Numerical experiments for turbulent flows

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Abstract. The aim of the work is to explore the possibilities of modelling transonic flows in the internal and external aerodynamics. Several configurations were analyzed and calculations were performed using both inviscid and viscous models of flow. Viscous turbulent flows have been simulated using either zero equation algebraic Baldwin-Lomax model and two equation $k - \omega$ model in its basic version and improved TNT variant. The numerical solution was obtained using Lax-Wendroff scheme in the MacCormack form on structured non-orthogonal grids. Artificial dissipation was added to improve the numerical stability. Achieved results are compared with experimental data.

1 Mathematical models

1.1 Navier-Stokes equations

The two-dimensional laminar flow of a viscous compressible liquid is described by the system of Navier-Stokes equations

$$W_t + F_x + G_y = R_x + S_y, \quad (1)$$

where

$$W = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ e \end{pmatrix}, F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (e + p)u \end{pmatrix}, G = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (e + p)v \end{pmatrix} \quad (2)$$

and

$$R = \begin{pmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ u\tau_{xx} + v\tau_{xy} + \lambda T_x \end{pmatrix}, S = \begin{pmatrix} 0 \\ \tau_{xy} \\ \tau_{yy} \\ u\tau_{xy} + v\tau_{yy} + \lambda T_y \end{pmatrix} \quad (3)$$

with shear stresses given for the laminar flow by equations

$$\tau_{xx} = \frac{2}{3}\eta(2u_x - v_y), \tau_{xy} = \eta(u_y + v_x), \tau_{yy} = \frac{2}{3}\eta(-u_x + 2v_y). \quad (4)$$

This system is enclosed by the equation of state

$$p = (\kappa - 1) \left[e - \frac{1}{2}\rho(u^2 + v^2) \right]. \quad (5)$$

In the above given equations, ρ denotes density, u, v are components of velocity in the direction of axis x, y , p is pressure, e is total energy per a unit volume, T is temperature, η is dynamical viscosity and λ is thermal conductivity coefficient. The parameter $\kappa = 1.4$ is the adiabatic exponent.

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1.2 Reynolds averaged Navier-Stokes equations

For the modelling of a turbulent flow, the system of RANS (Reynolds Averaged Navier-Stokes) equations enclosed by a turbulence model is used. Two different turbulence models with the turbulent viscosity were tested, one algebraic, Baldwin-Lomax and the two-equation $k - \omega$ model according to Wilcox. The system of averaged Navier-Stokes equations is formally the same as (1), but this time the flow parameters represent only mean values in the Favre sense, see [3]. The shear stresses are given for the turbulent flows by equations

$$\begin{aligned} \tau_{xx} &= \frac{2}{3}(\eta + \eta_t)(2u_x - v_y), \\ \tau_{xy} &= (\eta + \eta_t)(u_y + v_x), \\ \tau_{yy} &= \frac{2}{3}(\eta + \eta_t)(-u_x + 2v_y), \end{aligned} \quad (6)$$

where η_t denotes the turbulent dynamic viscosity according to the Boussinesq hypothesis. The Reynolds number is defined by $Re = \frac{u_\infty L}{\eta_\infty}$ and the Mach number by $M = \sqrt{\frac{q}{a}}$ where $q = \sqrt{u^2 + v^2}$ and a is the local speed of sound.

All the computations were carried out using dimensionless variables with reference variables given by inflow values. The reference length L is given by the width of the computational domain.

2 Turbulence models

2.1 Baldwin-Lomax model

Algebraic models are based on the model proposed for the boundary-layer flows by Cebeci and Smith. Baldwin-Lomax model is its modification applicable for general turbulent shear flows. The boundary layer is divided into two regions. In the inner (nearest to the wall) part, the turbulent viscosity is given by

$$\eta_t = \rho F_D^2 \kappa^2 y^2 |\Omega|, \quad (7)$$

where Ω is the vorticity, which is in the 2D flow determined by

$$\Omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \quad (8)$$

and

$$F_D = 1 - \exp\left(-\frac{y^+}{A^+}\right). \quad (9)$$

$y^+ = \frac{u_* y}{\nu}$ denotes dimensionless distance from the wall, $\nu = \frac{\mu}{\rho}$ is kinematic viscosity, $u_* = \sqrt{\left(\frac{\tau_w}{\rho}\right)}$ is so called friction velocity and $\tau_w = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0}$.

The turbulent viscosity in the outer region is given by

$$\eta_{to} = \alpha \rho C_{cp} F_w F_k, \quad (10)$$

where C_{cp} is a constant. Function F_w is determined by the relation

$$F_w = y_{max} F_{max} \quad (11)$$

for F_w being the maximum of the function

$$F = y F_D |\Omega| \quad (12)$$

and y_{max} the distance from the wall in which $F(y_{max}) = F_{max}$ holds and

$$F_k = \left[1 + 5.5 \left(C_{KL} \frac{y}{y_{max}} \right)^6 \right]^{-1}. \quad (13)$$

The Baldwin-Lomax model (1978) contains following values of the constants: $\kappa = 0.4$, $A^+ = 26$, $\alpha = 0.0168$, $C_{cp} = 1.6$, $C_{KL} = 0.3$.

2.2 $k - \omega$ model

Two-equation models are based on transport equations for two characteristic scales of turbulent motion, mostly for the turbulent energy k and dissipation rate ϵ , often used in the form of specific dissipation rate $\omega = \epsilon/k$. These characteristics are computed from transport equations. Turbulent viscosity is defined as

$$\eta_t = \rho \frac{k}{\omega}. \quad (14)$$

The standard Wilcox $k - \omega$ model is formed by the equations

$$\frac{\partial}{\partial t}(\rho k) + \frac{\partial}{\partial x_j}(\rho u_j k) = P_k + \frac{\partial}{\partial x_j} \left[(\eta + \sigma^* \eta_t) \frac{\partial k}{\partial x_j} \right] - \beta^* \rho k \omega, \quad (15)$$

$$\frac{\partial}{\partial t}(\rho \omega) + \frac{\partial}{\partial x_j}(\rho u_j \omega) = \gamma \frac{\omega}{k} P_k + \frac{\partial}{\partial x_j} \left[(\eta + \sigma \eta_t) \frac{\partial \omega}{\partial x_j} \right] - \beta \rho \omega^2, \quad (16)$$

where $P_k = \tau_{ij} \partial u_i / \partial x_j$ represents the production of turbulent energy. Model coefficients are given by values: $\alpha = 5/9$, $\beta = 3/40$, $\beta^* = 9/100$, $\sigma = 1/2$ and $\sigma^* = 1/2$, $i, j \in \{1, 2\}$.

So called TNT modification of $k - \omega$ model:

$$\frac{\partial(\rho k)}{\partial t} + \frac{\partial(\rho u_j k)}{\partial x_j} = P_k + \frac{\partial}{\partial x_j} \left[(\mu + \sigma_k \mu_t) \frac{\partial k}{\partial x_j} \right] - \beta^* \rho k \omega, \quad (17)$$

$$\frac{\partial(\rho \omega)}{\partial t} + \frac{\partial(\rho u_j \omega)}{\partial x_j} = \alpha \frac{\omega}{k} P_k + \frac{\partial}{\partial x_j} \left[(\mu + \sigma_\omega \mu_t) \frac{\partial \omega}{\partial x_j} \right] - \beta \rho k \omega^2 + C_D, \quad (18)$$

where

$$C_D = \sigma_d \frac{\rho}{\omega} \max \left(\frac{\partial k}{\partial x_i} \frac{\partial \omega}{\partial x_i}, 0 \right) \quad (19)$$

$$\text{and } \alpha_\omega = \frac{\beta}{\beta_*} - \frac{\sigma_\omega \kappa^2}{\sqrt{\beta_*}}, \quad \kappa = 0.41, \quad \beta = \frac{3}{40}, \quad \beta^* = \frac{9}{100}, \\ \sigma_\omega = 0.5, \quad \sigma_k = 0.666.$$

3 Numerical methods

For the modelling of the flow cases Lax-Wendroff finite volume method scheme was used on non-orthogonal structured grids of quadrilateral and hexahedral cells $D_{ij(k)}$.

– Predictor step:

$$W_{i,j}^{n+1/2} = W_{i,j}^n - \frac{\Delta t}{\mu_{i,j}} \sum_{k=1}^4 \left[\left(\tilde{F}_k^n - \frac{1}{Re} R_k^n \right) \Delta y_k - \left(\tilde{G}_k^n - \frac{1}{Re} S_k^n \right) \Delta x_k \right]. \quad (20)$$

– Corrector step:

$$W_{i,j}^{n+1} = \frac{1}{2} (W_{i,j}^n + W_{i,j}^{n+1/2}) - \frac{\Delta t}{2\mu_{i,j}} \sum_{k=1}^4 \left[\left(\tilde{F}_k^{n+1/2} - \frac{1}{Re} R_k^{n+1/2} \right) \Delta y_k - \left(\tilde{G}_k^{n+1/2} - \frac{1}{Re} S_k^{n+1/2} \right) \Delta x_k \right] + AD(W_{i,j}^n). \quad (21)$$

The Mac Cormack scheme in the cell centered form was applied to solving the system of RANS equations. Convective terms F , G are considered in predictor step in forward form and in the corrector step in upwind form of the first order of accuracy, dissipative terms in central form of the second order of accuracy. To indicate this we denote their numerical approximation as $\tilde{F}$, $\tilde{G}$.

The value $\mu_{ij(k)}$ represents the surface (volume) of the cell. Figure 14 shows evaluation of derivatives on edges of cells: We imagine a virtual cell as shown in the picture. We know the values in the centers of the cells and we define the other two as a mean value of its surrounding cells. From that we extrapolate to its edges and then we apply Green's formula.

The scheme was extended to include Jameson's artificial dissipation because of the stability of the method

$$AD(W_{i,j}^n) = C_1 \psi_1 (W_{i-1,j}^n - 2W_{i,j}^n + W_{i+1,j}^n) + C_2 \psi_2 (W_{i,j-1}^n - 2W_{i,j}^n + W_{i,j+1}^n), \quad (22)$$

where

$$\psi_1 = \frac{|p_{i-1,j}^n - 2p_{i,j}^n + p_{i+1,j}^n|}{|p_{i-1,j}^n| + |p_{i,j}^n| + |p_{i+1,j}^n|}, \quad (23) \\ \psi_2 = \frac{|p_{i,j-1}^n - 2p_{i,j}^n + p_{i,j+1}^n|}{|p_{i,j-1}^n| + |p_{i,j}^n| + |p_{i,j+1}^n|}.$$

Inlet boundary conditions IBC $\mathcal{A}$ were used for inviscid compressible flows; IBC $\mathcal{B}$ were used for inviscid compressible flows. A nonzero angle of attack α_1 was used only in the case of flows through DCA cascade.

On the **outlet** we prescribed only pressure $p_2 = p_1$ and the other values were extrapolated from the flow field.

Further on there are three other types of boundary conditions: solid wall, symmetry axis and periodicity. These conditions are implemented by using virtual cells. Such cells adjoin from outside on the boundary cells and we prescribe values of unknowns inside of them to obtain the desired effect.

Solid wall (an inviscid flow): velocity components prescribed so that the sum of velocity vectors equals to zero in its tangential component. **Solid wall (a viscous flow):** velocity components were prescribed so that the sum of velocity vectors equals zero. In both cases the rest of unknowns is the same in both the virtual and the boundary cell.

Symmetry axis: this condition was realized by the same way as the wall condition for an inviscid flow.

Periodicity condition: taking two corresponding segments of boundary we prescribe into virtual cells of the first segment the values of unknowns contained in the boundary cells of the second and vice-versa.

Initial conditions were prescribed to comply with the inlet conditions.

5 Results

The following pictures illustrate results obtained during the calculations. First three of them present simulation of the flow through the DCA cascade with the inlet Mach number $M_\infty = 0.832$ and angle of attack $\alpha = 0^\circ$. Result obtained by employing TNT scheme shows very good agreement with experiment. Then similar configuration with inlet Mach number $M_\infty = 0.863$ and angle of attack $\alpha = 0^\circ$ follows. In this case for the modelling of turbulence the Baldwin-Lomax model was used. The last configuration of this type is for inlet Mach number $M_\infty = 0.982$ with TNT model and again there is a good agreement. Second type domain DEIW calculations follow. First there is an inviscid flow in the same configuration as the reference result complemented by the viscous flow computation on a symmetrical domain. But the value of Reynolds number is 10 times bigger, which might be the cause why the separation is more distinct.

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Fig. 3. DCA cascade, inviscid flow, $M_\infty = 0.88$, $\alpha = 0.6^\circ$, Mach number isolines

Fig. 4. DCA cascade, viscous flow, $M_\infty = 0.88$, $\alpha = 0.6^\circ$, $Re = 10^7$, $k - \omega$ model, Mach number isolines

Fig. 5. DCA cascade, viscous flow, $M_\infty = 0.89$, $\alpha = 0.5^\circ$, $Re = 10^7$, $k - \omega$ model (TNT variant), Mach number isolines

Fig. 6. Experimental data, $M_\infty = 0.832$, $\alpha = 0^\circ$

Fig. 7. DCA cascade, inviscid flow, $M_\infty = 0.98$, $\alpha = 3.0^\circ$, Mach number isolines

Fig. 8. DCA cascade, viscous flow, $M_\infty = 0.98$, $\alpha = 3.0^\circ$, $Re = 10^7$, Baldwin-Lomax model, Mach number isolines

Fig. 9. Experimental data, $M_\infty = 0.863$, $\alpha = 0^\circ$

Fig. 10. DCA cascade, viscous flow, $M_\infty = 1.07$, $\alpha = 2.3^\circ$, $Re = 10^7$, $k - \omega$ model (TNT variant), Mach number isolines

Fig. 11. Experimental data, $M_\infty = 0.982$, $\alpha = 0^\circ$

Fig. 12. DEIW configuration, inviscid flow, $M_\infty = 0.775$, $\alpha = 0^\circ$, Mach number isolines

Fig. 13. symmetric DEIW configuration, viscous flow, $M_\infty = 0.775$, $\alpha = 0^\circ$, $Re = 10^6$, $k-\omega$ model (TNT variant), Mach number isolines

Fig. 14. DEIW configuration reference result, viscous flow, $M_\infty = 0.775$, $\alpha = 0^\circ$, $Re = 10^7$, simple mixing length model, Mach number isolines