\pi_a=\text{const}}, \tag{7}$$

it is possible to obtain for the TDP $\Omega(\sigma, \pi_a)$ of the system:

$$\Omega(M, \Delta) = \frac{M^2 + \Delta^2}{4G} + i\int \frac{d^2 p}{(2\pi)^2} \ln P_4(p_0), \quad P_4(p_0) = \eta^4 - 2a\eta^2 - b\eta + c. \tag{8}$$

we use the notations: $\eta = p_0 + \mu$ and

$$
\begin{aligned}
a &= M^2 + \Delta^2 + p_1^2 + \nu^2 + \nu_5^2; \quad b = 8p_1\nu\nu_5; \\
c &= a^2 - 4p_1^2(\nu^2 + \nu_5^2) - 4M^2\nu^2 - 4\Delta^2\nu_5^2 - 4\nu^2\nu_5^2.
\end{aligned} \tag{9}
$$

To simplify the task, due to the fact that the model is invariant under $U_{I_3}(1) \times U_{AI_3}(1)$ and the TDP depends effectively only on the two combinations $\sigma^2 + \pi_3^2$ and $\pi_1^2 + \pi_2^2$ of the bosonic fields, without loss of generality, we put $\pi_2 = \pi_3 = 0$ in (8), and the TDP (8) is a function of only two variables, $M \equiv \sigma$ and $\Delta \equiv \pi_1$. Now let us discuss the propeties of the TDP. One can see that the TDP is invariant with respect to the so-called duality transformation

$$
\mathcal{D}: \quad M \longleftrightarrow \Delta, \quad \nu \longleftrightarrow \nu_5. \tag{10}
$$

The TDP in the form of (8) and (16) is ultraviolet divergent and we need to renormalize it.

The *unrenormalized* TDP (8) can be presented in the following form,

$$
\Omega(M, \Delta) \equiv \Omega^{un}(M, \Delta) = \frac{M^2 + \Delta^2}{4G} - \int_{-\infty}^{\infty} \frac{dp_1}{4\pi} \Big(|p_{01}| + |p_{02}| + |p_{03}| + |p_{04}| \Big). \tag{11}
$$

where the roots p_{01}, p_{02}, p_{03} and p_{04} of this polynomial $P_4(p_0)$ are the energies of quasiparticle or quasiantiparticle excitations of the system.

First of all, let us obtain a finite, i.e. renormalized, expression for the TDP (11) at $\mu = 0$, $\nu = 0$ and $\nu_5 = 0$, i.e. in vacuum.

This procedure consists of two steps: (i) First of all we need to regularize the divergent integral, i.e. we suppose there that $|p_1| < \Lambda$. (ii) Second, we must suppose also that the bare coupling constant G depends on the cutoff parameter Λ in such a way that in the limit $\Lambda \to \infty$ one obtains a finite expression for the effective potential.

It can be established that the bare coupling $G \equiv G(\Lambda)$ has the following Λ dependence:

$$
\frac{1}{4G(\Lambda)} = \frac{1}{\pi} \ln \frac{2\Lambda}{m}, \tag{12}
$$

where m is a new free mass scale of the model, which appears instead of the dimensionless bare coupling constant G (dimensional transmutation) and, evidently, does not depend on a normalization point, i.e. it is a renormalization invariant quantity. We obtain in the limit $\Lambda \to \infty$ the finite and renormalization invariant expression for the effective potential,

$$
V_0(M, \Delta) = \frac{M^2 + \Delta^2}{2\pi} \left[\ln \left(\frac{M^2 + \Delta^2}{m^2} \right) - 1 \right]. \tag{13}
$$

It is possible to show that in the general case the expression for the TDP has two terms, the first is the vacuum effective potential and the second one is the correction due to non-zero chemical potentials. Only the first term is divergent and has to be renormalized.

Based on a numerical algorithm, it can be shown numerically that the global minimum point of the TDP can never be of the form $(M_0 \neq 0, \Delta_0 \neq 0)$. Hence, in order to establish the phase portrait of the model, it is enough to study the projections $F_1(M) \equiv \Omega^{ren}(M, \Delta = 0)$ and $F_2(\Delta) \equiv \Omega^{ren}(M = 0, \Delta)$ of the TDP to the M and Δ axes, correspondingly.

3 NJL$_4$ model

Now let us describe the two flavored (3+1)-dimensional NJL model that can be used as effective model for QCD with several chemical potentials. Its Lagrangian has the form

$$
L = \bar{q} \Big[\gamma^\nu i \partial_\nu + \frac{\mu_B}{3} \gamma^0 + \frac{\mu_I}{2} \tau_3 \gamma^0 + \frac{\mu_{I5}}{2} \tau_3 \gamma^0 \gamma^5 + \mu_5 \gamma^0 \gamma^5 \Big] q + \frac{G}{N_c} \Big[(\bar{q}q)^2 + (\bar{q}i\gamma^5 \vec{\tau} q)^2 \Big] \tag{14}
$$

and describes dense baryonic matter with two massless u and d quarks, i.e. q in is the flavor doublet, $q = (q_u, q_d)^T$, where q_u and q_d are four-component Dirac spinors as well as color N_c-plets (the summation in (14) over flavor, color, and spinor indices is implied); τ_k ($k = 1, 2, 3$) are Pauli matrices. The Lagrangian contains baryon μ_B-, isospin μ_I-, chiral isospin μ_{I5}- chemical potentials as in the NJL$_2$ case.

The quantities n_B, n_I and n_{I5} are densities of conserved charges, which correspond to the invariance of Lagrangian (1) with respect to the abelian $U_B(1)$, $U_{I_3}(1)$ and $U_{AI_3}(1)$ groups

As in the (1+1) dimensional case to find the TDP of the system, we use a semibosonized version of the Lagrangian (14) and in the leading order of $1/N_c$ expansion it is possible to obtain the following expression for the TDP

$$\Omega(M, \Delta) = \frac{M^2 + \Delta^2}{4G} + i \int \frac{d^4p}{(2\pi)^4} \ln\left(P_+(\eta)P_-(\eta)\right), \tag{15}$$

$$P_+(\eta)P_-(\eta) \equiv (\eta^4 - 2a_+\eta^2 + b_+\eta + c_+)(\eta^4 - 2a_-\eta^2 + b_-\eta + c_-),$$

where $\eta = p_0 + \mu$, $|\vec{p}| = \sqrt{p_1^2 + p_2^2 + p_3^2}$ and

$$
\begin{aligned}
a_\pm &= M^2 + \Delta^2 + (|\vec{p}| \pm \mu_5)^2 + \nu^2 + \nu_5^2; \quad b_\pm = \pm 8(|\vec{p}| \pm \mu_5)\nu\nu_5; \\
c_\pm &= a_\pm^2 - 4\nu^2\left(M^2 + (|\vec{p}| \pm \mu_5)^2\right) - 4\nu_5^2\left(\Delta^2 + (|\vec{p}| \pm \mu_5)^2\right) - 4\nu^2\nu_5^2.
\end{aligned}
\tag{16}
$$

It has been examined numerically and demonstrated that there is no mixed phase in the massless NJL model as in the (1+1) dimensional case. This circumstance significantly simplifies the investigation of the phase diagram of the model, since in this case it is enough to study only the projections $F_1(M) \equiv \Omega(M, \Delta = 0)$ and $F_2(\Delta) \equiv \Omega(M = 0, \Delta)$ of the TDP on the M and Δ axes, correspondingly.

Let us talk a little bit more about dualities in this case.

It is clear that the TDP of the system is invariant with respect to the transformation

$$\mathcal{D}: \quad M \longleftrightarrow \Delta, \quad \nu \longleftrightarrow \nu_5 \tag{17}$$

The same duality was discussed in (1+1)-dimensional case.

Let us elaborate a little more on the duality notion. Here it is a symmetry relation between condensates (phases) and matter content (chemical potentials). But the notion of duality is more widespread and it is a very powerful concept that is used in different domains ranging from string theory to condensed matter physics. For example, there is a class of dualities called strong-weak dualities that connect weak coupling regime of one theory with strong coupling regime of the other. To this class belongs famous AdS/CFT (or gauge/gravity) duality [27]. There is another class of dualities, they are called strong-strong dualities or usually bear another name large-N_c orbifold equivalences [28, 29]. Orbifold equivalences connect gauge theories with different gauge groups and matter content in the large-N_c limit. In the framework of orbifold equivalence formalism in [29] there have been also obtained a similar (to our) dualities.

In addition to G the cutoff parameter Λ is introduced. In the following we will use a special set of the model parameters,

$$G = 15.03 \, GeV^{-1}, \qquad \Lambda = 0.65 \, GeV.$$

The same parameter set has been used, e.g., in Refs [4, 11]. There can exist no more than three different phases in the model (1). The first one is the symmetric phase, where ($M_0 = 0, \Delta_0 = 0$), the CSB phase, ($M_0 \neq 0, \Delta_0 = 0$) and the charged PC phase, ($M_0 = 0, \Delta_0 \neq 0$).

4 Phase structure

Let us consider the phase portrait of the dense baryon matter with isospin and chiral asymmetry in terms of NJL_2 and NJL_4 models. First of all, we will study the phase structure of the NJL_2 model at different fixed values of the chiral isospin chemical potential ν_5. Moreover, it is possible to find the quark number density n_q or baryon density n_B (note that $n_q = 3n_B$). As a result, at Figs 2 left and 3 left we have drawn several (ν, μ)-phase portraits, corresponding to $\nu_5 = 0$ and $\nu_5 = m$, respectively. Recall that m is a free renormalization invariant mass scale parameter, which appears in the vacuum case of the model after renormalization.

It is clear from Fig. 2 left that at $\nu_5 = 0$ the charged PC phase with *nonzero baryon density n_B* (it is denoted by the symbol PC_d) is not realized in the model under consideration. Only the charged PC phase with *zero baryon density* can be observed at rather small values of μ. (Physically, it means that at $\nu_5 = 0$ the model predicts the charged PC phenomenon in the medium with $n_B = 0$ only. For example, it might consist of charged pions, etc. But in quark matter with nonzero baryon density the charged PC is forbidden.) Instead, at large values of μ there exist two phases, the chiral symmetry breaking and the symmetrical one, both with nonzero baryon density, i.e. the model predicts the CSB phase of dense quark matter. However, as we can see from Fig. 3 left, at rather high values of ν_5 there might appear on the phase portrait a charged PC phase with *nonzero baryon density*. Hence, in chirally asymmetric, i.e. for $\nu_5 > 0$, and dense quark matter the charged PC phenomenon is allowed to exist in the framework of the toy model (1). Thus, we see that $\nu_5 \neq 0$ is a factor which promotes the charged PC phenomenon in dense quark matter. Note that the compact region of the (ν, μ)-plane, which is occupied by the PC_d phase (see, e.g., Fig. 3 left at $\nu_5 = m$), continues to move up along the μ-axis, when ν_5 increases above the value $\nu_5 = m$.

Now let us consider the (ν, ν_5)-phase diagrams of the NJL_2 model. The phase diagrams is presented at Fig. 4 left and once more support the main conclusion that the charged PC phase with nonzero baryon density, i.e. the phase denoted as PCd, might be realized in the framework of the NJL_2 model only at $\nu_5 > 0$. It is also clear that this diagram is a self-dual, i.e. the CSB and charged PC phases are arranged symmetrically with respect to the line $\nu = \nu_5$ of the (ν, ν_5)-plane.

Schematic plot of the (ν, ν_5, μ)-parameter space phase diagram of the model is presented at Fig. 1. One can see that chiral isospin ν_5 promotes the charged PC phenomenon in dense quark matter.

Now let us talk about duality $\mathcal{D}$ of the TDP and the phase diagram. One can see that under the duality transformation $(\nu \leftrightarrow \nu_5, \text{CSB} \leftrightarrow \text{charged PC})$ (ν, ν_5, μ)- phase portrait is mapped to itself, i.e. the most general (ν, ν_5, μ)-phase portrait is self-dual.

Now let us confabulate about the phase diagram of the NJL4 model. The (ν, μ)-phase diagrams of the model at different typical values of ν_5 are presented at Figs 2 right and 3 right. One can see that this phase diagrams look very similar to the ones corresponding to the NJL_2 model (Fig 2 left and 3 left). In the NJL_4 model case there appears a bar of CSB phase that starts from PC phase and goes along the line $\mu = \nu$. This bar-like phase can be observed at the figure of NJL_2 model case. At bigger values of $\nu_5 > 0$, CSB phase starts to appear from zero values of ν, and the phase transition from it to the charged PC phase takes place at $\nu = \nu_5$ (see Fig. 2 right). As a result, the charged PC_d phase is shifted to greater values of ν as one increase ν_5. The same happens in the case of NJL_2 model, the Fig 2 left shows the (ν, μ)- phase diagram at $\nu_5 = 0$, if $\nu_5 \neq 0$ then the bar of CSB phase will shrink a little and there appears the CSB phase from zero values of ν and the phase transition from it to the charged PC phase will take place at $\nu = \nu_5$. Then, at even larger values of ν_5 the bar of CSB phase disappears (as in the NJL_2 model case, see Fig. 3 left) and charged PC_d phase shifts to the region of larger μ (see Fig. 3 right). In this region of ν_5 the shape of the charged PC_d
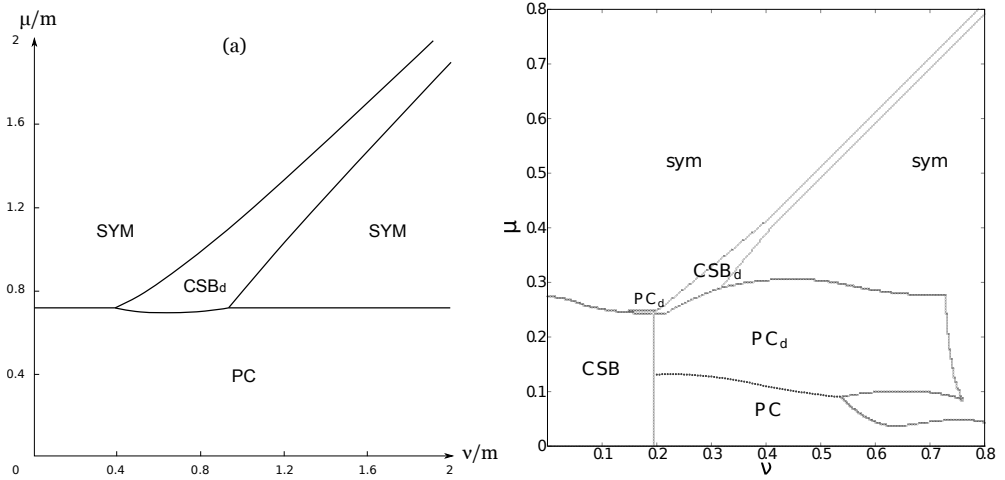
Figure 1. The (ν, μ)-phase portrait of the NJL_2 and NJL_4 models: left – the NJL_2 model case at $\nu_5 = 0$. right – the NJL_4 model case at $\nu_5 = 195$ MeV.

phase resembles again a sole of a boot and points towards the value of $\mu = \nu_5$. Qualitatively same behavior can be seen in the NJL_2 model case, see Fig. 3 left. The only qualitative difference is that in NJL_4 model the charged PC_d phase is realized only for nonzero values of ν (see Fig. 3 right), whereas in NJL_2 model it takes place even for $\nu = 0$ (see Fig. 3 left). But overall qualitatively behavior is the same. So one can say that in both models chiral imbalance generates charged PC phase in dense ($n_B \neq 0$) quark matter.

Now let us turn to the (ν, ν_5)-phase in the NJL4 model, it is depicted at Figs 4 right. It can be noted that charged PC_d and CSB_d phases at this figure take the form of soles of boots that look towards each other and points at values of $\nu_5 = \mu$ and $\nu = \mu$, correspondingly. Moreover, with an increase of μ charged PC_d phase and CSB_d phase move to the region of larger ν_5 and ν, correspondingly. So one can see that nonzero ν_5 also generates the charged pion condensation phase in dense, $n_B \neq 0$, quark matter in the case of NJL_4 model consideration. In (1+1)-dimensional case the charged PC_d phase also goes to higher values of ν_5 with increase of μ but this phase starts at zero values of ν (see Fig. 4 left), whereas in (3+1)-dimensional case the phase starts at some nonzero value of ν around 0.1 GeV (see Fig. 4 right). So in order to realize the charged PC_d phase in NJL_4 model, besides chiral imbalance there has to be the isotopic imbalance in the system. In that respect one can say that charged PC_d phase is generated by both isospin and chiral isospin chemical potentials. This is a feature that exists only in the NJL_4 model, in the NJL_2 model charged pion condensate phase with nonzero baryon density could be realized just by chiral isospin chemical potential even at $\nu = 0$.

5 Summary and conclusions

In this paper, the phase structure of the NJL_2 and NJL_4 models with two quark flavors is investigated in the large-N_c limit in the presence of baryon μ_B, isospin μ_I and chiral isospin μ_{I5} chemical potentials.

1) It was demonstrated in the framework of both models that chiral asymmetry (i.e. $\mu_{I5} \neq 0$) of dense quark matter can serve as a factor promoting there a charged pion condensation phenomenon. So this phenomenon is predicted in two models, in NJL_2 and NJL_4, and might be the property of real QCD.
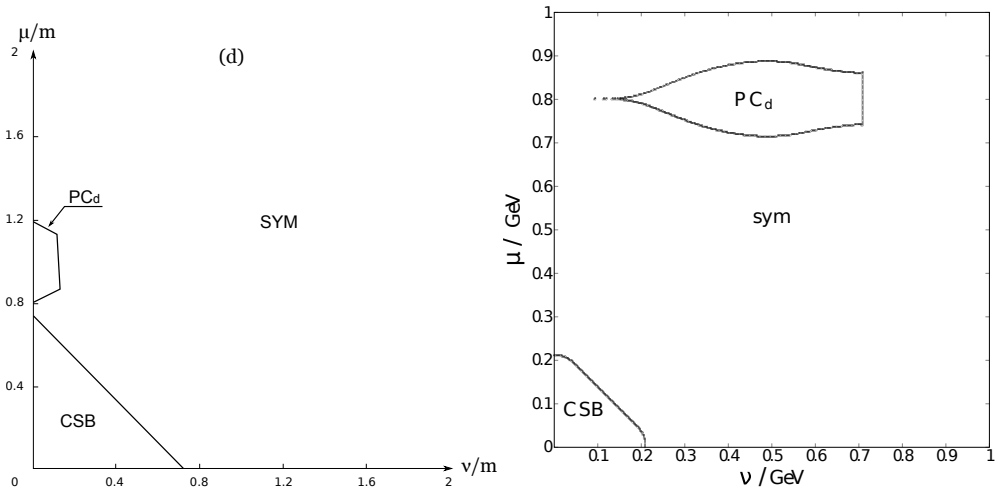
Figure 2. The (ν, μ)-phase portrait of the NJL$_2$ and NJL$_4$ models: left – the NJL$_2$ model case at $\nu_5 = m$. right – the NJL$_4$ model case at $\nu_5 = 0.8$ GeV. The notations are the same.
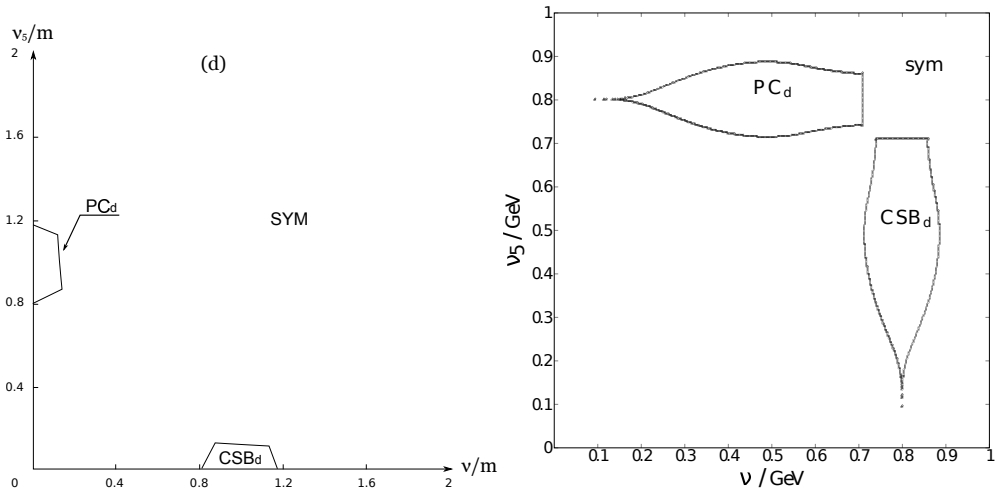
Figure 3. The (ν, ν_5)-phase portrait of the NJL$_2$ and NJL$_4$ models: left – the NJL$_2$ model case at $\mu = m$. right – the NJL$_4$ model case at $\mu = 0.8$ GeV. The notations are the same.

2) We have shown in the leading order of the large-N_c approximation that in the framework of the NJL$_2$ and NJL$_4$ models there is a duality correspondence between CSB and charged PC phenomena.

3) In contrast to NJL$_2$ model, where the generation of the PC$_d$ phase occurs even at very small values of isospin chemical potential, the generation of the PC$_d$ phase in NJL$_4$ model requires not very large but nonzero isospin chemical potential.

Studies in the framework of NJL$_2$ and more realistic (3+1)-dimensional NJL model in a sense complement each other. One can consider the NJL$_2$ model at arbitrary high values of

chemical potentials, whereas the NJL$_4$ model is a more realistic theory for QCD in the region of its validity.

Moreover, we hope that our results might shed new light on phase structure of dense quark matter with isotopic and chiral imbalance and hence could be of importance for describing physics in the heavy ion collision experiments and interior of neutron stars.

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