EPJ Web of Conferences **3**, 02008 (2010)
DOI:10.1051/epjconf/20100302008


# Causality and universality in low-energy scattering

Dean Lee[1,2,a] and H.-W. Hammer[2]

[1] Department of Physics, North Carolina State University, Raleigh, NC 27695, USA
[2] Helmholtz-Institut für Strahlen- und Kernphysik (Theorie) and Bethe Center for Theoretical Physics, Universität Bonn, D-53115 Bonn, Germany

**Abstract.** We discuss the generalization of Wigner's causality bounds and Bethe's integral formula for the effective range parameter to arbitrary dimension and arbitrary angular momentum. We consider the impact of these constraints on the separation of low- and high-momentum scales and universality in low-energy scattering.

## 1 Introduction

Causality in quantum mechanics requires that no scattered wave propagates before the incident wave first reaches the scatterer. For the case of finite-range interactions the constraints of causality on elastic phase shifts were first investigated by Wigner [1]. To illustrate the underlying physics we consider a wavepacket of outgoing spherical waves in $d$ spatial dimensions,

$$f_{\rm out}(r) = \int_0^\infty dp\, e^{ipr} \tilde{f}_{\rm out}(p). \tag{1}$$

We have absorbed normalization factors and the $r^{-(d-1)/2}$ dependence into the definition of $f_{\rm out}(r)$. We now include scattering effects. The $S$-matrix multiplies asymptotic outgoing states by a phase factor $e^{2i\delta(p)}$, where $\delta(p)$ is the elastic phase shift. We assume the momentum distribution $\tilde{f}_{\rm out}(p)$ is sharply peaked around some nonzero value $\bar{p}$. If $f^{\delta}_{\rm out}(r)$ is the scattered wavepacket, then

$$\begin{aligned} f^{\delta}_{\rm out}(r) &= \int_0^\infty dp\, e^{ipr} e^{2i\delta(p)} \tilde{f}_{\rm out}(p) \\ &\approx e^{2i\delta(\bar{p})} e^{-2i\delta'(\bar{p})\bar{p}} f_{\rm out}\left[r + 2\delta'(\bar{p})\right]. \end{aligned} \tag{2}$$

The wavepacket is shifted forward by $\Delta r = -2\delta'(\bar{p})$ relative to the wavepacket with no scattering. If we consider the wavepacket as a function of time, the same shift can be interpreted as a time shift or delay for the scattered wavepacket,

$$\Delta t = 2\left.\frac{d\delta}{dE}\right|_{\bar{E}}, \tag{3}$$

where $\bar{E}$ is the energy corresponding with $\bar{p}$. The radius shift and time delay are sketched in Fig. 1. A classical analysis of particle trajectories suggests that if the interactions have a finite range $R$, then causality requires $-\delta'(\bar{p}) \leq R$. While this argument is qualitatively correct, it ignores the quantum mechanical spread of the wavepacket in space. The exact causality constraint is the content of this proceedings article. The material presented here is drawn from Ref. [2] as well as a paper currently being written [3].

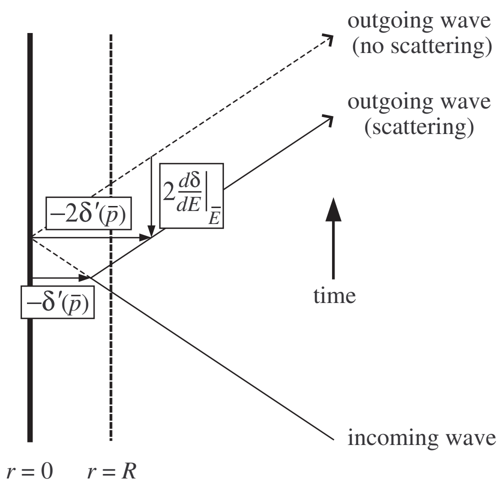


**Fig. 1.** Time versus radius diagram. The scattered wavepacket is shifted in distance by $-2\delta'(\bar{p})$ and shifted in time by $2\left.\frac{d\delta}{dE}\right|_{\bar{E}}$.

## 2 Low-energy universality

Universality at low energies can appear when there is a large separation between the short-distance scale of the interaction and the long-distance scales relevant to the physical system. One example of low-energy universality is the unitarity limit, which refers to an idealized system where the range of the interaction is zero and the $S$-wave scattering length is infinite. In nuclear physics, cold dilute neutron matter is close to the unitarity limit. Unitarity-limit physics is also probed in experiments with cold $^6$Li and $^{40}$K atoms using magnetically-tuned Feshbach resonances. For

[a] e-mail: dean_lee@ncsu.edu

reviews of recent cold atom experiments, see Refs. [4,5]. Theoretical overviews of ultracold Fermi gases and their numerical simulations are given in [6,7]. A general review of universality at large scattering length can be found in [8].

Several experiments have also investigated strongly-interacting $P$-wave Feshbach resonances in $^6$Li and $^{40}$K [9–13]. A key question is whether the physics of these strongly-interacting $P$-wave systems is universal, and if so, what are the relevant low-energy parameters. A resolution of these issues would provide a connection between the atomic physics of $P$-wave Feshbach resonances and the nuclear physics of $P$-wave alpha-neutron interactions in halo nuclei. Some progress in addressing these questions has been made with low-energy models of $P$-wave atomic interactions [14–19] and $P$-wave alpha-neutron interactions [20–23]. A renormalization group study showed that scattering should be weak in higher partial waves unless there is a fine tuning of multiple parameters [24].

We answer the question of universality and the constraints of causality for arbitrary dimension $d$ and arbitrary angular momentum $L$. The analysis applies to any finite-range interaction that is energy independent, non-singular, and spin independent. We present generalizations of Bethe's integral formula for the effective range [25] and causality bounds for arbitrary $d$ and $L$. Our results can be viewed as a generalization of the analysis of Phillips and Cohen [26], who derived a Wigner bound for the $S$-wave effective range for short-range interactions in three dimensions.

## 3 Radial equation

We consider two non-relativistic spinless particles in $d$ dimensions with a rotationally-invariant two-body interaction. We analyze the two-body system in the center-of-mass frame. Let $\mu$ be the reduced mass and $p^2/(2\mu)$ be the total energy. For $d > 1$ angular momentum is specified by $d-1$ integer quantum numbers [27]

$$\mathbf{L} = \{M_1, \cdots, M_{d-1}\}, \tag{4}$$

satisfying

$$|M_1| \leq M_2 \leq \cdots \leq M_{d-2} \leq M_{d-1}. \tag{5}$$

We let $L$ label the absolute value of the top-level quantum number, $|M_{d-1}|$. For example when $d = 3$, $M_1$ is $L_z$ and $M_2 = |M_2| = L$ is the total angular momentum. In one spatial dimension continuous rotations do not exist. Therefore we must treat $d = 1$ as a special case. Instead of rotational invariance, the key symmetry in one dimension will be invariance under parity. We assume a parity-symmetric interaction and write $\mathbf{L} = L = 0$ for even parity and $\mathbf{L} = L = 1$ for odd parity. In the following all results for rotationally-invariant interactions in $d > 1$ are also valid for parity-symmetric interactions in $d = 1$.

We analyze the two-body system in the center-of-mass frame. With reduced mass $\mu$ and energy $p^2/(2\mu)$, we rescale the radial wavefunction $R^{(p)}_{L,d}(r)$ as

$$u^{(p)}_{L,d}(r) = (pr)^{(d-1)/2} R^{(p)}_{L,d}(r). \tag{6}$$

The interaction is assumed to be energy independent and have a finite range $R$ beyond which the particles are non-interacting. Writing the interaction as a real symmetric operator with kernel $W(r, r')$, we obtain the radial equation

$$p^2 u^{(p)}_{L,d}(r) = \left[-\frac{d^2}{dr^2} + \frac{(2L+d-1)(2L+d-3)}{4r^2}\right] u^{(p)}_{L,d}(r) + 2\mu \int_0^R dr' W(r, r') u^{(p)}_{L,d}(r'). \tag{7}$$

The normalization of $u^{(p)}_{L,d}(r)$ is chosen so that for $r \geq R$,

$$u^{(p)}_{L,d}(r) = \sqrt{\frac{pr\pi}{2}}\, p^{L+d/2-3/2} \times \left[\cot\delta_{L,d}(p) J_{L+d/2-1}(pr) - Y_{L+d/2-1}(pr)\right]. \tag{8}$$

Here $J_\alpha$ and $Y_\alpha$ are Bessel functions of the first and second kind, and $\delta_{L,d}(p)$ is the phase shift for partial wave $L$. The phase shifts are directly related to the elastic scattering amplitude $f_{L,d}(p)$, where

$$f_{L,d}(p) \propto \frac{p^{2L}}{p^{2L+d-2}\cot\delta_{L,d}(p) - ip^{2L+d-2}}. \tag{9}$$

In addition to having finite range, we assume also that the interaction is not too singular at short distances. Specifically, we require that the effective range expansion defined below in Eq. (10) converges for sufficiently small $p$ and that $\frac{d}{dr}u^{(p)}_{L,d}$ is finite and $u^{(p)}_{L,d}$ vanishes as $r \to 0$. As an example these short-distance regularity conditions are satisfied for a local potential, $W(r, r') = V(r)\delta(r - r')$, provided that $V(r) = O(r^{-2+\epsilon})$ as $r \to 0$ for positive $\epsilon$ [28]. In our discussion, however, we make no assumption that the interactions arise from a local potential. The treatment of spin-dependent interactions with partial wave mixing is beyond the scope of this analysis. For coupled-channel dynamics without partial wave mixing the analysis can proceed by first integrating out higher-energy contributions to produce a single-channel effective interaction. In order to satisfy our condition of energy-independent interactions, this should proceed using a technique such as the method of unitary transformation described in Ref. [29–31].

## 4 Effective range expansion

The effective range expansion is

$$p^{2L+d-2}\left[\cot\delta_{L,d}(p) - \delta_{(d \bmod 2),0}\frac{2}{\pi}\ln(p\rho_{L,d})\right] = -\frac{1}{a_{L,d}} + \frac{1}{2}r_{L,d}p^2 + \sum_{n=0}^{\infty}(-1)^{n+1}\mathcal{P}^{(n)}_{L,d}p^{2n+4}. \tag{10}$$

The term $\delta_{(d \bmod 2),0}$ is 0 for odd $d$ and 1 for even $d$. $a_{L,d}$ is the scattering parameter, $r_{L,d}$ is the effective range parameter, and $\mathcal{P}^{(n)}_{L,d}$ are the $n^{\text{th}}$-order shape parameters. $\rho_{L,d}$ is an arbitrary length scale that can be scaled to any positive value. The rescaling results in a shift of the dimensionless coefficient of $p^{2L+d-2}$ on the right-hand of Eq. (10), and we define $\bar{\rho}_{L,d}$ as the special value for $\rho_{L,d}$ where this coefficient is zero.

Let $u^{(p)}_{L,d}$ and $u^{(p')}_{L,d}$ be radial solutions of the Schrödinger equation for two different momenta. We construct the Wronskian of the two solutions,

$$u^{(p)}_{L,d}\frac{d}{dr}u^{(p')}_{L,d} - u^{(p')}_{L,d}\frac{d}{dr}u^{(p)}_{L,d}, \tag{11}$$

and evaluate at some radius $r \geq R$. Taking the limits $p' \to 0$ and then $p \to 0$, we find that for any $r \geq R$,

$$r_{L,d} = b_{L,d}(r) - 2\lim_{p\to 0}\int_0^r dr'\left[u^{(p)}_{L,d}(r')\right]^2, \tag{12}$$

where $b_{L,d}(r)$ is defined as follows. For $2L + d = 2$, we have

$$b_{L,d}(r) = \frac{2r^2}{\pi}\left\{\left[\ln\left(\frac{r}{2\rho_{L,d}}\right) + \gamma - \frac{1}{2} + \frac{\pi}{2a_{L,d}}\right]^2 + \frac{1}{4}\right\}, \tag{13}$$

where $\gamma$ is the Euler-Mascheroni constant and for $2L+d = 4$,

$$b_{L,d}(r) = \frac{4}{\pi}\left[\ln\left(\frac{r}{2\rho_{L,d}}\right) + \gamma\right] - \frac{4}{a_{L,d}}\left(\frac{r}{2}\right)^2 + \frac{\pi}{a^2_{L,d}}\left(\frac{r}{2}\right)^4. \tag{14}$$

For the generic case of $2L + d$ any positive odd integer or any even integer $\geq 6$:

$$\begin{aligned} b_{L,d}(r) = &-\frac{2\Gamma(L+\frac{d}{2}-2)\Gamma(L+\frac{d}{2}-1)}{\pi}\left(\frac{r}{2}\right)^{-2L-d+4} \\ &-\frac{4}{L+\frac{d}{2}-1}\frac{1}{a_{L,d}}\left(\frac{r}{2}\right)^2 \\ &+\frac{2\pi}{\Gamma(L+\frac{d}{2})\Gamma(L+\frac{d}{2}+1)}\frac{1}{a^2_{L,d}}\left(\frac{r}{2}\right)^{2L+d}. \end{aligned} \tag{15}$$

The formula in Eq. (15) for $L = 0$ in three dimensions was first derived by Bethe [25] and extended by Madsen for general $L$ [32]. The results presented here give the generalization to arbitrary $d$ and arbitrary $L$.

## 5 Causality bounds

Since the integrand in Eq. (12) is positive semi-definite, $r_{L,d}$ satisfies the upper bound

$$r_{L,d} \leq b_{L,d}(r) \tag{16}$$

for any $r \geq R$. For $d = 3$ our results are equivalent to the causality bound derived by Wigner [1]. For $S$-wave interactions in three dimensions the upper bound on the effective range was discussed in Ref. [26]. It was observed

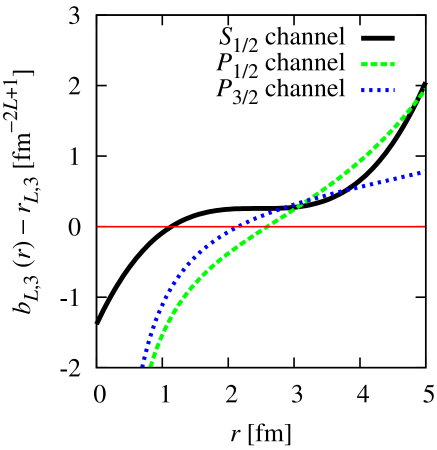


**Fig. 2.** Plot of $b_{L,3}(r) - r_{L,3}$ as a function of $r$ for alpha-neutron scattering in the $S_{1/2}$, $P_{1/2}$, and $P_{3/2}$ channels. Causality requires this function to be non-negative for $r \geq R$.

that for fixed $a_{L,d}$ the zero-range limit $R \to 0$ is possible only when $r_{L,d}$ is negative. The constraint becomes more severe for larger $2L + d$. For $2L + d \geq 4$, the limit $R \to 0$ at fixed $a_{L,d}$ produces a divergence in the effective range, $r_{L,d} \leq b_{L,d}(R) \to -\infty$.

Our results are exact only for the case where the interaction vanishes for $r \geq R$. For exponentially-bounded interactions of $O(e^{-r/R})$ at large distances, the results should still be accurate with only exponentially small corrections. For an exponentially-bounded but otherwise unknown interaction, the non-negativity requirement for $b_{L,d}(r) - r_{L,d}$ can be used to determine the minimum value for $R$ consistent with causality.

As an example we consider three-dimensional scattering of an alpha particle and neutron in the $S$-wave and $P$-wave channels. For $S$-wave scattering,

$$b_{0,3}(r) = 2r - \frac{2r^2}{a_{0,3}} + \frac{2r^3}{3a^2_{0,3}}, \tag{17}$$

and for $P$-wave scattering,

$$b_{1,3}(r) = -\frac{2}{r} - \frac{2r^2}{3a_{1,3}} + \frac{2r^5}{45a^2_{1,3}}. \tag{18}$$

In Fig. 2 we plot $b_{L,3}(r) - r_{L,3}$ for the $S_{1/2}$, $P_{1/2}$, and $P_{3/2}$ channels. A qualitatively similar plot was shown for nucleon-nucleon scattering in the $S$-wave spin-singlet channel [33]. We use the values $a_{0,3} = 2.464(4)$ fm and $r_{0,3} = 1.39(4)$ fm for $S_{1/2}$; $a_{1,3} = -13.82(7)$ fm$^3$ and $r_{1,3} = -0.42(2)$ fm$^{-1}$ for $P_{1/2}$; and $a_{1,3} = -62.951(3)$ fm$^3$ and $r_{1,3} = -0.882(1)$ fm$^{-1}$ for $P_{3/2}$ [34]. The non-negativity condition for $b_{L,3}(r) - r_{L,3}$ gives $R \geq 1.1$ fm for $S_{1/2}$, $R \geq 2.6$ fm for $P_{1/2}$, and $R \geq 2.1$ fm for $P_{3/2}$. For comparison, the alpha root-mean-square radius and pion Compton wavelength are both about 1.5 fm. Since the minimum values for $R$ are not small when compared with these, some caution is required when choosing the cutoff scale for an effective theory of alpha-neutron interactions.

We briefly comment on the requirement that the interactions are energy independent. For energy-dependent interactions it is possible to generate any energy dependence for the phase shifts even when the interaction $W(r, r'; E)$ vanishes beyond some finite radius $R$ for all $E$. Under these more general conditions there are no longer any causality bounds. However, it is misleading to regard interactions of this more general type as having finite range. As noted in the introduction, the scattering time delay is proportional to the energy derivative of the phase shift. The energy dependence of the interaction can by itself generate large negative time delays and thereby reproduce the scattering of long-range interactions. In this sense the range of the interaction as observed in scattering is set by the dependence of $W(r, r'; E)$ on the radial coordinates $r, r'$ as well as the energy $E$. For this case the bound in Eq. (16) can be viewed as an estimate for the minimum value of this interaction range.

We now consider the scattering amplitude in the low-energy limit $p \to 0$ while keeping the interaction range $R$ fixed. In the limit $p \to 0$, the hierarchy of terms in the effective range expansion depends on the value of $2L + d$. This is sketched in the diagram in Fig. 3.

In the low-energy limit the scattering amplitude depends on just one dimensionful parameter when $2L+d \leq 3$. For $2L+d = 1$ and $2L+d = 3$ the relevant parameter is $a_{L,d}^{-1}$, and for $2L+d = 2$ it is $\bar{\rho}_{L,d}$. When $2L+d \geq 4$ a second dimensionful parameter appears in the non-perturbative low-energy limit. In the limit $a_{L,d}^{-1} \to 0$, the upper bounds on the effective range reduce to the form $\bar{\rho}_{L,d} \leq \frac{r}{2} e^{\gamma}$ for $2L+d = 4$, and

$$r_{L,d} \leq -\frac{2\Gamma(L+\frac{d}{2}-2)\Gamma(L+\frac{d}{2}-1)}{\pi}\left(\frac{r}{2}\right)^{-2L-d+4} \tag{19}$$

for $2L+d \geq 5$. There is no way to suppress the $p^2 \ln(p\bar{\rho}_{L,d})$ term in the effective range expansion for $2L + d = 4$ by fine-tuning parameters because the bound forbids tuning the argument of the logarithm to 1 as $p \to 0$. Similarly, for $2L+d \geq 5$ the upper bound in Eq. (19) and the negative coefficient on the right-hand side prevent setting $r_{L,d}$ to zero to eliminate the term $\frac{1}{2} r_{L,d} p^2$. Hence in both cases we are left with two relevant parameters in the non-perturbative low-energy limit. This corresponds with two relevant directions near a fixed point of the renormalization group, and the universal behavior is characterized by two low-energy parameters. For the case of $P$-wave neutron-alpha scattering in three dimensions, this issue was already discussed in [20]. Proper renormalization of an effective field theory for $P$-wave scattering requires the inclusion of field operators for the scattering volume and the effective range at leading order. In the renormalization group study of [24], the emergence of two relevant directions around a fixed point was observed for various model potentials.

For $2L + d \geq 4$ this second dimensionful parameter has a straightforward physical interpretation for shallow bound states. Consider a bound state at $p = ip_I$ in the zero binding-energy limit $p_I \to 0^+$. Let $P_>(r)$ be the probability of finding the constituent particles with separation larger than $r$:

$$P_>(r) = \int_r^\infty dr' \left[\hat{u}_{L,d}^{(ip_I)}(r')\right]^2, \tag{20}$$

where $\hat{u}_{L,d}^{(ip_I)}$ is the normalized wave function. For $2L + d \leq 3$, the probability $P_>(r)$ is 1 in the limit $p_I \to 0^+$ for any $r$. At sufficiently low energies the physics at short distances is irrelevant, and the bound state wavefunction extends over large distances. For $2L + d \geq 4$, however, the situation changes. For $2L + d = 4$ the probability is logarithmically dependent on $\bar{\rho}_{L,d}$ and can be tuned to any value between 0 and 1. Similarly for $2L + d \geq 5$,

$$P_>(r) \to \frac{2\Gamma(L+\frac{d}{2}-2)\Gamma(L+\frac{d}{2}-1)}{(-r_{L,d})\pi}\left(\frac{r}{2}\right)^{-2L-d+4} \tag{21}$$

for $r \geq R$. For this case the characteristic size of the bound state wavefunction is $(-r_{L,d})^{1/(-2L-d+4)}$.

For $P$-wave Feshbach resonances in alkali atoms this analysis must be modified to take into account long-range van der Waals interactions of the type

$$W(r, r') = -C_6 r^{-6} \delta(r - r') \tag{22}$$

for $r, r' \geq R$. We briefly comment on potentials with a van der Waals tail in three dimensions. It is convenient to re-express $C_6$ in terms of the length scale $\beta_6 = (2\mu C_6)^{1/4}$. In the following, we set $d = 3$ and drop the $d$ subscript. Instead of free Bessel functions, scattering states should be compared with exact solutions of the attractive $r^{-6}$ potential [35,36]. The effect of the interactions for $r < R$ are described by a finite-range $K$-matrix $K_L(p^2)$ which is analytic in $p^2$ [37], $K_L(p^2) = \sum_{n=0,1,\cdots} K_L^{(2n)} p^{2n}$. When phase shifts are measured relative to free spherical Bessel functions, the effective range expansion is no longer analytic in $p^2$. For $L = 0$, the leading non-analytic term is proportional to $p^3$. For $L = 1$ the non-analytic term is proportional to $p^1$, thereby voiding the usual definition of the effective range parameter.

For a pure van der Waals tail, however, one can still obtain useful information from our approach. The zero-energy resonance limit is reached by tuning the lowest-order $K$-matrix coefficient $K_L^{(0)}$ to zero. It turns out that for $L = 1$ in this limit the $p^1$ coefficient in the effective range expansion also vanishes, and one can define an effective range parameter for both $S$- and $P$-waves [36,38],

$$\begin{aligned} r_0 &= [\Gamma(1/4)]^2 \left(\beta_6 + 3K_0^{(2)}\beta_6^{-1}\right)/(3\pi), \\ r_1 &= -36\,[\Gamma(3/4)]^2 \left(\beta_6^{-1} - 5K_1^{(2)}\beta_6^{-3}\right)/(5\pi). \end{aligned} \tag{23}$$

For the case of single-channel scattering of alkali atoms, the coefficients $K_L^{(2)}$ are negligible compared with $\beta_6^2$. This is also true for some multi-channel Feshbach resonance systems [39]. In these cases we observe that the upper bounds for $r_L$ in Eq. (19) are satisfied for $L = 0$ and $L = 1$ when we naively take $R \sim \beta_6$. In general, there may be multi-channel systems where the coefficients $K_L^{(2)}$ cannot be neglected. Nevertheless, the coefficients $K_L^{(2)}$ should satisfy causality bounds similar to those derived here for the effective range parameter.

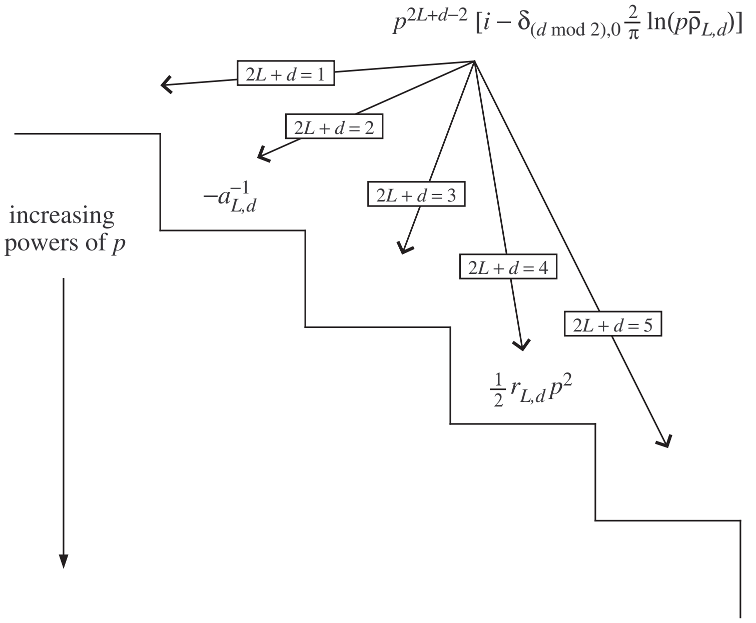


**Fig. 3.** Hierarchy of terms in the effective range expansion as $p \to 0$.

## 6 Summary

We have addressed the question of universality and the constraints of causality for arbitrary dimension $d$ and arbitrary angular momentum $L$. For finite-range interactions we have shown that causal wave propagation can have significant consequences for low-energy universality and scale invariance. In certain cases two relevant low-energy parameters are required in the non-perturbative low-energy limit. In the language of the renormalization group, this corresponds with two relevant directions in the vicinity of a fixed point. In particular, we confirm earlier findings in the case three dimensions for $P$-wave scattering [20] based on renormalization arguments and for higher partial waves in general [24] in the framework of the renormalization group.

## References


1. E. P. Wigner, Phys. Rev. **98**, (1955) 145.
2. H.-W. Hammer and D. Lee, Phys. Lett. **B681**, (2009) 500.
3. H.-W. Hammer and D. Lee, *in preparation*.
4. T. Koehler, K. Goral, and P. S. Julienne, Rev. Mod. Phys. **78**, (2006) 1311.
5. C. Regal and D. S. Jin (2006), cond-mat/0601054.
6. S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **80**, (2008) 1215.
7. D. Lee, Prog. Part. Nucl. Phys. **63**, (2009) 117.
8. E. Braaten and H.-W. Hammer, Phys. Rept. **428**, (2006) 259.
9. C. A. Regal, C. Ticknor, J. L. Bohn, and D. S. Jin, Phys. Rev. Lett. **90**, (2003) 053201.
10. C. Ticknor, C. A. Regal, D. S. Jin, and J. L. Bohn, Phys. Rev. **A69**, (2004) 042712.
11. J. Zhang, E. G. M. van Kempen, T. Bourdel, L. Khaykovich, J. Cubizolles, F. Chevy, M. Teichmann, L. Tarruell, S. J. J. M. F. Kokkelmans, and C. Salomon, Phys. Rev. **A70**, (2004) 030702(R).
12. C. H. Schunck, M. W. Zwierlein, C. A. Stan, S. M. F. Raupach, W. Ketterle, A. Simoni, E. Tiesinga, C. J. Williams, and P. S. Julienne, Phys. Rev. **A71**, (2005) 045601.
13. J. P. Gaebler, J. T. Stewart, J. L. Bohn, and D. S. Jin, Phys. Rev. Lett. **98**, (2007) 200403.
14. Y. Ohashi, Phys. Rev. Lett. **94**, (2004) 050403.
15. F. Chevy, E. G. M. van Kempen, T. Bourdel, J. Zhang, L. Khaykovich, M. Teichmann, L. Tarruell, S. J. J. M. F. Kokkelmans, and C. Salomon, Phys. Rev. **A71**, (2005) 062710.
16. V. Gurarie, L. Radzihovsky, and A. V. Andreev, Phys. Rev. Lett. **94**, (2005) 230403.
17. J. Levinsen, N. Cooper, and V. Gurarie, Phys. Rev. Lett. **99**, (2007) 210402.
18. K. B. Gubbels and H. T. C. Stoof, Phys. Rev. Lett. **99**, (2007) 190406.
19. M. Jona-Lasinio, L. Pricoupenko, and Y. Castin, Phys. Rev. **A77**, (2008) 043611.
20. C. A. Bertulani, H.-W. Hammer, and U. Van Kolck, Nucl. Phys. **A712**, (2002) 37.
21. P. F. Bedaque, H.-W. Hammer, and U. van Kolck, Phys. Lett. **B569**, (2003) 159.
22. K. M. Nollett, S. C. Pieper, R. B. Wiringa, J. Carlson, and G. M. Hale, Phys. Rev. Lett. **99**, (2007) 022502.
23. R. Higa, Mod. Phys. Lett. **A24**, (2009) 915.
24. T. Barford and M. C. Birse, Phys. Rev. **C67**, (2003) 064006.

25. H. A. Bethe, Phys. Rev. **76**, (1949) 38.
26. D. R. Phillips and T. D. Cohen, Phys. Lett. **B390**, (1997) 7.
27. J. Avery and Z.-Y. Wen, Int. J. Quantum Chem. **22**, (1982) 717.
28. D. Bolle and F. Gesztesy, Phys. Rev. **A30**, (1984) 1279.
29. N. Fukuda, K. Sawada, and M. Taketani, Prog. Theor. Phys. **12**, (1954) 156.
30. S. Okubo, Prog. Theor. Phys. **12**, (1954) 603.
31. E. Epelbaum, W. Glockle, and U.-G. Meißner, Nucl. Phys. **A637**, (1998) 107.
32. L. B. Madsen, Am. J. Phys. **70**, (2002) 811.
33. K. A. Scaldeferri, D. R. Phillips, C. W. Kao, and T. D. Cohen, Phys. Rev. **C56**, (1997) 679.
34. R. A. Arndt, D. D. Long, and L. D. Roper, Nucl. Phys. A **209**, (1973) 429.
35. B. Gao, Phys. Rev. **A58**, (1998) 1728.
36. B. Gao, Phys. Rev. **A58**, (1998) 4222.
37. B. Gao, Phys. Rev. **A80**, (2009) 012702.
38. V. V. Flambaum, G. F. Gribakin, and C. Harabati, Phys. Rev. **A59**, (1999) 1998.
39. T. M. Hanna, E. Tiesinga, and P. S. Julienne, Phys. Rev. **A79**, (2009) 040701(R).