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Acceleration of an improved linear scheme for the neutron transport

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Abstract. This paper presents an accelerated version of an improved linear method of characteristics scheme that preserves first order spatial moments of the flux. This scheme has slow convergence rate and this is why an acceleration method has been investigated. This acceleration is obtained by using the double P_N (DP_N) method described in previous studies and used in other codes developed by the APOLLO3® team. While the improved linear MoC has a high memory cost and is difficult to implement, the DP_N acceleration is shown to save a significant amount of time on a benchmark problem, even at the zeroth order treated in this paper.

INTRODUCTION

The neutron transport equation describes how neutrons move, scatter, and interact with their surroundings. Solving this equation yields valuable insights into neutron flux, power distributions, and other important parameters that govern reactor performance. To solve it, numerous numerical techniques have been developed, each with its own set of advantages and limitations. This paper focuses on the method of characteristics (MoC), a technique characterized by the utilization of the directional derivative term for the resolution of the equation as a first-order ordinary differential equation on a segment of the phase space, known as the characteristic. The specific assumption made in this paper is that each of the angular moments of the flux are developed as a polynomial along each characteristic, which allow simple computation. Classical MoC uses the stepwise constant (SC) or stepwise linear (SL) approximation [2]. In a more recent work, a new linear surface scheme (LS-xy) has been developed that preserved the linear spatial moments of the flux [1], which guarantees higher precision and open the door to other higher order schemes. The main problem with this scheme is that its rate of convergence is too slow to compete with previous methods, which leads to use acceleration methods. The method used here is double P_N method at zeroth order (DP_0), chosen in other TDT codes [2] for its stability [3]. In a first section, the basics of MoC for neutron transport are recalled. Then, the LS-xy method is presented, as well as its acceleration. Finally, the results on a simple heterogeneous geometry are presented, benchmarked in [4]. Results are discussed in a last section.

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1 METHOD OF CHARACTERISTICS

In this section, the main equations of the MoC for neutron transport are recalled. In addition, the work here is made after the discretization in energy groups and for one group only.

1.1 Steady state transport problem

The one velocity steady state neutron transport equation takes the form:

$$\begin{aligned}(\vec{\Omega}\cdot\vec{\boldsymbol{\nabla}}+\Sigma)\psi^{(k)}(\vec{r},\vec{\Omega}) &= q^{(k-1)}(\vec{r},\vec{\Omega}), \vec{r}\in D, \vec{\Omega}\in(4\pi)\\ \psi^{(k)}(\vec{r},\vec{\Omega}) &= \beta\psi^{(k-1)}(\vec{r},\vec{\Omega})+\psi_0(\vec{r},\vec{\Omega}), \vec{r}\in\partial D, \vec{\Omega}\in(2\pi)^-,\end{aligned} \tag{1}$$

with D and ∂D the spatial domain and its border, (4π) and $(2\pi)_-$ the unit sphere and its half negative part, being the directional domain, k an iteration index, $\vec{r}$ the position vector, $\vec{\Omega}$ the direction vector, ψ the unknown angular neutron flux, Σ the total macroscopic cross section, β an albedo operator and q the density source term, defined as:

$$q(\vec{r},\vec{\Omega}) = q_{ext}(\vec{r},\vec{\Omega}) + \int_{4\pi} \mathrm{d}\vec{\Omega}\,\Sigma_s(\vec{r},\vec{\Omega}\cdot\vec{\Omega}')\psi(\vec{r},\vec{\Omega}') = q_{ext}(\vec{r},\vec{\Omega}) + \vec{A}(\vec{\Omega})\boldsymbol{\Sigma_s}(\vec{r})\cdot\vec{\phi}(\vec{r}), \tag{2}$$

with $\boldsymbol{\Sigma_s}$ the diagonal matrix of angular moments of the differential scattering cross section with respect to the Legendre polynomials, $\vec{A}(\vec{\Omega}) = \left(A^l_m(\vec{\Omega})\right)_{l,m} = \left(A_\rho(\vec{\Omega})\right)_{\rho\in[\![1,N_m]\!]}$, the vector of ordonned angular real spherical harmonics with N_m the number of moments taken into account [2], q_{ext} the external source term (that can contain fission terms for instance) and $\vec{\phi}$ the angular flux moments:

$$\vec{\phi}(\vec{r}) = \int_{4\pi} \mathrm{d}\vec{\Omega}\,\vec{A}(\vec{\Omega})\psi(\vec{r},\vec{\Omega}), \tag{3}$$

that will be approximated by

$$\vec{\phi}(\vec{r}) \simeq \sum_{n=1}^{N} \omega_n\vec{A}(\vec{\Omega}_n)\psi(\vec{r},\vec{\Omega}_n), \tag{4}$$

with $(\vec{\Omega}_n,\omega_n)$ the couple angle and weight of an angular quadrature. The assumption of a Σ constant per mesh region is made, thus considering a characteristic t directed by $\vec{\Omega}$, it is possible to solve Eq. (1) as a first order differential equation:

$$\psi_t(s) = \psi_t^{in}e^{-\Sigma(s+\frac{l_t}{2})} + \int_{-l_t/2}^{s} ds' q_t(\vec{r}_t^{\,m}+s'\vec{\Omega})e^{-\Sigma(s-s')} \tag{5}$$

With ψ_t (resp. q_t) the flux (resp. source) along the characteristic t, s the local position on t, l_t the total length of t inside the considered region and $\vec{r}_t^{\,m}$ the midpoint of the characteristic t in the region. For the spatial integration, transversal characteristics weights $\omega_t^\perp$ are introduced such as:

$$V_D \simeq \sum_t \omega_t^\perp \int_{-l_t/2}^{l_t/2} \mathrm{d}s\,. \tag{6}$$

The main challenge of MoC scheme resulting in hindering the propagation of uncertainties. In order to avoid cumbersome notations, we will consider here $\vec{\phi} = (\phi^\rho)_{\rho\in[\![1,N_m]\!]}$.

2 IMPROVED LINEAR SCHEME

The work described here uses the classical assumptions of linear MoC scheme. The notations from previous studies [1] and [2] are used here, for the sake of readability.
The goal of this numerical scheme is to achieve the conservation of spatial flux moments, defined as:

$$\int_{V_i} \mathrm{d}\vec{r}\, \phi^{\rho}(\vec{r}) m_K(\vec{r}), \tag{7}$$

where $m_K \in P_k(D)$, where $P_k(D)$ the set of polynomial functions from D to $\mathbb{R}$ of degree inferior or equal to K. When speaking of conservation of a spatial moment, this means that for a chosen K, Eq. (1) is verified for Eq. (7) being the new unknown.
The main issue with the classic conservative linear surface scheme (CLS) is that, apart from the average constant moment of the flux (i.e for K = 0), it does not conserve spatial moments. In this work we use another form of the CLS noted LS-xy [1], that preserves the first order spatial moments (i.e for $K = 1$). The gist of the classical CLS scheme stems from the assumed linear evolution of the moment of the flux along a characteristic t between two surfaces of a mesh region D_i:

$$\phi^{\rho}_t(s) = \frac{1}{2}\left(\phi^{\rho}_{\alpha^{out},t} + \phi^{\rho}_{\alpha^{in},t}\right) + \frac{s}{l_t}\left(\phi^{\rho}_{\alpha^{out},t} - \phi^{\rho}_{\alpha^{in},t}\right) + \vec{\delta\phi}^{\rho}(s). \tag{8}$$

The difference from the LS-xy lays in the free parameter $\vec{\delta\phi}$ that now has a dependency on x and y:

$$\vec{\delta\phi}^{\rho}(s) = \vec{\delta\phi}^{\rho,1} + \vec{\delta\phi}^{\rho,xy} \cdot ((\vec{r} - \vec{r}_c) \oslash \vec{r^*}) = \vec{\delta\phi}^{\rho,1} + \vec{\delta\phi}^{\rho,xy} \cdot ((\vec{r}^{\,m}_t - \vec{r}_c) \oslash \vec{r^*}) + s\vec{\delta\phi}^{\rho,xy} \cdot (\vec{\Omega} \oslash \vec{r^*}), \tag{9}$$

where $\oslash$ is the Hadamard division, $\vec{r^*} = \frac{1}{2}\left[x_{max} - x_{min}, y_{max} - y_{min}\right]^T$ a scaling parameter that is computed using the characteristic dimensions of D_i.

$$\begin{aligned} \vec{\delta\phi}^{\rho,xy} \cdot ((\vec{r}^{\,m}_t - \vec{r}_c) \oslash \vec{r^*}) &= \delta\phi^{\rho,x} \cdot ((r^m_{t,x} - r_{C,x})/r^*_x) + \delta\phi^{\rho,y} \cdot ((r^m_{t,y} - r_{C,y})/r^*_y), \\ \vec{\delta\phi}^{\rho,xy} \cdot (\vec{\Omega} \oslash \vec{r^*}) &= \delta\phi^{\rho,x} \cdot (\Omega_x/r^*_x) + \delta\phi^{\rho,y} \cdot (\Omega_y/r^*_y)s, \end{aligned} \tag{10}$$

with $\vec{r}_C$ the barycenter of the D_i. This improved scheme is referred to as "LS-xy" due to this dependency. The total algebraic form of these corrective terms are given in the next sections.

2.1 Conservation equation

2.1.1 Corrective terms

$\delta\phi^1$ and $\vec{\delta\phi}^{xy}$ are corrective terms which allow the conservation of the zeroth order spatial moment and the first order spatial moment respectively. In order to determine these quantities, the conservation relations for Eq. (7) are needed. To get the gist of it, let say that one has a first approximate of the flux in a mesh region. This approximate solution does not fully satisfy Eq. (1), but it can be adjusted to "almost" meet this equation. This is the concept behind the corrective terms: they adjust the angular moments of the flux so that Eq. (1), once integrated for the zeroth and first-order spatial moments, is satisfied. The approximated term is referred to as the *geometric* average (i.e., Eq. (7)), and the term that satisfies Eq. (1) when

integrated for the appropriate spatial moment is called the *conservative* average [1][2]. The *conservative* average ϕ_C over the region D_i of border ∂D_i and volume V_i is:

$$\phi_C^{\rho,1} = \frac{1}{4\pi\Sigma V_i}\int_{4\pi} \mathrm{d}\vec{\Omega} \int_{D_i} \mathrm{d}\vec{r} A_\rho(\vec{\Omega})\left(q - \vec{\Omega}\cdot\vec{\nabla}\psi\right), \tag{11}$$

and should be equal to the *geometrical* average ϕ_G term over the same region:

$$\phi_G^{\rho,1} = \frac{1}{V_i}\int_{D_i} \mathrm{d}\vec{r} A_\rho \phi(\vec{r}), \tag{12}$$

for the zeroth order spatial moment. Moreover, for the first order spatial moment:

$$\begin{cases} \phi_C^{\rho,x} = \dfrac{1}{4\pi\Sigma V_i}\displaystyle\int_{4\pi} \mathrm{d}\vec{\Omega} \int_{D_i} \mathrm{d}\vec{r} A_\rho \frac{r_x - r_{Cx}}{r_x^*}\left(q - \vec{\Omega}\cdot\vec{\nabla}\psi\right) \\ \phi_C^{\rho,y} = \dfrac{1}{4\pi\Sigma V_i}\displaystyle\int_{4\pi} \mathrm{d}\vec{\Omega} \int_{D_i} \mathrm{d}\vec{r} A_\rho \frac{r_y - r_{Cy}}{r_y^*}\left(q - \vec{\Omega}\cdot\vec{\nabla}\psi\right) \end{cases}, \tag{13}$$

which should be equal to the first order *geometrical* average ϕ_G^w (for $w \in \{x, y\}$) term over the same region:

$$\begin{cases} \phi_G^{\rho,x} = \dfrac{1}{V_i}\displaystyle\int_{D_i} \mathrm{d}\vec{r} \frac{r_x - r_{C,x}}{r_x^*}\phi^\rho(\vec{r}) \\ \phi_G^{\rho,y} = \dfrac{1}{V_i}\displaystyle\int_{D_i} \mathrm{d}\vec{r} \frac{r_y - r_{C,y}}{r_y^*}\phi^\rho(\vec{r}) \end{cases} \tag{14}$$

with $\vec{r}_C$ the barycentre of the region D_i. By using Eqs. (11), (12), (13) and (14) , it is possible to compute the corrective terms as [1]:

$$\begin{cases} \delta\phi^{\rho,1} = \phi_C^{\rho,1} - \displaystyle\sum_\alpha \mathbb{M}_\alpha \phi_\alpha^\rho \\ \begin{pmatrix}\delta\phi^{\rho,x} \\ \delta\phi^{\rho,y}\end{pmatrix} = \mathbb{M}_V^{-1}\left[\begin{pmatrix}\phi_C^{\rho,x} \\ \phi_C^{\rho,y}\end{pmatrix} - \displaystyle\sum_\alpha \begin{pmatrix}\mathbb{M}_\alpha^x \phi_\alpha^\rho \\ \mathbb{M}_\alpha^y \phi_\alpha^\rho\end{pmatrix}\right] \end{cases}, \tag{15}$$

where $\left(\phi_\alpha^\rho\right)_{\rho\in[\![1,N_m]\!]}$ the angular moments of the flux averaged over a surface α, $\mathbb{M}_\alpha$ an average operator over the extruded volume orthogonal to α, $\mathbb{M}_\alpha^w$ the first order spatial moment operator orthogonal to α (for $w \in \{x, y\}$) and $\mathbb{M}_V$ an average operator over volume V_i of a region D_i.$\mathbb{M}_\alpha$, $\mathbb{M}_\alpha^w$ and $\mathbb{M}_V$ are properly defined in [1].
In order to clearly explain how the DP_0 acceleration is carried out, it is detailed in the following as in [1] how the zeroth order and first order terms are expressed.

2.1.2 Detailed zeroth order

To obtain the final formulation of the corrective terms, one needs to express fully the geometric and conservative average:

$$\phi_G^{\rho,1} = \sum_\alpha \mathbb{M}_\alpha \phi_\alpha^\rho + \delta\phi^{\rho,1}, \tag{16}$$

$$\phi_C^{\rho,1} = \frac{1}{\Sigma} Q_C^{\rho,1} - \frac{1}{\Sigma V_i}\Delta J^{\rho,1}, \tag{17}$$

where,

$$\begin{aligned} Q_C^{\rho,1} &= \Sigma_s \phi_G^{\rho,1} + \vec{q^{\rho}}_{ext,G} \\ -\Delta J^{\rho,1} &= \sum_n \omega_n A_\rho(\vec{\Omega}_n) \sum_{t \| \mathbf{\Omega}_n, t \cap \partial D_i} \omega_t^{\perp} (\psi_t^{in} - \psi_t^{out}). \end{aligned} \tag{18}$$

2.1.3 Detailed first order

The following equations are carried out as for the zeroth order but with the inclusion of the first-order terms, resulting in:

$$\begin{bmatrix} \phi_G^{\rho,x} \\ \phi_G^{\rho,y} \end{bmatrix} = \sum_\alpha \begin{bmatrix} \mathbb{M}_\alpha^x \phi_\alpha^\rho \\ \mathbb{M}_\alpha^y \phi_\alpha^\rho \end{bmatrix} + \begin{bmatrix} \mathbb{M}_V^{x,x} & \mathbb{M}_V^{x,y} \\ \mathbb{M}_V^{y,x} & \mathbb{M}_V^{y,y} \end{bmatrix} \begin{bmatrix} \delta\phi^{\rho,x} \\ \delta\phi^{\rho,y} \end{bmatrix}, \tag{19}$$

$$\begin{cases} \phi_C^{\rho,x} = \dfrac{1}{\Sigma} Q_C^{\rho,x} + \dfrac{1}{\Sigma V} \left(-\Delta J_{1,B}^{\rho,x} - \dfrac{\mathbb{Z}_0^{\rho,x}}{r_x^*} \Delta\Psi \right) \\ \phi_C^{\rho,y} = \dfrac{1}{\Sigma} Q_C^{\rho,y} + \dfrac{1}{\Sigma V} \left(-\Delta J_{1,B}^{\rho,y} - \dfrac{\mathbb{Z}_0^{\rho,y}}{r_y^*} \Delta\Psi \right) \end{cases}, \tag{20}$$

where:

$$\begin{aligned} -\Delta J_{1,B}^{\rho,w} &\simeq \sum_n \omega_n A_\rho(\vec{\Omega}_n) \sum_{t \| \vec{\Omega}_n, t \cap \partial D_i} \omega_t^{\perp} \frac{r_{t,w}^m - r_{C,w}}{r_w^*} (\psi_t^{in} - \psi_t^{out}) \\ -\frac{\mathbb{Z}_0^{\rho,w}}{r_w^*} \Delta\Psi &\simeq -\sum_n \omega_n A_\rho(\vec{\Omega}_n) \sum_{t \| \vec{\Omega}_n, t \cap \partial D_i} \omega_t^{\perp} \frac{l_t \Omega_{w,n}}{r_w^*} \Delta\psi_t \\ w &\in \{x, y\} \end{aligned} .$$

2.2 Acceleration using DP_N

This paper only treats the acceleration using the zeroth order of DP_N method. Thus, only the so-called DP_0 will be detailed in the next section.

2.2.1 Synthetic problem

To recall the general idea of the scheme used to solve the transport equation, it is possible to rewrite Eq. (1) :

$$\mathcal{L}\psi^{k+1/2} = \mathcal{H}\psi^k + q_{ext}, \tag{21}$$

with $\mathcal{L}$ the leakage operator, $\mathcal{H}$ the scattering operator, k the iteration index and $k + 1/2$ the iteration where the DP_N acceleration takes place. Its converged form is:

$$\mathcal{L}\psi^{\infty} = \mathcal{H}\psi^{\infty} + q_{ext}. \tag{22}$$

By introducing $\Delta\psi^{k+1/2} = \psi^{\infty} - \psi^{k+1/2}$, a new problem appears:

$$\begin{aligned}(\mathcal{L} - \mathcal{H})\Delta\psi^{k+1/2} &= \mathcal{H}(\psi^{k+1/2} - \psi^{k}) \\ \psi^{k+1} &= \psi^{k+1/2} + \Delta\psi^{k+1/2},\end{aligned} \tag{23}$$

which could be called the *synthetic problem*. This equation is indeed as hard to solve as the initial problem, the difference being the residual $\Delta\psi^{k+1/2}$. The DP_0 acceleration method intend to solve equation Eq. (23) using a lower order method than the one used for Eq. (21).

2.2.2 DP_0 components

The resolution of Eq. (23) with a lower order operator is done by firstly replacing every assumption made for the MoC by low order ones (number of spherical harmonics and characteristics considered) and secondly by expanding the flux on each surface α as in [3]. Moreover because here only the DP_0 expansion is made, one have $\vec{A}(\vec{\Omega}) = A_0(\vec{\Omega}) = 1$, which gives:

$$\begin{cases} \psi(\vec{r} \in \alpha, \vec{\Omega}) = \delta(\vec{\Omega}, \vec{n}_{\alpha^+})\psi_{\alpha^+} + \delta(\vec{\Omega}, \vec{n}_{\alpha^-})\psi_{\alpha^-} \\ \phi_\alpha(\vec{r} \in \alpha) = A_{\alpha,+}\psi_{\alpha^+} + A_{\alpha,-}\psi_{\alpha^-} \end{cases}, \tag{24}$$

where $\vec{n}_{\alpha^+}$ and $\vec{n}_{\alpha^-}$ are the normal vectors of α respectively exiting and entering region D_i, and:

$$\begin{aligned} A_{\alpha,+/-} &= \frac{1}{4\pi S_\alpha}\int_\alpha \mathrm{d}S \int_{2\pi^\pm} \mathrm{d}\vec{\Omega} \\ \delta(\vec{\Omega}, \vec{n}) &= \begin{cases} 1 & \text{if } \vec{\Omega}\cdot\vec{n} > 0, \\ 0 & \text{if } \vec{\Omega}\cdot\vec{n} < 0 \end{cases}, \end{aligned} \tag{25}$$

and ψ_{α^+} and ψ_{α^-} stand for surface averaged angular zeroth order moment, that will be called DP_0 components. Then the general idea of the following sections is to express the corrective terms of Eq. (15) using exclusively these DP_0 components.

Zeroth order terms

Beginning with writing:

$$-\Delta J^1 = \sum_\alpha \left(J_{\alpha,-} - J_{\alpha,+}\right),$$

by using Eq. (5):

$$\begin{aligned} J_{\alpha,-} &= \hat{A}_{\alpha,-}\psi_{\alpha,-} \\ J_{\alpha,+} &= \hat{A}_{\alpha,+}\psi_{\alpha,+} \\ &= T_\alpha\vec{\psi}_{i,-} + \frac{1}{\Sigma}\left(E_\alpha\vec{q}_i + \Sigma_s\left(\vec{\delta\phi}^{xy} + \frac{1}{\Sigma}\vec{\delta q}^{xy}_{ext}\right)\cdot\vec{R}_\alpha\right). \end{aligned} \tag{26}$$

The quantities $\vec{\psi}_{i,-}$ and $\vec{q}_i$ are the concatenated values of the flux and source on each surface of the N_S surface of region D_i:

$$\begin{aligned} \vec{\psi}_{i,-} &= (\psi_{1,-} \dots \psi_{N_s,-})^T \\ \vec{q}_i &= (q_1 \dots q_{N_s})^T \end{aligned}. \tag{27}$$

Moreover:

$$\begin{aligned}
\hat{A}_{\alpha,+/-} &= \frac{1}{4\pi S_\alpha}\int_\alpha \mathrm{d}S_\perp \int_{2\pi^{+/-}} \mathrm{d}\vec{\Omega} \\
&= \frac{1}{S_\alpha}\int_\alpha \mathrm{d}S \int_{2\pi^{+/-}} \mathrm{d}\vec{\Omega}\,|\vec{\Omega}\cdot\vec{n}_\alpha| \\
T_{\alpha\beta} &= \frac{1}{4\pi S_\alpha}\int_\alpha \mathrm{d}S \int_{(\beta\to\alpha)} \mathrm{d}\vec{\Omega}\,|\vec{\Omega}\cdot\vec{n}_\alpha| e^{-\tau_t} \\
E_{\alpha\beta} &= \frac{1}{4\pi S_\alpha}\int_\alpha \mathrm{d}S \int_{(\beta\to\alpha)} \mathrm{d}\vec{\Omega}\,|\vec{\Omega}\cdot\vec{n}_\alpha|(\beta_0(\tau_t)-\beta_1(\tau_t)) \\
&\quad + \delta_{\alpha\beta}\int_\alpha \mathrm{d}S \int_{2\pi^+} \mathrm{d}\vec{\Omega}\,|\vec{\Omega}\cdot\vec{n}_\alpha|\beta_1(\tau_t) \\
\beta_0(\tau_t) &= 1-e^{-\tau_t} \\
\beta_1(\tau_t) &= 1-\frac{1}{\tau_t}\left(1-e^{-\tau_t}\right) \qquad .
\end{aligned} \tag{28}$$

$(\beta \to \alpha)$ indicates that the integration is made over the direction such that β is the entering surface and α is the exiting one. As for the flux and source, one have use the notation:

$$\begin{aligned}
T_\alpha &= (T_{\alpha,1}\dots T_{\alpha,N_s})^T \\
E_\alpha &= (E_{\alpha,1}\dots E_{\alpha,N_s})^T .
\end{aligned} \tag{29}$$

Then with

$$\begin{aligned}
Q_C^1 &= \Sigma_s\left(\sum_\alpha \mathbb{M}_\alpha\phi_\alpha + \delta\phi^1\right) + \vec{q}^{\,1}_{ext,G} \\
&= \Sigma_s M\left(A_{i,-}\vec{\psi}_{i,-} + A_{i,+}\vec{\psi}_{i,+}\right) + \Sigma_s\delta\phi^1 + M\vec{q}_{i,ext}, \\
M &= (\mathbb{M}_\alpha)_{\alpha\in\partial D_i} .
\end{aligned} \tag{30}$$

Then by using the flat source approximation: $q_{\alpha,t} = A_0(\vec{\Omega_n})(\Sigma_s\phi_\alpha + \vec{q}_{\alpha,ext})$ and Eqs. (15),(16),(17) and (18), it is possible to express the zeroth order corrective term as:

$$\begin{aligned}
\delta\phi^1 = \left(I-\tilde{\Sigma}_s\right)^{-1}\Bigg\{&\left[\frac{1}{\Sigma V_i}(\hat{A}_{i,-} - T - \tilde{\Sigma}_s E A_{i,-}) + (\tilde{\Sigma}_s - I)M A_{i,-}\right]\vec{\psi}_{i,-} \\
&+\left[(\tilde{\Sigma}_s - I)M - \frac{1}{\Sigma_i V_i}\tilde{\Sigma}_s E\right]A_{i,+}\vec{\psi}_{i,+} \\
&-\frac{1}{\Sigma_i V_i}E\vec{\tilde{q}}_{i,ext} + M\vec{q}_{i,ext} - \frac{1}{\Sigma V_i}\left[\tilde{\Sigma}_s\vec{\delta\phi}^{\,xy} + \vec{\delta\tilde{q}}^{\,xy}_{ext}\right]\cdot\vec{R}\Bigg\}.
\end{aligned} \tag{31}$$

Quantities marked with ~ are divided by Σ and:

$$\begin{aligned}
T &= \sum_\alpha T_\alpha \\
E &= \sum_\alpha E_\alpha .
\end{aligned} \tag{32}$$

First order terms

Using a variation of the zeroth order that includes the first spatial moment term $\frac{r^m_{t,w} - r_{C,w}}{r^*_w}$, for $w \in \{x, y\}$, it is possible to express the first order corrective terms as:

$$\begin{pmatrix} \delta\phi^x \\ \delta\phi^y \end{pmatrix} = \begin{pmatrix} G^x_- \vec{\psi}_{i,-} + G^x_+ \vec{\psi}_{i,+} + G^x_{ext} \vec{\tilde{q}}_{i,ext} + \mathbf{K}^x_{xy} \cdot \vec{\delta\tilde{q}}^{xy}_{ext} \\ G^y_- \vec{\psi}_{i,-} + G^y_+ \vec{\psi}_{i,+} + G^y_{ext} \vec{\tilde{q}}_{i,ext} + \mathbf{K}^y_{xy} \cdot \vec{\delta\tilde{q}}^{xy}_{ext} \end{pmatrix}, \tag{33}$$

with $\delta\phi^1$ of Eq. (9) being:

$$\delta\phi^1 = F_- \vec{\psi}_{i,-} + F_+ \vec{\psi}_{i,+} + F_{ext} \vec{\tilde{q}}_{i,ext} + \mathbf{F}^{xy}_{ext} \cdot \vec{\delta\tilde{q}}^{xy}_{ext}. \tag{34}$$

The implemented form of the transmission use a collision free matrix, by injecting Eqs. (33) and (34) in Eq. (26), which gives :

$$\vec{J}_{i,+} = T' \vec{J}_{i,-} + E' \vec{\tilde{q}}_{i,ext} + \delta E'^x \vec{\delta\tilde{q}}^x_{ext} + \delta E'^y \vec{\delta\tilde{q}}^y_{ext}. \tag{35}$$

With the matrices T', E', $\delta E'^x$ and $\delta E'^y$ that are not developped for the sake of brevity of the paper.
Arrived in the internal iterations, a flux is computed using Eq. (35), which is used to compute the corrective terms through Eqs. (33) and (34) in order to ensure the conservation of constant and first order spatial moments.

3 NUMERICAL RESULTS

In this section, we benchmark the accelerated LS-xy scheme with a well-known test case, the C5G7. The goals were on the one hand to shed light on the scope of the DP_0 for the LS-xy scheme, and on the other hand to observe the differences between the LS-xy and the CLS when accelerated using DP_0. The results are obtained using TDT, with an angular quadrature with 20 uniformly distributed horizontal angles, 3 vertical angles of the Bickley-Naylor type [5] and a transversal integration step of 0.01. The mesh is the following showed on Fig. 1:

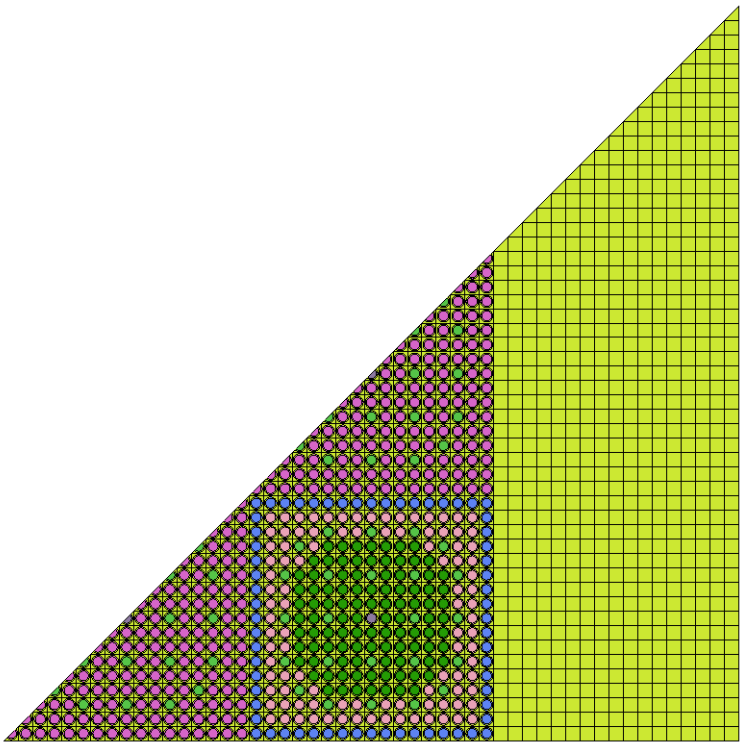

Figure 1. Spatial mesh on the C5G7, using 2516 regions

The initial observation is that the chosen mesh is coarser than the one required for CLS convergence [2]. Firstly, for the CLS convergence an additional mesh region in the water medium of each pin is needed. Furthermore, a flux gradient exists at the core/reflector interface and an additional mesh is needed for the CLS to achieve convergence.
The obtained results are compared with a Monte Carlo computation of the k_{eff} provided in [4]. Tab.1 shows the differences between the computation using LS-xy with and without DP_0 acceleration.

MoC Scheme	$\Delta k_{eff}/k_{eff}$ (pcm)	Execution Time (s)	Number Inner	Number Outer
LS-xy	51	17013	8466	150
CLS DP_0	228	1688	1271	37
LS-xy DP_0	50	2983	1318	38

Table 1. C5G7 results between classical and accelerated LS-xy

Tab. 1 underlines that compared with the classical CLS, the LS-xy scheme gives results closer to the reference value and that the DP_0 acceleration effectively accelerates the convergence. A larger scaling of the acceleration for DP_1 can be hoped for.

4 CONCLUSIONS

This paper discusses the improvements that have been developed for the LS-xy higher MOC scheme in the APOLLO3® code system. In particular we have developed an acceleration scheme, both for the internal and for the external iterations, allowing to use the transport scheme for industrial applications. We have implemented the DP_0 synthetic hypothesis onto this scheme and solved the technical issues related to this approach. The final results we present show that this method is likely to remain more efficient than its predecessors once implemented in DP1 and has a full maturity to be used industrially. In future we aim to test it for lager scale application as full 2D core calculations. Finally, this development also enables the integration of other higher-order methods integrating development of cross sections with suitable acceleration techniques in the future.

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