QCD at finite temperature and density on the lattice

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Abstract. In the first lecture we briefly summarize the basics of field theory thermodynamics and critical phenomena. We then introduce the lattice gauge field theory approach to QCD at finite temperature and density, which is a non-perturbative scheme allowing first principle calculations using the QCD Lagrangian as a sole input. Some of the general concepts and idea introduced at the beginning are demonstrated by use of simple effective models of QCD. The second lecture is devoted to applications. We emphasize that current methods suffice to study the main phenomena at RHIC and LHC energies, and we discuss the ongoing theoretical efforts devoted to the solution of the sign problem which hampers the simulations of cold and dense matter. We conclude with short overview of the status of the field as of Summer 2008.

References

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MpL-QCD at finite T and μ

Thermodynamics Basics

Thermodynamical Ensembles

- Microcanonical: E, N, V fixed
- Canonical: T, N, V fixed
- Grandcanonical : T, μ , V fixed

In a GrandCanonical Ensemble define:

$$\begin{aligned}\hat{\rho} &= e^{-(H-\mu\hat{N})/T} \\ Z &= Tr\hat{\rho}\end{aligned}$$

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ρ e Z determine the system's state:

$$\langle O \rangle = \text{Tr} O \hat{\rho} / Z$$

$$P = T \frac{\partial \ln Z}{\partial V}$$

$$N = T \frac{\partial \ln Z}{\partial \mu}$$

$$S = \frac{\partial \ln Z}{\partial T}$$

$$E = -PV + TS + \mu N$$

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Bosons and fermions

- 1 bosonic degree of freedom

$$\hat{H} = 1/2 \omega a^\dagger a + 1/2 \omega = \omega (\hat{N} + 1/2)$$

$$Z = \text{Tr} e^{-\beta(H - \mu \hat{N})} = \text{Tr} e^{-\beta(\omega - \mu) \hat{N}}$$

$$= \sum_{n=0}^{\infty} \text{Tr} e^{-\beta(\omega - \mu)n} = (1 - e^{-\beta(\omega - \mu)})^{-1}$$

$$N = (e^{\beta(\omega - \mu)} - 1)^{-1}$$

- 1 fermionic degree of freedom

$$\hat{H} = 1/2 \omega a^\dagger a - 1/2 \omega = \omega (\hat{N} - 1/2)$$

$$Z = \text{Tr} e^{-\beta(H - \mu \hat{N})} = \text{Tr} e^{-\beta(\omega - \mu) \hat{N}}$$

$$= \sum_{n=0}^1 \text{Tr} e^{-\beta(\omega - \mu)n}$$

$$N = (e^{\beta(\omega - \mu)} + 1)^{-1}$$

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*Example 1 : Relativistic free fermionic gas*Non interacting particles: Z

$$Z = \prod Z^j$$

$$\ln Z = d^3x \int d^3p (\ln(e^{-\beta(\omega-\mu)} + 1) + \ln(e^{-\beta(\omega+\mu)} + 1))$$

Note! Relativistic particles!

- Fermions (+ μ) and antifermions ($-\mu$).
- $\omega = \sqrt{p^2 + m^2}$.
- Relativistic chemical potential

 Z can be exactly computed in the chiral limit $m = 0$:

$$\begin{aligned} T \ln Z &= V \mu^4 / (12\pi^2) + V \mu^2 T^2 / 6 + 7V \pi^2 T^4 / 180 \\ n &= 4V \mu^3 / 12\pi^2 + 2V \mu T^2 / 6 \end{aligned}$$

Ideally, this should be the state of the quarks at very high temperatures!

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APPROACH TO FREE FIELD : ANALYTIC RESULTS VS. LATTICE DATA

Based on A. Vuorinen, 2004

D'Elia, Di Renzo, MPL, Vuorinen, in progress

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More observables : Response Functions
Susceptibilities

$$\chi_{j_u,j_d}(T) = \left. \frac{\partial^{(j_u+j_d)} p(T, \mu_u, \mu_d)}{\partial \mu_u^{j_u} \partial \mu_d^{j_d}} \right|_{\mu_u=\mu_d=0}.$$

Test for fluctuations.
Taylor coefficients of the excess pressure:

$$\Delta p(T, \mu_u, \mu_d) \equiv p(T, \mu_u, \mu_d) - p(T, \mu_u = 0, \mu_d = 0)$$

$$\Delta p(T, \mu_u, \mu_d) = \sum_{j_u,j_d} \chi_{j_u,j_d}(T) \frac{\mu_u^{j_u}}{j_u!} \frac{\mu_d^{j_d}}{j_d!},$$

containing information about baryon density effects in the EoS.

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Susceptibilities Towards Free Field

RBC-Bielefeld Collaboration, 2008

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Example II : Interacting Fermions

3 d Gross Neveu Model

$$L = \bar{\psi}(\not{\partial} + m)\psi - g^2/N_f[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2]$$

Global Chiral Invariance

$$\begin{aligned}\psi_i &\rightarrow e^{i\alpha\gamma_5}\psi_i \\ \bar{\psi}_i &\rightarrow \bar{\psi}_i e^{i\alpha\gamma_5}\end{aligned}$$

Basic Properties

- At $T = \mu = 0$ and g 'large', spontaneous symmetry breaking, Goldstone mechanism.
- Rich particle spectrum
- Amenable to a lattice study at $T, \mu \neq 0$!!!!

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MPL-QCD at finite T and μ *Phase diagram of the 3d Gross Neveu model*

da S. Hands, 1998

Mean Field Solution vs Exact Lattice Results

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MPL-QCD at finite T and μ

-
- Grand Canonical Formalism at finite T, μ
 - Basic Observables : number density, susceptibilities
 - Free Fermions : Exact Solution
 - Simple model with interacting fermions : mean field solution
 - Phase Diagram at nonzero T and μ of a purely fermionic model
 - Simple calculations : can reproduce limiting behaviour and give generic information; in general inaccurate

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Lattice QCD
=
first principles calculations from the
QCD Lagrangian

$$\mathcal{L} = \mathcal{L}_{YM} + \bar{\psi}(i\gamma_{\mu}D_{\mu} + m + \mu\gamma_0)\psi$$

A vast phase space to be explored:

Real baryon chemical potential, temperature, isospin chemical potential.
And also: Imaginary chemical potential, number of color and number of flavor, bare masses

to address phenomenological issues as well as to study more theoretical questions

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Phenomenological challenges:
ab initio study of the QCD phase diagram

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Theoretical questions can (or, rather, should ?) be addressed in a larger phase space:

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■ *High T*

- Chiral symmetry pattern : order disorder, light baryons
- Deconfinement/screening: string breaking via recombination with light pairs
- Instanton molecules

■ *High μ*

- Chiral symmetry pattern : instability at the Fermi surface
- Deconfinement/screening : string breaking via recombination with real particles
- Instanton chains

Differences at high T and high μ in the *gauge dynamics* provide further motivation to study nonzero μ on a lattice.

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Lattice QCD at Finite Temperature and Density

Lecture I

I Formulation

II Computational Schemes

II.1 Effective Fermionic Models - Analytic approaches

II.2 Effective Gluonic Models - Numerical approaches

III QCD at Finite Baryon Density: Methods

III.1 Derivatives

III.2 Reweighting

III.3 Expanded Reweighting

III.4 Imaginary Chemical Potential

IV Results - Discussion : Tomorrow's lecture

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MpL-QCD at finite T and μ

1 Formulation

Grand Canonical Formalism and Path Integral

★ Chemical Potential for Conserved Charge $\hat{N}$

$$\begin{aligned}\rho &= e^{-(H - \mu \hat{N})/T} \\ Z &= \text{Tr} \rho = \int d\phi d\psi e^{-S(\phi, \psi)}\end{aligned}\tag{1}$$

★ Temperature: Reciprocal of Imaginary Time

$$S(\phi, \psi) = \int_0^{1/T} dt \int d^d x \mathcal{L}(\phi, \psi)$$

with boundary conditions for fermions and bosons

$$\begin{aligned}\phi(t=0, x) &= \phi(t=1/T, x) \\ \psi(t=0, x) &= -\psi(t=1/T, x)\end{aligned}$$

★ Z = partition function of a statistical system in $d+1$ dimension, where T is the reciprocal of the imaginary time.

★ Thermodynamics and spectrum properties are treated on the same footing.

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Lattice QCD at $T, \mu \neq 0$

★ **Temperature:** as in the continuum $T = 1/N_t * a$

★ **Density**

In the continuum: $L(\mu) = L_0 + \mu J_0$ $J_0 = \bar{\psi} \gamma_0 \psi \rightarrow N - \bar{N} = \int J_0$

On the lattice:

$$L(\mu) = \bar{\psi}_x \gamma_0 e^{\mu a} \psi_{x+\hat{0}} - \bar{\psi}_{x+\hat{0}} \gamma_0 e^{-\mu a} \psi_x$$

$$J_0 = -\partial_\mu L = \bar{\psi}_x \gamma_0 e^{\mu a} \psi_{x+\hat{0}} + \bar{\psi}_{x+\hat{0}} \gamma_0 e^{-\mu a} \psi_x$$

Time Forward propagation enhanced by $e^{\mu a}$

Time Backward propagation discouraged by $e^{-\mu a}$

Particles-antiparticles asymmetry!

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More on The Lattice (digression)

Path integral is a regulated on a four dimensional lattice

- Gauge fields: link variables $U_\mu(x)$ for parallel transport of field A from x to $x + \hat{\mu}a$

$$x \xrightarrow{U_\mu(x)} x + \hat{\mu}a$$

$$U_{x,\mu} = \text{Tr} \exp \left(ig \int_x^{x+\hat{\mu}a} dx^\mu A_\mu(x) \right)$$

- Gauge invariants and Yang Mill Action:

$$\begin{aligned} W_{n,\mu\nu}^{(1,1)} &= \frac{1}{3} \text{Re} \left[\text{Tr} \begin{array}{|c|} \hline \square \\ \hline \end{array} \right]_{n,\mu\nu} \\ &= \text{Re} \text{Tr} U_{n,\mu} U_{n+\hat{\mu},\nu} U_{n+\hat{\mu}+\hat{\nu},\mu}^\dagger U_{n,\nu}^\dagger \\ &= \frac{g^2 a^4}{2} F_{\mu\nu}^a F_{\mu\nu}^a + \mathcal{O}(a^6) \end{aligned}$$

- Lattice Yang Mill Action

$$\beta S_G = \beta \sum_{0 \leq \mu < \nu \leq 3} W_{n,\mu\nu}^{(1,1)} \rightarrow \int d^4x \mathcal{L}_{YM} + \mathcal{O}(a^2)$$

$$\beta = 5/g(a)^2.$$

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- Lattice fermions

Simply:

$$\psi(x) \rightarrow \psi(n) \quad !$$

$$\partial_\mu \psi_f(x) = (\psi(n+\hat{\mu}) - \psi(n-\hat{\mu}))/2a,$$

(doubling problem and chiral symmetry: staggered fermions, Wilson fermions, chiral fermions)

$\bar{\psi} \gamma_\mu \psi$ on the lattice
Naive discretisation:

$$\begin{aligned} \phi_{LATT}(n_1, n_2, n_3, n_4) &= \phi(n_1 a, n_2 a, n_3 a, n_4 a) \\ \Delta_\mu \phi_{LATT}(n_1, n_2, n_3, n_4) &= \\ \phi(n_1 a, (n_\mu + 1)a, n_3 a, n_4 a) &= \phi(n_1 a, n_3 a, n_4 a)/2a \end{aligned}$$

Problems with free fermions: the internal energy e diverges in the continuum limit $a \rightarrow 0$

$$L = \bar{\psi}_x \gamma_\mu \psi_{x+\hat{\mu}a} + m \bar{\psi}_x \psi_x + \mu \bar{\psi}_x \gamma_0 \psi_x$$

$$e \propto \frac{\mu^2}{a^2} \xrightarrow{a \rightarrow 0} \infty$$

Elegant solution : μ is an external field in the 0th direction

$$\bar{\psi} \gamma_\mu A_\mu \psi \longleftrightarrow i\mu \bar{\psi} \gamma_0 \psi$$

- External fields live on lattice link. (cfr. electrodynamics: $A \rightarrow \theta = e(iA)$)

- $L(\mu) = \bar{\psi}_x \gamma_0 e^{i\mu a} \psi_{x+\hat{0}} - \bar{\psi}_{x+\hat{0}} \gamma_0 e^{-i\mu a} \psi_x$

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MpL-QCD at finite T and μ

• Simple interpretation

- True Forward propagation enhanced by $e^{\mu a}$
- True Backward propagation discouraged by $e^{-\mu a}$

Particles-antiparticle asymmetry!

• $\lim_{a \rightarrow 0} J_0 = -\partial_\mu L = \bar{\psi}_x \gamma_0 e^{\mu a} \psi_{x+0} + \bar{\psi}_{x+0} \gamma_0 e^{-\mu a} \psi_x - \mu \bar{\psi} \gamma_0 \psi$

Via an unitary transformation for the field

$$L(\mu) = L(0)$$

+ boundary conditions
Explicit dependence on fugacity

Lattice QCD Thermodynamics at a Glance

$$\begin{aligned} \mathcal{L}_{QCD} = & \frac{6}{g^2} \text{Tr} U_{n,\mu} U_{n+\hat{\mu},\nu} U_{n+\hat{\nu},\mu}^\dagger U_{n,\nu}^\dagger \\ & + \sum_{i=1}^3 (\bar{\psi}_x \gamma_i U_i(x) \psi_{x+\hat{i}} - \bar{\psi}_{x+\hat{i}} \gamma_i U_i^\dagger(x) \psi_x) \\ & + \bar{\psi}_x \gamma_0 e^{\mu} U_0(x) \psi_{x+0} - \bar{\psi}_{x+0} \gamma_0 e^{-\mu} U_0^\dagger(x) \psi_x \\ & + m \bar{\psi} \psi \end{aligned}$$

II Computational Schemes

$$\mathcal{Z} = \int d\phi d\psi e^{-S(\phi,\psi)}, (\phi,\psi) = \int_0^{1/T} dt \int d^3x \mathcal{L}(\phi,\psi)$$
$$\mathcal{L}_{QCD} = \mathcal{L}_{YM} + \bar{\psi}(i\gamma_\mu D_\mu + m)\psi + \mu \bar{\psi}\gamma_4\psi$$

Two options:

1. *Integrate out gluons first:*

$$\mathcal{Z}(T,\mu,\bar{\psi},\psi,U) \simeq \mathcal{Z}(T,\mu,\bar{\psi},\psi) \rightarrow$$

effective approximate fermion models

2. *Integrate out fermions **exactly** as S is bilinear in $\psi, \bar{\psi}$*
 $S = S_{YM}(U) + \bar{\psi}M(U)\psi$

$$\mathcal{Z}(T,\mu,U) = \int dU e^{-(S_{YM}(U)-\log(\det M))}$$

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$$\mathcal{Z}(T,\mu,U) = \int dU e^{-(S_{YM}(U)-\log(\det M))}$$

starting point for numerical calculations

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 II. 1 Effective Fermionic Models on the Lattice

Lattice Strong Coupling Calculations:
Starting point : Yang Mill Action decouples
at $g = \infty \rightarrow \int dU$ exact
Work on two color QCD by Y. Nishida, K. Fukushima, and T. Hatsuda

Initial studies by F. Karsch, U. Wolff, and others.

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Work from the 80's on three color :
B. Petersson and collaborators, P. Damgaard, F. Karsch and many others.

$$\mathcal{L} = G \frac{1}{8N_c^2} [(\bar{\psi}\tau^-\psi)^2 + (\bar{\psi}\tau^-\gamma_5\psi)^2]$$

CSG/CFL Phase from Strong Coupling?

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The phase diagram of QCD at strong coupling *Kawamoto et al, 2005.*

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11.2 Importance Sampling and The Positivity Issue

$$Z(T, \mu, U) = \int dU e^{-(S_{YM}(U) - \log(\det M))}$$

$$M^\dagger(\mu_B) = -M(-\mu_B)$$

$\det M > 0 \rightarrow$ Importance Sampling

- $\mu = 0 \rightarrow \det M$ is *real*
Particles-antiparticles symmetry
- Imaginary $\mu \neq 0 \rightarrow \det M$ is *real*
(Real) Particles-antiparticles symmetry
- Real $\mu \neq 0$ Particles-antiparticles asymmetry
 $\rightarrow \det M$ is *complex* in QCD
- QCD with a real baryon chemical potential:
use information from the accessible region

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$$\text{Re}\mu = 0, \text{Im}\mu \neq 0$$

III QCD AT FINITE BARYON DENSITY-METHODS

QCD and a Complex μ_B

A map: complex $\mu \rightarrow$ complex μ^2 .

$Z(\mu^2)$ is real valued for real μ^2

Analogy with statistical models in external fields

*The Phase Diagram in the T, μ_B^2 Plane Region accessible to simulations:
 μ^2 real ≤ 0 .*

- $\mu = 0$ Derivatives, Reweighting, Expanded reweighting
- $\mu^2 \leq 0$ Imaginary chemical potential

The Roberge and Weiss analysis

$$Z(\nu) = \text{Tr} e^{-\beta H + i\beta \nu N} = e^{-\beta H + i\beta N}$$

- 1. $Z(\theta)$ has a periodicity 2π anyway.
- 2. If only color singlet are allowed, then $N = 0 \bmod (N_c)$ and periodicity becomes $2\pi/N_c$

However (Roberge Weiss (1986))

$Z(\theta)$ has always period $2\pi/N_c$

The imaginary chemical potential changes the preferred vacuum for the Polyakov loop from $\phi_P = 0$ to one of its Z_3 images

The strong coupling analysis shows that periodicity is smooth at low temperature, and *p.t.* theory suggests that it is sharp at high T

III.1 Derivatives at $\mu = 0.0$

TARO

MILC

QCD-

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MpL-QCD at finite T and μ III.2 Reweighting from $\mu = 0$

Ian Barbour's proposal, or

The Glasgow method: $Z(\mu)$ can be computed using simulations at $\mu = 0$:

$$Z = \left\langle \frac{|M(\mu)|}{|M(\mu=0)|} \right\rangle_{\mu=0}$$

$$Z = \frac{\int [dU][dU^\dagger] |M(\mu)| e^{-S_g[U, U^\dagger]}}{\int [dU][dU^\dagger] |M(\mu=0)| e^{-S_g[U, U^\dagger]}}$$

Needs overlap betweensimulation ensemble at $\mu = 0$ target ensemble at $\mu \neq 0$ At $T = 0$ the Glasgow procedure fails because of a poor overlap.

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Broken phase

Symm. Phase

Distributions of the $\langle \sigma \rangle - \langle \bar{\psi} \psi \rangle$ fields

Example of successful reweighting at $\mu \neq 0$: no conceptual problems
1-dim $SU(3)$ can be exactly solved (Bilic, Demeterfi, 1988)

Exact Z 's zeros in the Imaginary, Real μ plane
(diamonds) and the cloud of zeros obtained from reweighting with a very
poor statistics

Z 's zeros from an high statistics reweighting : OK

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Z.Fodor and F.Katz's proposal :
Multiparameter reweighting use *fluctuations around T_c* to explore the critical region

Picture taken from
S. Muroya, A. Nakamura, C. Nonaka and T. Takaishi

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III.3 Taylor Expanded Reweighting

Bielefeld-Swansea

Taylor expansion of the reweighting factor as a power series in $\lambda = \mu/T$, and similarly for any operator.

Computationally convenient: simplifies calculation of determinant.

Expectation values are then given by

$$\langle \mathcal{O} \rangle_{(\beta, \mu)} = \frac{\langle (\mathcal{O}_0 + \mathcal{O}_1 \lambda + \mathcal{O}_2 \lambda^2 + \dots) \exp(\mathcal{R}_1 \lambda + \mathcal{R}_2 \lambda^2 + \dots - \Delta S_g) \rangle_{\lambda=0, \beta_0}}{\langle \exp(\mathcal{R}_1 \lambda + \mathcal{R}_2 \lambda^2 + \dots - \Delta S_g) \rangle_{\lambda=0, \beta_0}}.$$

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III.IV Imaginary Baryon Chemical Potential

Bridge between Canonical and GrandCanonical ensembles

A. Hasenfratz, D. Toussaint, M. Alford, A. Karsch, P. Węlczek, ...

$$\mathcal{Z}(\mu) = \sum \mathcal{Z}(N) e^{\beta \mu B(N)}$$

$$\mathcal{Z}(\nu) = \text{Tr} e^{-\beta(H - i\nu B(N))}$$

$$\mathcal{Z}(N) = \frac{\beta}{2\pi} \int_0^{2\pi/\beta} d\nu \mathcal{Z}(\nu) e^{-i\beta \nu B(N)}$$

Idea: $\mu = 0$ fluctuations allow the exploration of $N_b \neq 0$ hence tell us about $\mu \neq 0$

Note: same argument suggests Glasgow reweighting might work

Practical Strategy:

$\mathcal{Z}(\mu, T)$ must be

1. analytic
2. non trivial

Rule of thumb:

$$\chi(T, \mu) = \partial \rho(\mu, T) / \partial \mu = \partial^2 \log \mathcal{Z}(\mu, T) / \partial \mu^2 > 0$$

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μ Imm.: Lessons from $g = \infty$

$$S_{QCD} = S_{YM} + S_F \rightarrow g \rightarrow \infty = S_F$$

$$Z = \left(\int V_{eff}(\langle \bar{\psi}\psi \rangle, d\langle \bar{\psi}\psi \rangle) \right)^{V_s}$$

$$V_{eff}(\langle \bar{\psi}\psi \rangle, \mu) = 2\cosh(rN_t N_c \mu) + \sinh[(N_t + 1)N_c \langle \bar{\psi}\psi \rangle] / \sinh(N_t \langle \bar{\psi}\psi \rangle)$$

$$V_{eff}(\langle \bar{\psi}\psi \rangle, i\mu) = 2\cos(rN_t N_c \mu) + \sinh[(N_t + 1)N_c \langle \bar{\psi}\psi \rangle] / \sinh(N_t \langle \bar{\psi}\psi \rangle)$$

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$\langle \bar{\psi}\psi \rangle$ as a function of real and imaginary μ , for $T \simeq 0$ and $T \simeq T_c$

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Two color QCD as a testbed for Imaginary μ_B

P. Giudice, Tesi di Laurea in Fisica, Universita della Calabria; Advisor A. Papa

courtesy of the Author

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MpL-QCD at finite T and μ

Gross Neveu Model

The critical line:

$$1 - \mu/\Sigma_0 = 2T/\Sigma_0 \ln(1 + e^{-\mu/T})$$

Reduces to:

$$T(T - T_c) + \mu^2/(8\ln 2) = 0$$

Second order approximation good up to $\mu \simeq T_c$

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From O. Philipsen and E. Laermann
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Lattice QCD at Finite Temperature and Density

Lecture I

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III.1 Derivatives

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III.4 Imaginary Chemical Potential

IV Results - Discussion : Tomorrow's lecture

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LATTICE FIELD THEORY : FIRST PRINCIPLE APPROACH TO
THE QCD PHASE DIAGRAM

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1. T_c , THE CRITICAL LINE AND THE CRITICAL POINT

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T_c :
RBC-BIELEFELD

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ImL-QCD at finite T and μ

T_c :
WUPPERTAL-JUELICH

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ImL-QCD at finite T and μ

T_c AT $\mu_B = 0$: STATUS AS OF QM2008

RBC-BIELEFELD :

$$T_c = 190(5) \text{ MeV}$$

WUPPERTAL-JUELICH :

$$T_c = 175(5) \text{ MeV (Glue)}$$

$$T_c = 151(6) \text{ MeV (Fermions)}$$

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THE CRITICAL
LINE IN THE
 μ_B, T PLANE AT
SMALL μ_B

$$T = T_c - K\mu_B^2$$

$$K \propto 1/N_f$$

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TOWARDS THE CONTINUUM LIMIT:
CURVATURE K OF THE CRITICAL LINE IN THE
 μ_B - T PLANE

Results for $N_f=2+1$

from gluonic and fermionic susceptibilities
Fodor, Guse, Katz, Kálmán K. Szabó, QM2008

THE CRITICAL ENDPOINT OF QCD

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MpL-QCD at finite T and μ

CHALLENGING THE CRITICAL ENDPOINT OF QCD

Forcrand-Philipsen : strategy

Scenario I or Scenario II ? To decide, measure slope K in

$$\frac{m_c(\mu)}{m_c(0)} = 1 + K \left(\frac{\mu}{T} \right)^2 + \dots$$

$K > 0$: **Scenario I , critical endpoint at small μ_B**

$K < 0$: **Scenario II, NO critical endpoint at small μ_B : favored at QM2006**

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Forcrand-Philipsen Lat07 : improve and confirm previous results

$$\frac{m_c(\mu)}{m_c(0)} = 1 - 3.3(5) \left(\frac{\mu}{T}\right)^2 + \dots$$

Kogut-Sinclair 2007 : QCD at finite isospin density

NB : $\mu_I \simeq \mu_B$ at $T \simeq T_c$ Toublan, Kogut, Sinclair 2004

$$\frac{m_c(\mu)}{m_c(0)} = 1 - 3.(1) \left(\frac{\mu_I}{T}\right)^2 + \dots$$

Current Results:

- Confirm unusual scenario for $N_t = 4$
- Suggest NO critical endpoint for $\mu_B < 600 \text{ MeV}$
- See talk by Philippe de Forcrand

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$SU(3)_L \times SU(3)_R$ NJL MODEL

Kovács and Szép, 2006 compute critical surface of the $SU(3)_L \times SU(3)_R$ chiral quark model at non-zero baryon density.

Conventional Scenario realized in NJL, $\mu_B^{CEP} \sim 900 \text{ MeV}$.

Qualitative difference between QCD and NJL at small μ_B .: Details of the dynamics important.

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2. EOS AND CRITICAL BEHAVIOUR

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LATTICE OBSERVABLES FOR THERMODYNAMICS

Number Density : accessible at **imaginary chemical potential**.

$$n_{u,d}(T, \mu_u, \mu_d, m_u, m_d) = \frac{\partial p(T, \mu_u, \mu_d)}{\partial \mu_{u,d}}; p(T, \mu_u, \mu_d) = \frac{T}{V} \ln Z(T, \mu_u, \mu_d)$$

Susceptibilities: accessible at **at $\mu = 0$**

$$\chi_{j_u, j_d}(T) = \frac{\partial^{(j_u + j_d)} p(T, \mu_u, \mu_d)}{\partial \mu_u^{j_u} \partial \mu_d^{j_d}} \Big|_{\mu_u = \mu_d = 0}.$$

Test for fluctuations.

Taylor coefficients of the excess pressure:

$$\Delta p(T, \mu_u, \mu_d) \equiv p(T, \mu_u, \mu_d) - p(T, \mu_u = 0, \mu_d = 0) = \sum_{j_u, j_d} \chi_{j_u, j_d}(T) \frac{\mu_u^{j_u}}{j_u!} \frac{\mu_d^{j_d}}{j_d!},$$

containing information about baryon density effects in the EoS.

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THERMODYNAMICS AND CRITICAL LINES in THE $T - \mu^2$ PLANE

Three regimes for thermodynamics:

- **Low Temperature,**

away from critical lines:

Hadron Gas QM2006

$$n(T, \mu) = K(T) \sin(N_c \mu / T)$$

- **In the critical region:**

$$p(T, \mu) = b(T)|t + a(T)(\mu^2 - \mu_c^2)^{(2-\alpha)}$$

implying

$$n(T, \mu) = A(T)\mu(\mu_c^2 - \mu^2)^{(2-\alpha)}$$

- **High Temperature,**

away from critical line

Approach to Free Field

$$n(T, \mu) \rightarrow n_{SB}(T, \mu)$$

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HADRON GAS; CRITICAL BEHAVIOUR; SB from SUSCEPTIBILITIES

2+1 Flavor, $m_\pi = 220$ MeV

**RBC-Bielefeld QM2008;
courtesy C. Schmidt**

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CRITICAL BEHAVIOR AND THERMODYNAMICS AT THE ENDPOINT OF THE RW TRANSITION

Critical behavior at imaginary μ
 $n(\mu_I) = A(T)\mu_I(\mu_I^2 - \mu_I^2)^{(2-\alpha)}$

Continued to real μ
 $n(\mu) = A(T)\mu(\mu^2 + \mu^2)^{(2-\alpha)}$
 $n_{SB}(\mu) = A\mu + B\mu^3 \rightarrow \alpha = 1$

D'Elia, Di Renzo, Lombardo, 2007, QM2008

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CRITICAL BEHAVIOR, THERMODYNAMICS, QUASIPARTICLE MODELS

Kämpfer, Bluhm Proposal

amenable to an easy comparison with lattice data.

I. Quasiparticlemodel vs Imaginary Chemical Potential Lattice Data,
and analytic continuation to Real Chemical Potential

Kämpfer, Bluhm, 2007, QM2008

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II. Quasiparticle model vs Taylor Coefficients

Crucial ingredients:

- Explicit dependence of the self-energy parts on $\mu_i = \mu_{u,d}$ and T
- Implicit dependence via the effective coupling $G^2(T, \mu_u, \mu_d)$.

$$\omega_i^2 = k^2 + m_i^2 + \Pi_i, \quad \Pi_i = \frac{1}{3} \left(T^2 + \frac{\mu_i^2}{\pi^2} \right) G^2(T, \mu_u, \mu_d).$$

Kämpfer-Bluhm : see Poster

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APPROACH TO SB : ANALYTIC RESULTS VS. LATTICE DATA

Based on A. Vuorinen, 2004

D'Elia, Di Renzo, Lombardo, Vuorinen, in progress

Lattice Cutoff Effects : Karsch, Laermann, 2008

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3. TOWARDS FAIR

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STRONG COUPLING
New Results on The Mass Spectrum

Kawamoto, Miura 2007

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HEAVY QUARK EFFECTIVE MODEL

Double limit: $M \rightarrow \infty$, $\mu \rightarrow \infty$, $\zeta \equiv \exp(\mu - \ln M)$: Fixed

Evolved 'quenched approximation' in the presence of charged matter

Polyakov Loop

Results for $N_f = 3$:

- Identified phase transition
- Identified Ridge in the T, μ plane
- Studies of diquark in progress

Di Pietro, Feo, Seiler, Stamatescu 2008

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CANONICAL FORMALISM

$$\mathcal{Z}_C(T, N) =$$

$$\frac{3}{2\pi} \int_0^{2\pi T} d(\mu_I/T) e^{-N i \mu_I/T} \mathcal{Z}_{GC}(T, i \mu_I)$$

Forcrand and Kratovchila, 2006,2007

Canonical partition a la Hasenfratz-Toussaint:

$$\begin{aligned} & \frac{\mathcal{Z}_C(B, \beta)}{\mathcal{Z}_{GC}(\beta_0 = \beta, \mu = i \mu_{I_0})} \\ &= \left\langle \frac{1}{2\pi} \int_{-\pi}^{\pi} d\left(\frac{\mu_I}{T}\right) e^{-i 3 B \frac{\mu_I}{T} \frac{\det(U; i \mu_I)}{\det(U; i \mu_{I_0})}} \right\rangle_{\beta_0, i \mu_{I_0}} \\ &\equiv \left\langle \frac{\hat{\mathcal{Z}}_C(U; B)}{\det(U; i \mu_{I_0})} \right\rangle_{\beta_0, i \mu_{I_0}} \end{aligned}$$

Results for $N_f = 4$:

- First order line-coexistence region
- Critical μ and Critical densities $\rightarrow$

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CANONICAL APPROACH II : HISTOGRAMS

Proposed by Ejiri,2007

Alternative search for a critical point in $N_f = 2$ QCD, large masses

Alexandru, Li, Liu, 2007 :

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CANONICAL APPROACH III : Grand Partition Function via a Taylor Expansion.

Ejiri, 2007,2008

$$\frac{\mathcal{Z}_{GC}(T, \mu_q)}{\mathcal{Z}_{GC}(T, 0)} \equiv \left\langle e^{\left[N_f N_t V \sum_{n=1}^{\infty} D_n \left(\frac{\mu_q}{T} \right)^n \right]} \right\rangle_{(T, \mu_q=0)},$$

$$\frac{\mathcal{Z}_C(T, \bar{\rho} V)}{\mathcal{Z}_{GC}(T, 0)} \text{ at large } V \approx \left\langle \exp \left[V \left(N_f N_t \sum_{n=1}^{\infty} D_n z_0^n - \bar{\rho} z_0 \right) \right] e^{-i\alpha/2} \sqrt{\frac{1}{V|D''(z_0)|}} \right\rangle_{(T, \mu=0)}$$

Results for $N_f = 2$:

- Qualitative change at $T/T_c \simeq 0.8$
- Indication of First Order Line

Ejiri 2008

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DENSITY OF STATES

Luo, Azcoiti et al., Ambjorn et al., Anagnostopoulos and J. Nishimura,

$$\langle O \rangle = \int d\phi \langle O f(U) \rangle_{\phi} \rho(\phi) / \int d\phi \langle f(U) \rangle_{\phi} \rho(\phi), \phi \text{ fixed}$$

Density of states – ρ –

constrained partition function:

$$\rho(x) = \int \mathcal{D}U g(U) \delta(\phi - x).$$

Results for $N_f = 4$

- Signal of two phase transition lines
- Indication for a triple point

Fodor, Katz, Schmidt 2007

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4. COLD AND DENSE MATTER

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TWO COLOR QCD

- **Symmetry** : $qq \leftrightarrow \bar{q}q$
- **Symmetry** : $\mu_B \leftrightarrow \mu_I$
- **At T=0** $\mu_B^c = \mu_I^c = m_\pi/2$
- **Diquark Condensate** for $\mu_B > \mu_B^c$
- **Pion Condensate** for $\mu_B > \mu_I^c$
- **No sign problem**

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QCD at $\mu_I \neq 0$ and TWO COLOR QCD at $\mu_B \neq 0$

No sign problem, many studies!

- **Low Temperature, Fermionic Sector**
 - Strong coupling analysis [Dagotto, Karsch, Moreo, Wolff, 1986]
 - Laboratory for diquark condensation [Alford, Rajagopal and Wilczek; Rapp, Schäfer, Shuryak and Velkovsky, 1998]
 - Numerical studies of symmetries and spectrum [Hands, Kogut, Morrison, MpL 1998]
 - Lattice studies of diquark condensation [Hands, Kogut, Morrison, Sinclair; 1998; Aloisio, Azcoiti, di Carlo, Galante, 1999]
 - RMT analysis [Akemann, Splittorff, Toublan, Verbaarschot, 2001]
 - Studies of the Dirac spectrum [Bittner, Markum, Pullirsch, MpL, 2001]
 - $\mu_c(T=0) = m_\pi/2$: amenable to χ PT analysis.
 - [Kogut, Stephanov, Toublan, Verbaarschot and Zhitnitsky, 2001; Kogut, Toublan, Sinclair 2002]
 - EoS and Gluon Condensate from χ PT [Metlitsky, Toublan, Zhitnitsky, 2005, 2006]
 - Model studies of the vector sector [Lenaghan, Sannino, Splittorff 2002]
 - Vector spectroscopy on the lattice [Alles, D'Elia, Pepe, MpL, 2001; Muroya, Nakamura, Nonaka, 2003]
 - Test Bed for Imaginary μ [Cea, Cosmai, Giudice, D'Elia, Papa, 2006, 2007]
- **High T, Gluon Sector finite μ transition from hadron gas to QGP, similar to the T=0 transition**
 - Critical Line [Kogut, Sinclair, 2003-2008]
 - Topological Susceptibility [Alles, D'Elia, Pepe, MpL 2002]
 - Polyakov Loop [Muroya, Nakamura, 2004; Alles, D'Elia, MpL 2006]
 - Order Parameter for Confinement [D'Elia Conradi 2007]
 - Critical Line [Forcrand, Stephanov, Wenger, 2007]
- **Low T : Chiral Symmetry, Confinement, BEC Phase**
 - Confinement/Deconfinement, Screening [Hands, Kim, Skullerud 2006–2008]
 - Glueballs spectrum, screening [Lombardo, Paciello, Petrarca, Taglienti, 2007]
 - BEC Phase, interface BEC/QGP [Forcrand, Stephanov, Wenger, 2007]

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THE PHASE DIAGRAM FOR A FINITE ISOSPIN DENSITY

Forcrand, Stephanov, Wenger, 2007

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GLUONIC OBSERVABLES IN THE BEC PHASE of QC₂D

0^{++} Glueball : lighter in the BEC phase

Susceptibility: $\chi = \langle P^2 \rangle - \langle P \rangle^2$ peaks at μ_c

Normal Phase	
m_π/m_ρ	m_0^{++}/m_ρ
0.40	1.07
0.42	1.26
BEC	
0.64	0.80
0.80	0.23

Lombardo, Petrarca, Paciello, Taglienti, 2007

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BEC AND CONFINEMENT: deconfinement deep in the BEC phase

Hands, Kim, Skullerud 2006,2007

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DECONFINEMENT IN THE BEC PHASE – BEC/BCS crossover

Hands, Kim, Skullerud 2008; preliminary, courtesy S. Hands

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From Small $N_c = 2$ to
Large N_c : Proposed Phase Diagram
 McLerran, Pisarski, 2007

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QCD AT FINITE TEMPERATURE AND DENSITY : SUMMARY

T_c known within 20 % .

Fate of the critical endpoint of QCD???

Small μ : Mature calculations. Critical line towards continuum

Mass spectrum depends on μ_B

Critical line at lower temperature, larger μ 'bends down'

First evidences of a triple point in the T, μ_B plane

Cold Phases studied at $\mu_I \neq 0$ / QC_2D at $\mu_B \neq 0$

Examples of interesting patterns confinement/chiral symmetry.

Critical region needs exact dynamics, models do not suffice

Qualitative difference critical surface QCD / NJL , HRG fails around T_c

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