Relativistic BCS-BEC Crossover at Quark Level

Lianyi He, Shijun Mao, and Pengfei Zhuang
Physics Department, Tsinghua University, Beijing 100084, China

Abstract. The non-relativistic $G_0G$ formalism of BCS-BEC crossover at finite temperature is extended to relativistic fermion systems. The theory recovers the BCS mean field approximation at zero temperature and the non-relativistic results in a proper limit. For massive fermions, when the coupling strength increases, there exist two crossovers from the weak coupling BCS superfluid to the non-relativistic BEC state and then to the relativistic BEC state. For color superconductivity at moderate baryon density, the matter is in the BCS-BEC crossover region, and the behavior of the pseudogap is quite similar to that found in high temperature superconductors.

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References

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Pengfei Zhuang
Physics Department, Tsinghua University, Beijing 100084

1) Motivation
2) Mean Field Theory at $T = 0$
3) Fluctuations at $T \neq 0$
4) Application to QCD:
   Color Superconductivity and Pion Superfluid
5) Conclusions

1) Motivation
BCS (Barden, Cooper and Schrieffer, 1957): normal superconductivity
weak coupling, large pair size, \( k \)-space pairing, overlapping Cooper pairs

BEC (Bose-Einstein-Condensation, 1924/1925):
strong coupling, small pair size, \( r \)-space pairing, ideal gas of bosons,
first realization in dilute atomic gas with bosons in 1995.

BCS-BEC crossover (Eagles, Leggett, 1969, 1980):
BCS wave function at \( T=0 \) can be generalized to arbitrary attraction: a smooth crossover from BCS to BEC!

\[
\begin{align*}
\psi_0 &= \exp\left(N_B^{1/2} \sum_k \phi_k c_k^+ c_{-k}^+ \right) |0\rangle \\
\psi_0 &= \prod_k (w_k + v_k c_k^+ c_{-k}^+)|0\rangle
\end{align*}
\]

pairings

in BCS, \( T_c \) is determined by thermal excitation of fermions,
in BEC, \( T_c \) is controlled by thermal excitation of collective modes
**BCS-BEC in QCD**

*In order to study the question of ‘vacuum’, we must turn to a different direction: we should investigate some ‘bulk’ phenomena by distributing high energy over a relatively large volume.*

**QCD phase diagram**

```
  \[ \text{pair dissociation line} \]
```

```
  \[ \mu \quad \text{BCS} \quad \text{BEC} \quad \text{sQGP} \]
```

```
  \[ (\bar{q}q) \neq 0 \]
```

```
  \[ \text{rich QCD phase structure at high density, natural attractive interaction in QCD, possible BCS-BEC crossover?} \]
```

**New phenomena in BCS-BEC crossover of QCD:**

- relativistic systems, anti-fermion contribution, rich inner structure (color, flavor), medium dependent mass, ……

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**Theory of BCS-BEC crossover**

1. *) Leggett mean field theory (Leggett, 1980)
2. *) NSR scheme (Nozieres and Schmitt-Rink, 1985)
3. Extension of of BCS-BEC crossover theory at \( T=0 \) to \( T \neq 0 \) (above \( T_c \))
5. Extension of non-relativistic NSR theory to relativistic systems, BCS-NBEC-RBEC crossover

\[
\Omega = \int \frac{d^4 q}{(2\pi)^3} \ln \left[ \frac{1}{G} \chi (q) \right], \quad \chi = \text{\bigg(} G_0 G \text{\bigg)}
\]

*) ) G\(_0\)G scheme (Chen, Levin et al., 1998, 2000, 2005)

- Asymmetric pair susceptibility

*) Bose-fermion model (Friedder, Lee, 1989, 1990)

- Extension to relativistic systems (Deng, Wang, 2007)

**Kitazawa, Rischke, Shovkovy, 2007, NJL+phase diagram**

**Brauner, 2008, collective excitations** ……
2) **Leggett Mean Field Theory at \( T = 0 \)**


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### non-relativistic mean field theory

- **Fermion number**: \( n = n_F = \frac{k_F^3}{3\pi^2} \)
- **Fermi momentum**: \( k_F = \frac{\pi}{\sqrt{g_{12}}} \)
- **Fermi energy**: \( e_F = \frac{k_F^2}{2m} \)

\[
L = i \hbar \psi^\dagger \left( i \frac{\nabla^2}{2m} - \sigma_3 h \right) \psi + g \psi^\dagger \psi^\dagger \psi \psi^\dagger
\]

**Order parameter**

\[
\Delta = 2g \left( \psi^\dagger \psi \right) = 2g \left( \psi^\dagger \psi^\dagger \right)
\]

**Thermodynamics in mean field approximation**

\[
\Omega = \frac{\Delta^2}{g} - \int \frac{d^3k}{(2\pi)^3} \left[ E_k - \Delta^2 - \frac{\Delta^2}{4E_k} \right]
\]

**Quasi-particle energy**

\[
E_k = \sqrt{x_k^2 + D^2}, \quad x_k = e_k - m, \quad e_k = \frac{k^2}{2m}
\]

**Renormalization to avoid the integration divergence**

\[
g \rightarrow a_s, \quad \frac{m}{4\pi a_s} = \frac{1}{g} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k}
\]

\(a_s < 0\) for attractive coupling, \(a_s > 0\) for repulsive coupling
**universality**

**gap equation**
\[ \frac{\partial \Omega}{\partial \Delta} = 0 \]
\[ \Rightarrow \Delta(n), \quad \mu(n) \]

**number conservation**
\[ n = -\frac{\partial \Omega}{\partial \mu} \]
\[ -\frac{m}{4\pi a_s} = \int \frac{d^3k}{(2\pi)^3} \left( \frac{1}{2E_k} - \frac{1}{2E_k} \right) \]
\[ n = \int \frac{d^3k}{(2\pi)^3} \left( 1 - \frac{\bar{\epsilon}_k}{E_k} \right) \]

**universality**
\[ \bar{\mu} = \mu / \epsilon_F, \quad \bar{\Delta} = \Delta / \epsilon_F, \quad \eta = \frac{1}{k_F a_s} \]

**effective coupling**
\[ \begin{aligned}
\bar{\eta} &= \frac{\eta}{2} \int_0^{\eta} dx x^2 \left( \frac{1}{(x^2 - \bar{\mu})^2 + \bar{\Delta}^2} + \frac{1}{x^2} \right) \\
\bar{\Lambda} &= \frac{2}{3} \int_0^{\eta} dx x^2 \left( 1 - \frac{x^2 - \bar{\mu}}{(x^2 - \bar{\mu})^2 + \bar{\Delta}^2} \right) \\
\end{aligned} \]
\[ \Rightarrow \bar{\Lambda}(\eta), \quad \bar{\mu}(\eta) \]

---

**non-relativistic BCS-BEC crossover**

**BCS limit**
\[ \eta \to -\infty, \quad \bar{\Lambda} = \frac{8}{e^{\eta}} e^{2\eta / \pi}, \quad \bar{\mu} = 1 \]

**BEC limit**
\[ \eta \to \infty, \quad \bar{\Lambda} = \sqrt{\frac{16\eta}{3\pi}}, \quad \bar{\mu} = -\eta^2 \]
\[ \mu = \frac{\epsilon_b}{2}, \quad \epsilon_b = \frac{1}{m a_s^2} \]
\[ n(p) = \frac{1}{e^{(e-\mu)/T} - 1} \quad \Rightarrow \quad \mu \leq 0 \]

**BCS-BEC crossover**
\[ \eta < 0 \to \eta > 0, \]
\[ \text{small } \Delta \to \text{large } \Delta, \quad \mu > 0 \to \mu < 0 \]
### relativistic BCS-BEC crossover

\[ E^\pm_k = \sqrt{(x^\pm_k)^2 + D^2} , \quad x^\pm_k = \sqrt{k^2 + m^2} \pm m \]

- \( m^- = m \) plays the role of non-relativistic chemical potential

**BCS-BEC crossover around**

\[ m^- = m = 0, \quad m = m \]

**pair binding energy**

\[ \epsilon_b < 2m \]

\[ m^- > -\frac{\xi^+}{2} - m, \quad m < 0 \]

at \( m \to 0 \) fermion and anti-fermion degenerate, relativistic effect

<table>
<thead>
<tr>
<th>RBEC limit</th>
<th>RBEC-NBEC crossover</th>
<th>NBEC limit</th>
<th>NBEC-BCS crossover</th>
<th>BCS limit</th>
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<tbody>
<tr>
<td>0</td>
<td>( m )</td>
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### relativistic mean field theory

- extremely high \( T \) and high density: \( pQCD \)
- finite \( T \) (zero density): lattice \( QCD \)
- moderate \( T \) and density: models like 4-fermion interaction (NJL)

\[ L = \bar{\psi} \left( i \gamma^\mu \partial_\mu - m \right) \psi + \frac{g}{4} \left( \bar{\psi} i \gamma_5 C \psi \right) \left( \psi^T i C \gamma_5 \psi \right) \]

\[ C = i \gamma_0 \gamma_2 \] charge conjunction matrix

**order parameter**

\[ \Delta = \frac{g}{2} \left( \psi^T i C \gamma_5 \psi \right) \]

**mean field thermodynamic potential**

\[ \Omega = \frac{\Delta^2}{g} - \int \frac{d^3 k}{(2\pi)^3} \left[ E_k^+ + E_k^- - \xi_k^+ - \xi_k^- \right] \]

- antifermion
- fermion

the most important thermodynamic contribution from the uncondensed pairs is from the Goldstone modes, \( \epsilon \square T^4 \), at \( T=0 \), the fluctuation contribution disappears, and MF is a good approximation at \( T=0 \).
broken universality

gap equation and number equation:
\[
\begin{align*}
\frac{1}{g} &= \int \frac{d^3k}{(2\pi)^3} \left( \frac{1}{2E_k} - \frac{1}{2E_k^+} \right), \\
\eta &= \int \frac{d^3k}{(2\pi)^3} \left[ \left( 1 - \frac{E_k^-}{E_k^+} \right) - \left( 1 - \frac{E_k^+}{E_k^-} \right) \right]
\end{align*}
\]

renormalization to avoid the integration divergence
\[
-\frac{1}{U} = \frac{1}{g} - \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left( \frac{1}{\varepsilon_k - m} + \frac{1}{\varepsilon_k + m} \right), \quad U = \frac{4\pi a_s}{m}
\]

the ultraviolet divergence can not be completely removed, and a momentum cutoff \( \Lambda \) still exists in the theory.

broken universality:
\[
\begin{align*}
-\frac{\pi}{2} \eta &= \int_0^\infty dx x^2 \left[ \left( \frac{1}{E_k^+} - \frac{1}{\varepsilon_k - 2\zeta} \right) + \left( \frac{1}{E_k^-} - \frac{1}{\varepsilon_k + 2\zeta} \right) \right] \\
\frac{2}{3} &= \int_0^\infty dx x^2 \left[ \left( 1 - \frac{E_k^-}{E_k^+} \right) - \left( 1 - \frac{E_k^+}{E_k^-} \right) \right]
\end{align*}
\]
\[
\eta = \frac{1}{k_f a_s}, \quad \zeta = \frac{k_f}{m}
\]

explicit density dependence!

non-relativistic limit

non-relativistic limit:
1) \( \zeta = k_f / m \ll 1 \) non-relativistic kinetics
2) \( |\mu - m| < m \) negligible anti-fermions

\[
m(h,z) = m \quad \text{IO} \quad h^1_c(z) \quad \bullet \quad \text{NBEC}
\]
\[
m(h,z) \hat{=} 0 \quad \text{IO} \quad h^2_c(z) \quad \bullet \quad \text{RBEC}
\]
**density induced crossover**

In non-relativistic case, only one dimensionless variable, $\eta = 1/k_Fa_s$, changing the density of the system can not induce a BCS-BEC crossover. However, in relativistic case, the extra density dependence $z = k_F/m$ may induce a BCS-BEC crossover.

![Graph showing density induced crossover](image)

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3) Fluctuations at $T \neq 0$
\textbf{Introduction}

- the \textit{Landau mean field theory} is a good approximation only at $T=0$ where there is no thermal excitation, \textbf{one has to go beyond the mean field at finite temperature.}

- an \textit{urgent question in relativistic heavy ion collisions}

\begin{equation}
L = \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi + \frac{g}{4} \left( \bar{\psi} i \gamma^\mu \gamma^\nu \right) \left( \psi^T i \gamma^\nu \psi \right)
\end{equation}

gap and number equations at finite $T$

\begin{equation}
\begin{split}
\frac{1}{g} = \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1 - 2 f (E^-_k)}{2 E^-_k} + \frac{1 - 2 f (E^+_k)}{2 E^+_k} \right] \\
n = \int \frac{d^3k}{(2\pi)^3} \left[ 1 - \frac{\bar{\psi} \gamma^\mu \psi}{E^-_k} \right] \left( 1 - 2 f (E^-_k) \right) - \left( 1 - \frac{\bar{\psi} \gamma^\mu \psi}{E^+_k} \right) \left( 1 - 2 f (E^+_k) \right)
\end{split}
\end{equation}

\textit{fermion propagator with diagonal and off-diagonal elements in Nambu-Gorkov space}

\begin{equation}
S(k, \mu) = \begin{pmatrix}
G(k, \mu) & F(k, \mu) \\
F(k, -\mu) & G(k, -\mu)
\end{pmatrix}
\end{equation}

\begin{equation}
G^{-1}(k, \mu) = G_0^{-1}(k, \mu) - \Sigma_m(k)
\end{equation}

\begin{equation}
F(k, \mu) = -G(k, \mu) i \gamma^\mu \Delta G_0(k, -\mu)
\end{equation}

\textit{bare propagator and condensate induced self-energy}

\begin{equation}
G_0^-(k, \mu) = (k_0 + \mu) \gamma_0 - \vec{p} \cdot \vec{k} - m
\end{equation}

\begin{equation}
\Sigma_m(k) = -\Delta G_0(k, -\mu)
\end{equation}

gap and number equations in terms of the fermion propagator

\begin{equation}
\Delta \left[ 1 + i \frac{g}{2} \sum_k Tr \left[ G(k, \mu) G_0^-(k, -\mu) \right] \right] = 0,
\end{equation}

\begin{equation}
n = -i \sum_k Tr \left[ \gamma_0 G(k, \mu) \right]
\end{equation}

\textit{Matsubara frequency summation}

\begin{equation}
\sum_n = \int \frac{d^3k}{(2\pi)^3}, \ k_n \rightarrow i\omega_n, \ \omega_n = \frac{2n\pi T}{(2n+1)\pi T}, \ \text{bosons}
\end{equation}

\begin{equation}
\text{fermions}
\end{equation}
mean field theory in $G_0G$ scheme

introducing a condensed-pair propagator

$$t_{mf}(q) = i \frac{\Delta^2}{T} \delta(q)$$

$$\Sigma_{mf}(k) = \sum q t_{mf}(q) G_0(q-k, \mu) = \frac{t_{mf}}{G_0}$$

defining an uncondensed pair propagator

the name $G\tilde{G}$ scheme, $G$ is the mean field fermion propagator

$$t(q) = \frac{ig}{1 - g \chi(q)}$$
$$\chi(q) = -\frac{i}{2} \sum_k \text{Tr}[G(k, \mu)G_0(q-k, \mu)]$$

gap equation in condensed phase is determined by uncondensed pairs

$$t^{-1}(0) = 0 \quad \Delta(T, \mu)$$

problem: there is no feedback of the uncondensed pairs on the fermion self-energy

going beyond mean field in $G_0G$ scheme

full pair propagator

$$t(q) = t_{mf}(q) + t_{pg}(q)$$

$$t_{mf}(q) = i \frac{\Delta^2}{T} \delta(q)$$

$$t_{pg}(q) = \frac{ig}{1 - g \chi(q)} = \frac{i g}{1 - g}$$

with full susceptibility and full propagator

$$\chi(q) = -\frac{i}{2} \sum_k \text{Tr}[G(k, \mu)G_0(q-k, \mu)]$$

\begin{align*}
G^{-1}(k, \mu) &= G_0^{-1}(k, \mu) - \Sigma(k) \\
\Sigma(k) &= \sum_{q} t(q) G_0(q-k, \mu) \\
&= \Sigma_{mf}(k) + \Sigma_{pg}(k) + \ldots
\end{align*}

full fermion self-energy

fermions and pairs are coupled to each other

new gap equation

$$t_{pg}^{-1}(0) = 0 \quad \text{a new order parameter } \Delta(T, \mu) \text{ which is different from the mean field one}$$

all the formulas look the same as the mean field ones, but we do not know the expression of the full fermion propagator $G$. 

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He, Zhuang, 2007

Dense Matter In Heavy Ion Collisions and Astrophysics (DM2008)

01007-p.11
approximation in condensed phase

1) \( t_{pg}^{-1}(0) = 0 \rightarrow t_{pg}(q) \) peaks at \( q = 0 \)

\[
\Sigma_{pg}(k) = \sum_{q>0} t_{pg}(q) G_q(q-k,\mu) - \sum_{q<0} t_{pg}(q) G_q(-k,\mu) = -\Delta_{pg}^2 G_q(-k,\mu)
\]

\[
\Delta_{pg}^2 = -\sum_{q>0} t_{pg}(q)
\]

The pseudogap is related to the uncondensed pairs and does not change the symmetry!

full self-energy \( \Sigma(k) = \Sigma_{mf}(k) + \Sigma_{pg}(k) \) and \( \Delta + \Delta_{pg}^2 \) gap equation

\( t_{pg}^{-1}(0) = 0 \rightarrow (\text{mean field gap equation with} \) \( \Delta \rightarrow \sqrt{\Delta^2 + \Delta_{pg}^2} \)

2) \( t_{pg}^{-1}(0) = 0 \rightarrow \) expansion around \( q = 0 \)

\[
t_{pg}(q) = \frac{ig}{1-g\chi(q)} = \frac{-i}{Z_{q_0} + Z_{q_0} - \xi q^2} \quad \text{with} \quad Z_{q_0} = 1 \frac{\partial^2 \chi}{\partial q_0^2} \mid_{q_0 = 0}, \quad \xi = \frac{1}{2} \frac{\partial^2 \chi}{\partial q^2} \mid_{q = 0}
\]

\[
\begin{aligned}
\Delta^2_{pg} &= \frac{1}{Z_2} \frac{d^3q}{(2\pi)^3} f_B(\omega_q - \nu) + f_B(\omega_q + \nu) \\
\Lambda(T,\mu), \quad \Delta_{pg}(T,\mu)
\end{aligned}
\]

\( \nu = Z_1 / 2Z_2, \quad \omega_q = \sqrt{\xi^2 q^2 + \nu^2} \)

thermodynamics

\[
\Omega = \Omega_{mf} + \Omega_B
\]

\[
\Omega_{mf} = \frac{\Delta^2}{g} + \int \frac{d^3 k}{(2\pi)^3} \left[ (\xi^+ + \xi^- - E^+ - E^-) - \frac{1}{\beta} \ln \left( 1 + e^{-\beta E^+} \right) \left( 1 + e^{-\beta E^-} \right) \right]
\]

\[
\Omega_B = \sum_q \ln \left[ 1 - g\chi(q) \right] - \frac{1}{\beta} \int \frac{d^3 q}{(2\pi)^3} \ln \left( 1 - e^{-\beta q^2/2\omega^2} \right)
\]

\[
= \frac{1}{\beta} \int \frac{d^3 q}{(2\pi)^3} \ln \left( 1 - e^{-\beta q^2/2m^2} \right) \quad Z_1 >> Z_2, \quad \text{non-relativistic boson gas}
\]

\[
= \frac{2}{\beta} \int \frac{d^3 q}{(2\pi)^3} \ln \left( 1 - e^{-\beta q^2} \right) \quad Z_2 >> Z_1, \quad \text{relativistic boson gas}
\]

number of bosons

\[
n_B = \frac{n}{2} - \int \frac{d^3 k}{(2\pi)^3} \left[ f \left( \frac{\xi^{-}}{\xi_{k}} \right) - f \left( \frac{\xi^{+}}{\xi_{k}} \right) \right] = Z_1 \left( \Delta^2 + \Delta_{pg}^2 \right) = n_B^{mf} + n_B^{B}
\]

fraction of condensed pairs

\[
P_c = \frac{n_B^{mf}}{n/2} = \frac{Z_1 \Delta^2}{n/2}
\]
**BCS-NBEC-RBEC crossover**

- $T_c$: critical temperature
  - $T < T_c$: $\Delta \neq 0$, condensed phase
  - $T > T_c$: $\Delta = 0$, normal or pseudogap phase

- $T^*$: pair dissociation temperature
  - $T_c < T < T^*$: $\Delta_{pg} \neq 0$, pseudogap phase
  - $T > T^*$: $\Delta_{pg} = 0$, normal phase

- **BCS**: $\eta < 0$, $\mu > m$, no pairs
- **NBEC**: $0 < \eta < m/k_F$, $0 < \mu < m$, heavy pairs, no anti-pairs
- **RBEC**: $\eta > m/k_F$, $\mu \leq 0$, light pairs, almost the same number of pairs and anti-pairs

**discussion on $G_0 G$**

**Can the symmetry be restored in the pseudogap phase?**

**Fermion propagator including fluctuations (to the order of $\Delta^2 / \Lambda^2$):**

$$S^{-1}(k) = \begin{pmatrix}
G_0^{-1}(k, \mu) - \Sigma_{pg}(k, \mu) & i\gamma_5 \Delta \\
 i\gamma_5 \Delta & G_0^{-1}(k, \mu) - \Sigma_{pg}(k, -\mu)
\end{pmatrix}$$

The pseudogap appears in the diagonal elements of the propagator and does not break the symmetry of the system.

**Kadanoff and Martin:**

The scheme $\chi \sqcap GG$ cannot give a correct symmetry restoration picture and the specific heat $C_v \sqcap T^2$ is wrong.
BCS-BEC in Bose-fermion model  

\[ \mathcal{L}_0 = \mathcal{L}_{f0} + \mathcal{L}_{b0} + \mathcal{L}_{10}, \]

\[ \Omega = -\sum_{\epsilon = \pm} \int \frac{d^3k}{(2\pi)^3} \left\{ \epsilon_k^2 + 2T \ln \left[ 1 + \exp \left( -\frac{\epsilon_k}{T} \right) \right] \right\} + \frac{(m_0^2 - \mu_0^2)\Delta^2}{4g^2} \]

\[ \mathcal{L}_{f0} = \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi, \]

\[ \mathcal{L}_{b0} = \partial_\mu \overline{\phi}^\dagger \partial^\mu \phi - m_0^2 \phi^\dagger \phi, \]

\[ \mathcal{L}_{10} = g(\overline{\psi}_C \gamma_5 \psi + \phi^\dagger \psi \gamma_5 \psi_C) \]

mean field thermodynamics \[ \Delta = 2g\phi, \]

fluctuation changes the phase transition to be first-order.

4) Application to QCD:
Color Superconductivity and Pion Superfluid
motivation

* QCD phase transitions like chiral symmetry restoration, color superconductivity, and pion superfluid happen in non-perturbative temperature and density region, the coupling is strong.

* relativistic BCS-BEC crossover is controlled by $\mu - m$, the BEC-BCS crossover would happen when the light quark mass changes in the QCD medium.

* effective models at hadron level can only describe BEC state, they can not describe BEC-BCS crossover. One of the models that enables us to describe both quarks and mesons and diquarks is the NJL model at quark level.

```
L_{NJL} = \bar{\psi} \left( i \gamma^\mu \partial_\mu \psi - m_0 + \mu \gamma_0 \right) + G_S \left( \left( \bar{\psi} \psi \right)^2 + \left( \bar{\psi} i \gamma_5 \psi \right)^2 \right) + G_D \left( \bar{\psi}^a \gamma^\mu \epsilon^{\alpha \beta \mu \nu} i \gamma_5 \psi^a \right) \left( \bar{\psi}^a \gamma^\nu \epsilon^{\alpha \beta \mu \nu} i \gamma_5 \psi^a \right)
```

disadvantage: no confinement

* there is no problem to do lattice simulation for real QCD at finite isospin density

color superconductivity

order parameters of spontaneous chiral and color symmetry breaking

$\sigma = \langle \bar{\psi} \psi \rangle$

$\Delta = \Delta^3 = \langle \bar{\psi}^a \epsilon^{\alpha \beta} \gamma_5 \psi^a \rangle$ color breaking from SU(3) to SU(2)

leading order of $1/N_c$ for quarks, and next to leading order for mesons & diquarks

quark propagator in 12D Nambu-Gorkov space

```
\Psi = \left( \begin{array}{c}
\psi_{a_1} \\
\psi_{b_1} \\
\psi_{c_1} \\
\psi_{d_1} \\
\psi_{a_2} \\
\psi_{b_2} \\
\psi_{c_2} \\
\psi_{d_2} \\
\psi_{a_3} \\
\psi_{b_3} \\
\psi_{c_3} \\
\psi_{d_3}
\end{array} \right) \\
S = \left( \begin{array}{cccc}
S_A & S_B & \cdots & S_E \\
S_B & S_C & \cdots & S_D \\
\vdots & \vdots & \ddots & \vdots \\
S_E & S_D & \cdots & S_A
\end{array} \right)
```

diquark & meson propagators at RPA

```
\Psi_RPA \approx \oplus \Psi + \oplus \Psi \oplus \Psi + \cdots = \frac{\Psi}{1-\Xi}
```

Dense Matter In Heavy Ion Collisions and Astrophysics (DM2008)
**BCS-BEC and color neutrality**

He, Zhuang, 2007–

**Gap equations for chiral and diquark condensates at $T=0$**

\[
\begin{align*}
 m - m_0 &= 8G_j m \left\{ \frac{d^3 k}{(2\pi)^3} \frac{1}{E^k} \right\} \frac{E_k - \mu_g}{E^k_{\Delta}} + \frac{E_k + \mu_g}{E^k_{\Delta}} + \Theta\left( E_k - \mu_g / 3 \right) \\
 \Delta &= 8G_j \Delta \left\{ \frac{d^3 k}{(2\pi)^3} \frac{1}{E^k_{\Delta}} \right\}
\end{align*}
\]

To guarantee color neutrality, we introduce color chemical potential:

\[
\mu_r = \frac{\mu_g}{3} + \frac{\mu_b}{3}, \quad \mu_b = \frac{\mu_g}{3} - \frac{2\mu_b}{3}
\]

to guarantee color neutrality, we introduce color chemical potential:

\[
\mu_r = \frac{\mu_g}{3} + \frac{\mu_b}{3}, \quad \mu_b = \frac{\mu_g}{3} - \frac{2\mu_b}{3}
\]

there exists a BCS-BEC crossover

color neutrality speeds up the chiral restoration and reduces the BEC region

---

**Vector meson coupling and magnetic instability**

Vector-meson coupling

\[
L_V = -G_V \left[ (\bar{\psi} \gamma^\mu \psi)^2 + (\bar{\psi} \gamma^\mu \gamma^5 \tau^a \psi)^2 \right]
\]

Vector condensate

\[
\rho_V = 2G_V \langle \bar{\psi} \gamma^\mu \psi \rangle
\]

Gap equation

\[
\rho_V = 8G_j \left\{ \frac{d^3 k}{(2\pi)^3} \frac{1}{E^k} \right\} \frac{E_k + \mu_g / 3}{E^k_{\Delta}} - \frac{E_k - \mu_g / 3}{E^k_{\Delta}} + \Theta\left( -E_k + \mu_g / 3 \right)
\]

\[
\eta = 1
\]

Vector meson coupling slows down the chiral symmetry restoration and enlarges the BEC region.

Meissner masses of some gluons are negative for the BCS Gapless CSC, but the magnetic instability is cured in BEC region.

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beyond men field

\[ \Delta_0 = \Delta(T = 0) \text{ is determined by the coupling and chemical potential} \]
\[ \Delta_0 = 100 - 200 \text{ MeV} \leftrightarrow \mu_q = 300 - 500 \text{ MeV} \]

- going beyond mean field reduces the critical temperature of color superconductivity
- pairing effect is important around the critical temperature and dominates the symmetry restored phase

pion superfluid

NJL with isospin symmetry breaking

\[ L_{NJL} = \bar{\psi} \left( i \gamma^\mu \partial_\mu - m_u + \mu \gamma_5 \right) \psi + G \left( (\bar{\psi} \psi)^2 + (\bar{\psi} i \gamma_5 \gamma_\mu \psi)^2 \right) \]

quark chemical potentials

\[ \mu = \begin{pmatrix} \mu_u & 0 \\ 0 & \mu_d \end{pmatrix} = \begin{pmatrix} \mu_u / 3 + \mu_d / 2 & 0 \\ 0 & \mu_u / 3 - \mu_d / 2 \end{pmatrix} \]

chiral and pion condensates with finite pair momentum

\[ \sigma = \langle \bar{\psi} \psi \rangle = \sigma_u + \sigma_d, \quad \sigma_u = \langle \bar{u} u \rangle, \quad \sigma_d = \langle \bar{d} d \rangle \]
\[ \pi_+ = \sqrt{2} \langle \bar{u} u \gamma_5 d \rangle = \frac{\pi}{\sqrt{2}} e^{2i\eta}, \quad \pi_- = \sqrt{2} \langle \bar{d} d \gamma_5 u \rangle = \frac{\pi}{\sqrt{2}} e^{-2i\eta} \]

quark propagator in MF

\[ S^{-1}(p, q) = \begin{pmatrix} \gamma^\mu p - \gamma^5 \frac{2iG}{2G} \pi_5 m & \frac{2iG}{2G} \pi_5 \\ \frac{2iG}{2G} \pi_5 & \gamma^\mu k - \gamma^5 \frac{2iG}{2G} \pi_5 m \end{pmatrix} \]
\[ m = m_0 - 2G \sigma \]

thermodynamic potential and gap equations:

\[ \Omega = G(\sigma^2 + \pi^2) - \frac{T}{V} \text{Tr} \ln S^{-1} \]
\[ \frac{\partial \Omega}{\partial \sigma_u} = 0, \quad \frac{\partial^2 \Omega}{\partial \sigma_u^2} \geq 0, \quad \frac{\partial \Omega}{\partial \sigma_d} = 0, \quad \frac{\partial^2 \Omega}{\partial \sigma_d^2} \geq 0, \quad \frac{\partial \Omega}{\partial \pi} = 0, \quad \frac{\partial^2 \Omega}{\partial \pi^2} \geq 0, \quad \frac{\partial \Omega}{\partial q} = 0, \quad \frac{\partial^2 \Omega}{\partial q^2} \geq 0 \]
mesons in RPA

**meson propagator** $D$ at RPA

\[
D \simeq \times + \times + \times + \ldots = \frac{\times}{1 - \times}
\]

considering all possible channels in the bubble summation

**meson polarization functions**

\[
\Pi_m(k) = i \frac{d^4 p}{(2\pi)^4} \text{Tr}\left( \Gamma_m^\dagger S(p+k)\Gamma_m S(p) \right)
\]

\[
\Gamma_m = \begin{cases} 
1, & m = \sigma \\
 \imath \gamma_\nu, & m = \pi \\
 \imath \gamma_\nu, & m = \pi_0
\end{cases}
\]

pole of the propagator determines meson masses $M_m$

\[
\det \begin{bmatrix}
1 - 2G\Pi_{\sigma\sigma}(k) & -2G\Pi_{\sigma\pi}(k) & -2G\Pi_{\sigma\pi_0}(k) \\
-2G\Pi_{\pi\sigma}(k) & 1 - 2G\Pi_{\pi\pi}(k) & -2G\Pi_{\pi\pi_0}(k) \\
-2G\Pi_{\pi_0\sigma}(k) & -2G\Pi_{\pi_0\pi}(k) & 1 - 2G\Pi_{\pi_0\pi_0}(k)
\end{bmatrix} = 0
\]

**mixing among normal** $\sigma, \pi_+^+, \pi_-$ **in pion superfluid phase,**

the new eigen modes $\bar{\sigma}, \pi_+^-, \pi_-$ are linear combinations of $\sigma, \pi_+^+, \pi_-^-$

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**phase diagram of pion superfluid**

**chiral and pion condensates** at $T = \mu_p = \tilde{q} = 0$

in NJL, Linear Sigma Model and Chiral Perturbation Theory, there is no remarkable difference around the critical point.

**analytic result:**

critical isospin chemical potential for pion superfluidity is exactly the pion mass in the vacuum:

\[
\mu_c^- = m_\pi
\]

**pion superfluidity phase diagram** in $\mu_\pi - \mu_\rho$ plane at $T=0$

$\mu_\pi$: average Fermi surface

$\mu_\rho (n_\rho)$: Fermi surface mismatch

homogeneous (Sarma, $\tilde{q} = 0$) and inhomogeneous pion superfluid (LOFF, $\tilde{q} \neq 0$)

**magnetic instability** of Sarma state at high average Fermi surface leads to the LOFF state
BCS-BEC crossover of pion superfluid

BCS-BEC crossover in asymmetric nuclear matter

- Transition from BCS pairing to BEC in low-density asymmetric nuclear matter, U. Lombardo, P. Nozières: PRC64, 064314 (2001)
- Spatial structure of neutron Cooper pair in low density uniform matter, Masayuki Matsuo: PRC73, 044309 (2006)

Asymmetric nuclear matter with both np and nn and pp pairings, Mao, Huang, Zhuang, 2008

Considering density dependent Paris potential and nucleon mass

\[ \Psi(\vec{r}) = \int \frac{d^3 p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{r}} \varphi(\vec{p}) \]

There exists a strong Friedel oscillation in BCS region, and it is washed away in BEC region.
near side $D\bar{D}$ correlation as a signature of sQGP

Zhu, Xu, Zhuang, PRL100, 152301(2008)

* c quark motion in QGP:
\[
\frac{dp}{dt} = -\gamma(T)\vec{p} + \vec{\eta},
\]
we take drag coefficient to be a parameter charactering the coupling strength

* QGP evolution: ideal hydrodynamics

for strongly interacting quark-gluon plasma:
- at RHIC, the back-to-back correlation is washed out,
- at LHC, c quarks are fast thermalized, the strong flow push the D and Dbar to the near side!

large drag parameter is confirmed by $R_{AA}$ and $v_2$ of non-photonic electrons (PHENEX, 2007; Moore and Teaney, 2005; Horowitz, Gyulassy, 2007).

conclusions

* BCS-BEC crossover is a general phenomena from cold atom gas to quark matter.

* BCS-BEC crossover is closely related to QCD key problems: vacuum, color symmetry, chiral symmetry, isospin symmetry ……

* BCS-BEC crossover of color superconductivity and pion superfluid is not induced by simply increasing the coupling constant of the attractive interaction but by changing the corresponding charge number.

* There are potential applications in heavy ion collisions (at CSR/Lanzhou, FAIR/GSI and RHIC/BNL) and compact stars.

thanks for your patience