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Modif ed effective range expansion for nucleon-nucleon scattering

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Abstract. The standard effective range expansion is commonly used in nucleon-nucleon scattering to encode the properties of the nuclear force model-independently in a small set of parameters. However, its applicability is limited by the longest-range part of the nuclear potential, i.e. by the one pion exchange, to the domain of momenta below half the pion mass. Therefore, it is not useful to study shorter-range parts of the interaction, e.g. two-pion exchange. To this aim a modification that explicitly takes a given long-range part of the interaction into account is required. This is known as modified effective range expansion. Here, we apply this approach to the nucleon-nucleon interaction to separate the known long-range interactions from the rest. To show the effectiveness of this technique, we consider a toy model with a two-range potential. We study the scaling behaviour of the parameters of the standard and modified effective range expansion in this two-scale problem and compare their convergence behaviour.

1 Introduction

The effective range expansion has been applied to the field of nuclear interactions from its discovery in 1944 by Landau and Smorodinsky [1] to the present day. Its success can be manifested by the possibility to encode the complex structure of a (nuclear) potential into a small set of parameters. When considering nuclear interactions from a theoretical point of view, one usually defines the effect of a specific potential model in terms of the phase shift it results in. However, a comparison between two different models is a nontrivial task, because one has to compare the phase shifts over the complete range of momenta where the model is applicable. Using the effective range expansion, one can express the structure of the potential in a small set of parameters that can be easily compared with other models. There are different mechanisms that contribute to the nuclear force whose significance depends on the energy at which the process is considered. If one considers a scattering process at low energy, i.e. near the threshold, the long-range part of the nucleon-nucleon interaction is clearly dominating the energy dependence of observables and the effective range expansion is suitable to describe the interaction. If one takes a look at the scattering process at higher momenta, the shorter-range parts of the interaction become more and more important. The longest-range part of the strong nucleon-nucleon interaction is well known to be due to one-pion exchange, the medium range is dominated by two-pion exchange processes and the short-range part is driven by higher-order pion exchange, heavy meson exchange and other mechanisms. For low-energy reactions, the short-range part of the nuclear force cannot be resolved and thus can be parametrized by generic contact interactions [2]. The application of the effective range expansion provides a severe limitation when considering scattering processes at momenta of the order of the pion mass and higher since the range of applicability of this method is limited by the longest-range part of the interaction. Given that the main interest at present is in analysing the shorter-range contributions beyond the one pion exchange, it is desirable to have an approach similar to the effective range expansion but with a larger range of applicability.

Landau and Smorodinky as well as Bethe had already worked on the inclusion of the Coulomb force which was necesseary when considering proton-proton scattering. In the 1960s, Cornille and Martin [3] and Lambert [4] carried these studies forward and developed methods to remove the limitations that were imposed on the range of convergence of the effective range expansion by the infinitely long-range Coulomb potential. This approach was further developed and extended by van Haeringen and Kok [5] to all partial waves and an arbitrary combination of long- and short-range potentials and is known as the modified effective range expansion.

A first approach of applying these methods to the nucleon-nucleon interaction was presented by Steele and Furnstahl [6]. Along similar lines and using the formalism of the modified effective range expansion that was developed for general potentials by van Haeringen and Kok [5], we here want to study in more detail the scaling behaviour

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of the effective range parameters for the standard and modified effective range expansion. To this aim, we need to have knowledge of both the short- and the long-range parts of the potential. Therefore, we at first restrict ourselves to a toy model resembling some properties of nuclear potentials. An application to realistic potentials is in principle possible, however, since the true nature of the short range potential is unknown one has to use a short range potential that is fitted to the experimental data, which is true for example for the Nijmegen potentials [7].

2 The effective range expansion

The effective range expansion was first proposed in 1944 by Landau and Smorodinsky as a semi-empirical formula for s-waves that gave an expansion of the center of mass momentum k times the cotangent of the phase shift $\delta(k)$ as a power series of k^2,

$$k\cot(\delta(k)) = -\frac{1}{a} + \frac{1}{2}r_0 k^2 + O(k^4), \tag{1}$$

where, for low energies, the series could basically be truncated after the first two terms, neglecting terms of the orders k^4 and higher. The parameter a was later called the *Fermi scattering length* at zero energy and r_0 is called the *effective range*. A convenient proof of this relation was given in 1949 by Bethe [8]. An earlier proof was given by Schwinger also in 1949, but it was never published since Bethe found his simpler proof only shortly after Schwinger. However, a summary of Schwinger's proof can be found in [10]. The effective range expansion can be extended to arbitrary partial waves with $\ell > 0$ and then takes the form

$$k^{2\ell+1}\cot(\delta_\ell(k)) = -\frac{1}{a} + \frac{1}{2}r_0 k^2 + \sum_{n=2}^{\infty} v_n k^{2n}. \tag{2}$$

In this context we call the left hand side of the equation

$$K_\ell(k^2) = k^{2\ell+1}\cot(\delta_\ell(k)) \tag{3}$$

the effective range function. One can show from the asymptotic behaviour of solutions of the radial Schrödinger equation that $K_\ell(k^2)$ is indeed an analytic function in k^2 near the origin. If one assumes a potential that falls of exponentially for large r,

$$V(r) \overset{r\to\infty}{\sim} e^{-mr}, \tag{4}$$

which is true for example for potentials of the Yukawa or Malfliet-Tjon type, the effective range expansion has only a finite radius of convergence. One can show that the (maximal) radius of convergence in the complex k-plane is constrained by

$$|k| < \frac{m}{2}. \tag{5}$$

If one now considers a two-range potential, such as

$$V(r) = \underbrace{A_S e^{-m_S r}}_{V_S(r)} + \underbrace{A_L e^{-m_L r}}_{V_L(r)}, \tag{6}$$

where we have a short-range part $V_S(r)$ and a long-range part $V_L(r)$ with $m_S > m_L$, we see that, as a consequence of the above, the radius of convergence of the effective range expansion is limited by

$$|k| < \frac{m_L}{2}, \tag{7}$$

which would also be the case if the short-range potential $V_S(r)$ was not present at all. If the long-range potential was absent, the radius of convergence would only be restricted by

$$|k| < \frac{m_S}{2}. \tag{8}$$

In the nucleon-nucleon potential the longest-range part of the potential is given by the one-pion exchange, which takes the form of a Yukawa potential and therefore limits the radius of convergence to very low momenta of

$$|k| < \frac{m_\pi}{2} \approx 70\ \text{MeV}, \tag{9}$$

To be able to encode the properties of the short-range interactions, which we are interested in, into the coefficients of the effective range expansion, we have to somehow remove the effects of the long-range potential from the expansion. This is where the modified effective range expansion comes into play.

3 The modif ed effective range expansion

The modified effective range expansion has its origin in works of Cornille and Martin [3] and Lambert [4] who were working on the application of the effective range expansion to proton-proton scattering. Since there the Coulomb force is present (which can be seen as equivalent to a Yukawa potential with vanishing mass m), the effective range expansion has zero radius of convergence. Therefore they were looking for a modification to the effective range function $K_\ell(k^2)$ which again would be an analytic function in k^2 with non-vanishing radius of convergence. They found, for s-waves, that this function had to take the form

$$K_0^C(k^2) = k\cot(\delta_0^C(k))C_0^2(\eta) + \alpha\mu h(\eta), \tag{10}$$

where

$$C_0^2(\eta) = \frac{2\pi\eta}{e^{2\pi\eta} - 1} \tag{11}$$

and

$$h(\eta) = \eta^2 \sum_{n=1}^{\infty} \frac{1}{n(n^2+\eta^2)} - \log\eta - \gamma_E \tag{12}$$

are functions of the Sommerfeld parameter $\eta = \alpha\mu/k$ with μ being the reduced mass of the nucleon-nucleon system. In this consideration the definition of the phase shift itself is also different than before. While before the phase shift $\delta_\ell(k)$ was defined as the shift in the phase of the wave function of the scattering solution with respect to the free solution, the phase shift $\delta_0^C(k)$ that occurs in the definition of $K_\ell^C(k^2)$ is the phase shift of the wave function corresponding to the full potential, i.e. the nuclear potential $V_n(r)$ plus the Coulomb potential $V_C(r)$, with respect to the Coulomb wave functions.

The approach of the Coulomb modified effective range function $K_\ell^C(k^2)$ has been extended in 1982 by van Haeringen and Kok [5] for a general potential that can be separated in a short- and long-range part,

$$V(r) = V_S(r) + V_L(r). \tag{13}$$

As for the case of the Coulomb modified effective range expansion, the effective range function $K_\ell(k^2)$ has been altered to be an analytic function of k^2 in a region that is only constrained by the short-range potential $V_S(r)$. For the formulation of this modified effective range function $K_\ell^M(k^2)$, we first need to introduce the so called Jost solution $f_\ell(k,r)$ which is defined as a solution of the radial Schrödinger equation for the long-range potential $V_L(r)$ only,

$$\left(\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - 2\mu V_L(r) + k^2\right) f_\ell(k,r) = 0 \tag{14}$$

with the boundary conditions

$$\lim_{r\to\infty} e^{-ikr} f_\ell(k,r) = 1. \tag{15}$$

In general, $f_\ell(k,r)$ is an irregular solution of the Schrödinger equation, which behaves at the origin as

$$\lim_{r\to 0} f_\ell(k,r) = d_\ell r^{-\ell} + O(r^{-\ell+1}). \tag{16}$$

We also introduce the Jost function $\mathcal{F}_\ell(k)$, which is defined as

$$\mathcal{F}_\ell(k) = \frac{\ell!}{(2\ell)!}(-2ik)^\ell \lim_{r\to 0}(r^\ell f_\ell(k,r)) \tag{17}$$

and is well defined due to the behaviour of the Jost solution $f_\ell(k,r)$ for $r \to 0$. Analogously to the Coulomb case, we also define a new phase shift $\delta_\ell^M(k)$ which is defined as the shift of the solution of the full Schrödinger equation with respect to the solution where only the long-range potential $V_L(r)$ is present. The modified phase shift can therefore be written as

$$\delta_\ell^M(k) = \delta_\ell(k) - \delta_\ell^L(k) \tag{18}$$

where $\delta_\ell(k)$ is the full phase shift and $\delta_\ell^L(k)$ the phase shift of the solution for the long-range potential only, with respect to the free solution in each case. We can now give the formulation for the modified effective range function as

$$K_\ell^M(k^2) = \mathrm{Re}(M_\ell(k)) + \frac{k^{2\ell+1}}{|\mathcal{F}_\ell(k)|^2}\cot(\delta_\ell^M(k)) \tag{19}$$

where

$$M_\ell(k) = \frac{(2\ell)!}{(2^\ell \ell!)^2} \frac{\lim_{r\to 0}\left(\frac{d}{dr}\right)^{2\ell+1} r^\ell f_\ell(k,r)}{\lim_{r\to 0} r^\ell f_\ell(k,r)}. \tag{20}$$

One can easily see that for $V_L(r) \to 0$ we recover the standard effective range function $K_\ell(k^2)$. In their work [5] van Haeringen and Kok find, that the function $M_\ell(k)$ and, therefore, the modified effective range function $K_\ell^M(k^2)$ is only well-defined if the long-range potential $V_L(r)$ is analytic at $r = 0$.

We now have a prescription to apply the effective range expansion to a potential composed of several parts without restricting the radius of convergence by the longest-range part. By choosing the long-range potential $V_L(r)$ in the modified effective range formalism such that this part is known and well under control, one can exhibit the residual short-range part by means of the effective range expansion as if the long-range part was removed from the interaction.

4 Application to a toy model problem

We now apply the formalism described in the last section to a toy potential to show the effectiveness of the approach. As a testing ground we choose a two exponential potential,

$$V(r) = \underbrace{A_S \frac{(m_S r)^2}{1+(m_S r)^2} e^{-m_S r}}_{V_S(r)} + \underbrace{A_L \frac{(m_S r)^2}{1+(m_S r)^2} e^{-m_L r}}_{V_L(r)} \tag{21}$$

with the parameters

$$A_L = -175\ \text{MeV}, \quad A_S = 1500\ \text{MeV} \tag{22}$$

$$m_L = 200\ \text{MeV}, \quad m_S = 750\ \text{MeV} \tag{23}$$

The factors of $(m_S r)^2/(1+(m_S r)^2)$ are not essential for our considerations but can be used to perform an expansion of the potential. Eventually we are interested in applying the formalism to realistic nuclear potentials in connection with chiral potentials for the long-range part, which are also organized in a series due to the chiral expansion. We compute the effective range function $K_\ell(k^2)$ for 20 equidistant momentum points in the range $\frac{1}{20}m_L < k < \frac{1}{10}m_L$ and then fit this to the series

$$K_\ell(k^2) = -\frac{1}{a} + \frac{1}{2}r_0 k^2 + \sum_{n=2}^{\infty} v_n k^{2n} \tag{24}$$

to determine the coefficients of the effective range expansion for $\ell = 0$. Since the effective range expansion, as stated above, is determined by the longest-range part of the potential, we assume that the underlying mass scale of the problem is given by the light mass m_L. Therefore, we expect the coefficients of the expansion to be of natural size, i.e. of the order of one, when expressed in terms of this mass scale. The results we obtain can be found in Table 1. We give the results in both their respective powers of fm and in terms of the two mass scales m_L and m_S. As anticipated, the coefficients of the effective range expansion are natural when expressed in terms of the light mass m_L. To further explore this behaviour, we use the obtained coefficients to re-compute the phase shifts $\delta_\ell(k)$ from the truncated series $K_\ell(k^2)$ including only terms up to $O(k^{2n})$, which we call $K_\ell^{(2n)}(k^2)$. Accordingly, we define the approximate phase shift by

$$\delta_\ell^{(2n)}(k) = \mathrm{arccot}\left(\frac{K_\ell^{(2n)}(k^2)}{k^{2\ell+1}}\right). \tag{25}$$

Table 1. Coefficients of the effective range expansion for s-waves

	a	r_0	v_2	v_3	v_4
in units of fm	5.458 fm	2.432 fm	0.113 fm^3	0.515 fm^5	-0.993 fm^7
in units of m_L	$5.532/m_L$	$2.465/m_L$	$0.118/m_L^3$	$0.551/m_L^5$	$-1.091/m_L^7$
in units of m_S	$20.746/m_S$	$9.244/m_S$	$6.194/m_S^3$	$408.7/m_S^5$	$-11375/m_S^7$

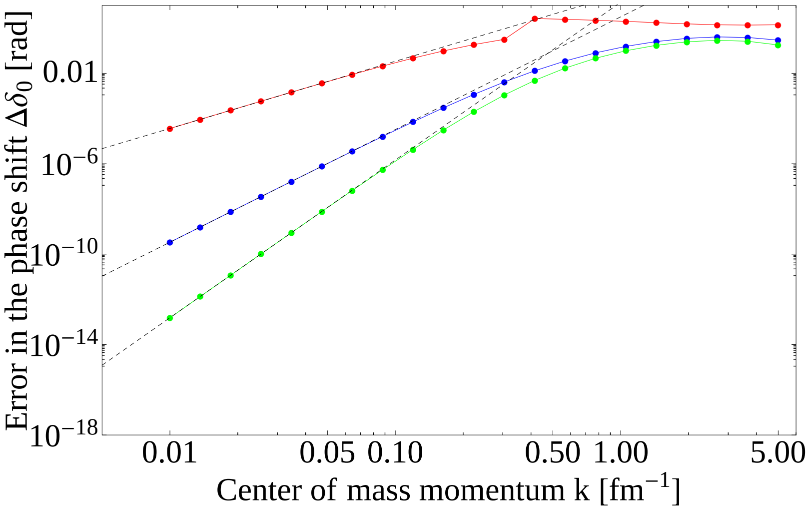

Fig. 1. Error plot for the effective range expansion

We then plot the error

$$\Delta\delta_\ell^{(2n)}(k) = \delta_\ell(k) - \delta_\ell^{(2n)}(k) \tag{26}$$

with respect to k for the orders $n = 0$ (using only the scattering length a), $n = 1$ (using a and the effective range r_0) and $n = 2$ (using a, r_0 and the first shape parameter v_2). The result can be found in Figure 1.

We have extrapolated the linear parts of the error curves to determine the radius of convergence of the expansion, which we define as the crossing points of these lines. This definition is meaningful, since taking even higher orders of the effective range expansion into account cannot reduce the error in the phase shift beyond this point. We find that the radius of convergence is about 0.5 fm^{-1} which is very much what we where expecting from half of the light mass of 200 MeV in the exponential of the long range potential. We also compute the modified effective range function in the same manner as for the effective range expansion and again fit this to the power series from above to determine the coefficients of the modified effective range expansion. This time we use 20 equidistant momentum points in the range $\frac{1}{20}m_S < k < \frac{1}{10}m_S$ for the fit. The results we obtain can be found in Table 2. Again we have given the coefficients in units of fm and in units of the two mass scales of the problem. This time we observe that the coefficients of the modified effective range expansion clearly only assume natural values when expressed in terms of the heavy mass scale m_S. We would also expect this scaling behaviour for the coefficients of the regular effective range expansion if the long-range V_L was not present. Therefore, the modified effective range expansion seems to have removed the limiting effects of the long-range potential from the expansion.

To underline this point we again use the obtained coefficients of the modified effective range expansion to predict phase shifts. For reasons of simplicity, we define an effective short-range phase shift $\delta_\ell^S(k)$ by setting

$$k^{2\ell+1}\cot(\delta_\ell^S(k)) = K_\ell^M(k^2) \tag{27}$$

and using the truncated series of the modified effective range expansion to predict this effective short-range phase shift. We could equally well predict the modified phase shift $\delta_\ell^M(k)$ or even the full phase shift $\delta_\ell(k)$ using Eqs. (18) and (19). But since this is numerically more involved, we restrict ourselves to the effective short-range phase shift defined above to circumstanciate our point. As in the case of the effective range expansion, we compute the approximate phase shift by

$$\delta_\ell^{S(2n)}(k) = \operatorname{arccot}\left(\frac{K_\ell^{M(2n)}(k^2)}{k^{2\ell+1}}\right) \tag{28}$$

and the corresponding error is given by

$$\Delta\delta_\ell^{S(2n)}(k) = \delta_\ell^S(k) - \delta_\ell^{S(2n)}(k). \tag{29}$$

If we again compute this error for our momentum grid for the orders $n = 0$, $n = 1$ and $n = 2$, we obtain the error plot depicted in Figure 2.

We again read off the radius of convergence from the plot and find a value of about 2 fm^{-1}. As expected, his corresponds to about half the mass of the exponential of the short-range potential $V_S(r)$ and is similar as for the regular effective range expansion for vanishing long-range potential. Again we observe, that the modified effective range

Table 2. Coefficients of the modified effective range expansion for s-waves

	a^M	r_0^M	v_2^M	v_3^M	v_4^M
in units of fm	0.450 fm	-0.280 fm	-0.00790 fm^3	-0.000858 fm^5	0.000223 fm^7
in units of m_L	$0.456/m_L$	$-0.283/m_L$	$-0.00822/m_L^3$	$-0.000917/m_L^5$	$0.000245/m_L^7$
in units of m_S	$1.710/m_S$	$-1.063/m_S$	$-0.434/m_S^3$	$-0.680/m_S^5$	$2.552/m_S^7$

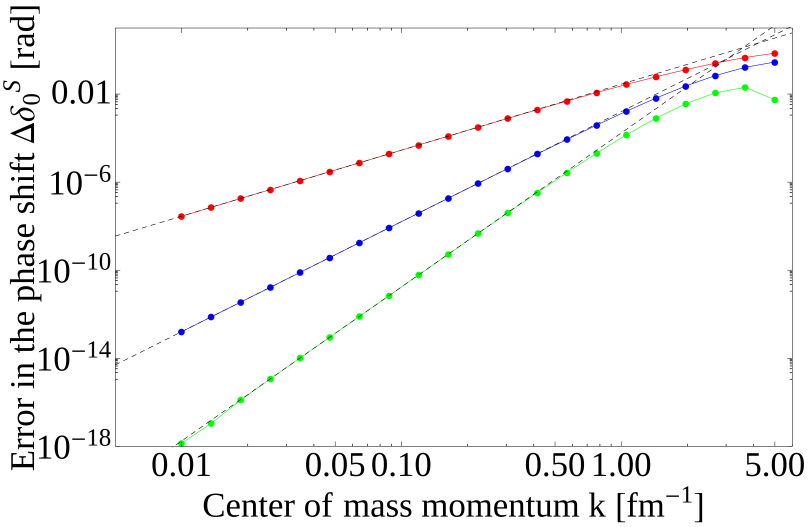

Fig. 2. Error plot for the modified effective range expansion

expansion removes the effect that the long-range potential $V_L(r)$ imposes on the unmodified effective range expansion and allows us to explore the short-range part of the potential by means of the effective range expansion.

5 Conclusion

We have seen that the use of the modified effective range expansion on a two-range potential is really able to remove the limiting effect of the long-range potential from the expansion. We saw this in the scaling behaviour of the coefficients as well as in the clearly visible improvement of the radius of convergence of the expansion. In principle our method is also applicable to realistic potentials, however, unfortunately the fitting method we use to extract the coefficients of the expansion needs an extremely high numerical precision to give stable results. This does not pose a problem for our toy potential, since we can use an arbitrarily high precision here, but as soon as we include experimental data in terms of phenomenological potentials such as [7], we need to improve our fitting method and find a way to take the experimental uncertainties into account which we are currently working on. In contrast to what has been proposed by van Haeringen and Kok in [5], we found that for higher partial waves ($\ell > 0$) the function $M_\ell(k)$ appearing in the modified effective range function $K_\ell(k^2)$ is not well-defined even for potentials that are regular at the origin such as our toy potential. However, we found that a meaningful expansion is still possible if one uses a suitable subtraction method to obtain finite results. Application of the modified effective range expansion to coupled channels is in progress. These extensions of the formalism will be discussed in a future work [9].

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References

1. L. Landau and J. Smorodinsky, Phys. U.S.S.R. **8**, (1944) 154
2. E. Epelbaum, H.-W. Hammer, U.-G. Meißner, Rev. Mod. Phys. 81 (2009) 1773; E. Epelbaum, Prog. Part. Nucl. Phys. **57**, 654 (2006)
3. H. Cornille and A. Martin, Nuovo Cimento **26**, (1962) 298
4. E. Lambert, Helvetica Physica Acta **42**, (1969) 667
5. H. van Haeringen and L.P. Kok, Phys. Rev. **A26**, (1982) 1218
6. J.V. Steele and R.J. Furnstahl, Nucl. Phys. **A645**, (1999) 439
7. V.G.J. Stoks, R.A.M. Klomp, M.C.M. Rentmeester, and J.J. de Swart, Phys. Rev. **C48**, (1993) 792
8. H.A. Bethe, Phys. Rev. **76**, (1949) 38
9. D. Minossi, M. Pavon Valderrama, E. Epelbaum and A. Nogga, in preparation.
10. J.M. Blatt and J.D. Jackson, Phys. Rev. **76**, (1949) 18