EPJ Web of Conferences **45**, 01097 (2013)
DOI: 10.1051/epjconf/20134501097

PERTURBATION ANALYSIS OF "$k-\omega$" AND "$k-\epsilon$" TURBULENT MODELS. WALL FUNCTIONS

Karel Vostruha[1,a] and Jaroslav Pelant[1,b]

[1]Výzkumný a Zkušební Letecký Ústav, VZLÚ,
Beranových 130, 199 05 Praha - Letňany, Czech Republic

Abstract. Presented article shows rigorous method how derive non-stationary turbulent boundary layer equations by perturbation analysis. The same method is used for analysing behaviour of "k-omega" and "k-epsilon" turbulent models. The analysis is divided into two parts: near wall behaviour - boundary conditions, and behaviour in "log-layer" - wall functions. Both parts have important place in CFD. Boundary conditions are important part of CFD. "k-omega" and "k-epsilon" are related by one simple formula, but they yield to different solutions. Exact values for k, omega and epsilon on a wall are evaluated and all theoretical results are compared with numerical solutions. Special treatment is dedicated to "k-epsilon" model and Dirichlet boundary condition for "epsilon, instead of standard Neumann boundary condition. "Log-Layer" is well known from experiments and it is used for setting constants in turbulent models. Standard equations are derived by perturbation analysis. In presented article are these equations derived with 3 more terms, than in standard case. This yields to sharper approximation. These new equations are solved and solution is a bit different, than in standard case. Due to 3 extra terms is possible to get better approximation for k and new view into problematic..

1 Introduction

Two equations turbulent models are widely used in CFD - they give "good" results in "good" time. The most common are "$k-\omega$" and "$k-\varepsilon$". This article is about turbulent boundary layer equations and perturbation analysis of turbulent models - on a wall and in "log-layer". Main goal is deriving Mathematical formulas for "k", "ω" and "ε". on a wall - boundary conditions, and in "log-layer" - wall functions.

System of the Navier-Stokes equations in 2D with turbulent models

$$\frac{\partial \varrho}{\partial t} + \frac{\partial(\varrho u)}{\partial x} + \frac{\partial(\varrho v)}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + \frac{\partial(\varrho u^2 + p)}{\partial x} + \frac{\partial(\varrho uv)}{\partial y} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} - \frac{2}{3}\frac{\partial(\varrho k)}{\partial x}$$

$$\frac{\partial v}{\partial t} + \frac{\partial(uv)}{\partial x} + \frac{\partial(\varrho v^2 + p)}{\partial y} = \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} - \frac{2}{3}\frac{\partial(\varrho k)}{\partial y}$$

$$\frac{\partial e}{\partial t} + \frac{\partial(e+p)u}{\partial x} + \frac{\partial(e+p)v}{\partial y} = \frac{\partial}{\partial x}(u\tau_{xx} + v\tau_{xy}) + \frac{\partial}{\partial y}(u\tau_{yx} + v\tau_{yy}) + \frac{\partial}{\partial x}\left(\left(\frac{\mu}{Pr} + \frac{\mu_T}{Pr_T}\right)\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\left(\frac{\mu}{Pr} + \frac{\mu_T}{Pr_T}\right)\frac{\partial T}{\partial y}\right)$$

Where

$$\tau_{xx} = (\mu + \mu_T)\left(\frac{4}{3}\frac{\partial u}{\partial x} - \frac{2}{3}\frac{\partial v}{\partial y}\right)$$

$$\tau_{yy} = (\mu + \mu_T)\left(-\frac{2}{3}\frac{\partial u}{\partial x} + \frac{4}{3}\frac{\partial v}{\partial y}\right)$$

$$\tau_{xy} = \tau_{yx} = (\mu + \mu_T)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$e = \varrho\varepsilon + \frac{\varrho}{2}(u^2 + v^2) \quad \varepsilon = \frac{p}{(\kappa - 1)\varrho}$$

We need to close the Navier-Stokes equations by a turbulent model.

"$k-\omega$" turbulent model:

$$\frac{\partial(\varrho k)}{\partial t} + \frac{\partial(\varrho ku)}{\partial x} + \frac{\partial(\varrho kv)}{\partial y} = P_k + \frac{\partial}{\partial x}\left((\mu + \sigma^*\mu_T)\frac{\partial k}{\partial x}\right) + \frac{\partial}{\partial y}\left((\mu + \sigma^*\mu_T)\frac{\partial k}{\partial y}\right) - \beta^*\varrho k\omega$$

$$\frac{\partial(\varrho\omega)}{\partial t} + \frac{\partial(\varrho\omega u)}{\partial x} + \frac{\partial(\varrho\omega v)}{\partial y} = \alpha\frac{\omega}{k}P_k + \frac{\partial}{\partial x}\left((\mu + \sigma\mu_T)\frac{\partial\omega}{\partial x}\right) + \frac{\partial}{\partial y}\left((\mu + \sigma\mu_T)\frac{\partial\omega}{\partial y}\right) - \beta\varrho\omega^2$$

$$\mu_T = \frac{\varrho k}{\omega}$$

"$k-\epsilon$" turbulent model:

[a] e-mail: `vostruha@vzlu.cz`
[b] e-mail: `pelant@vzlu.cz`

$$\frac{\partial(\varrho k)}{\partial t} + \frac{\partial(\varrho k u)}{\partial x} + \frac{\partial(\varrho k v)}{\partial y} = P_k + \frac{\partial}{\partial x}\left((\mu + \sigma^* \mu_T)\frac{\partial k}{\partial x}\right) + \frac{\partial}{\partial y}\left((\mu + \sigma^* \mu_T)\frac{\partial k}{\partial y}\right) - \beta^* \varrho \epsilon$$

$$\frac{\partial(\varrho \epsilon)}{\partial t} + \frac{\partial(\varrho \epsilon u)}{\partial x} + \frac{\partial(\varrho \epsilon v)}{\partial y} = C_1 \frac{\epsilon}{k} P_k + \frac{\partial}{\partial x}\left((\mu + \sigma \mu_T)\frac{\partial \epsilon}{\partial x}\right) + \frac{\partial}{\partial y}\left((\mu + \sigma \mu_T)\frac{\partial \epsilon}{\partial y}\right) - C_2 \varrho \frac{\epsilon^2}{k}$$

$$\mu_T = C_\nu \frac{\varrho k^2}{\epsilon}$$

Where

$$P_k = \tau_{xx}\frac{\partial u}{\partial x} + \tau_{xy}\frac{\partial u}{\partial y} + \tau_{yx}\frac{\partial v}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y}$$

Model parameters $\alpha.\beta, \beta^*, \sigma, \sigma^*.C_1.C_2$ are constants. k is called kinetic turbulent energy, ϵ is called dissipation of turbulent energy. ω is called specific dissipation and it was introduced by Wilcox. Relation between ϵ and ω is: $\omega = \frac{\epsilon}{k}$.

Frequently used term is friction velocity defined as $u_\tau = \sqrt{\nu \left|\frac{\partial u}{\partial y}\right|_w}$.

2 Turbulent boundary layer equations

We assume a smooth flat plate and a vicious, incompressible turbulent flow. We use neglecting method which is based on magnitudes of Reynolds number and distance from the wall.

$$Re = \frac{LV}{\nu} \quad Re_T = \frac{\mu_T}{\mu}$$

Re_T is function of coordinates, but we assume for purposes of neglecting method, that $Re_T = \frac{C}{\sqrt{Re}}$. This approximation fits only in some hight up to the wall. Our goal is studying near wall behaver and we will see, that value of the constant C isn't significant. We define the neglecting parameter:

$$\varepsilon^2 = \frac{1}{Re}$$

We introduce new variables

$$\xi = \frac{x}{L} \quad N = \frac{y}{L\varepsilon} \quad \tilde{t} = t\frac{L}{V}$$

Asymptotic expansion of (u, v):

$$\bar{u} = \frac{u}{V} = U_1 + \varepsilon U_2 + \cdots$$
$$\bar{v} = \frac{v}{V} = \varepsilon V_1 + \varepsilon^2 V_2 + \cdots$$

We put new variables and asymptotic expansions into incompressible turbulent the Navier-Stokes equations in nondimensional form. We neglect all elements containing ε, then we get turbulent boundary layer equations:

$$\frac{\partial U_1}{\partial \xi} + \frac{\partial V_1}{\partial N} = 0$$
$$\frac{\partial U_1}{\partial \tilde{t}} + U_1\frac{\partial U_1}{\partial \xi} + V_1\frac{\partial U_1}{\partial N} = -\frac{1}{\varrho}\frac{\partial p_w}{\partial \xi} + \frac{\partial}{\partial N}\left((1 + C)\frac{\partial U_1}{\partial N}\right)$$
$$p = p_w - \frac{2}{3}\varrho k$$

We should easily return to dimensional equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\varrho}\frac{\partial p_w}{\partial x} + \frac{\partial}{\partial y}\left((\nu + \nu_T)\frac{\partial u}{\partial y}\right) \tag{2}$$
$$p = p_w - \frac{2}{3}\varrho k \tag{3}$$

Equations (1) - (3) were derived without respect to used turbulent model. We can use the same method to "$k - \omega$" and "$k - \epsilon$" turbulent models and get behaviour of these models in the turbulent boundary layer

$$\frac{\partial k}{\partial t} + u\frac{\partial k}{\partial x} + v\frac{\partial k}{\partial y} = (\nu + \nu_T)\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left((\nu + \sigma^* \nu_T)\frac{\partial k}{\partial y}\right) - \beta^* k\omega$$

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \alpha\frac{\omega}{k}(\nu + \nu_T)\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left((\nu + \sigma \nu_T)\frac{\partial \omega}{\partial y}\right) - \beta\omega^2 \tag{4}$$

$$\frac{\partial \epsilon}{\partial t} + u\frac{\partial \epsilon}{\partial x} + v\frac{\partial \epsilon}{\partial y} = C\frac{\epsilon}{k}(\nu + \nu_T)\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left((\nu + \sigma^* \nu_T)\frac{\partial \epsilon}{\partial y}\right) - C_2\frac{\epsilon^2}{k} \tag{5}$$

3 Behaviour on a wall and boundary conditions

We assume in all cases a stationary, incompressible flow, $\frac{\partial}{\partial x} \ll \frac{\partial}{\partial y}$, $v \ll u$. and $\nu_T \ll \nu$. These assumptions are correct and we will see another reason to use these assumptions later. We starts with equation for "ω".

Comparing with DNS tells, that $\omega = O(y^{-\alpha})$, velocity profile on wall is linear: $u(y) = y\frac{u_\tau^2}{\nu}$. By this way we should neglect $\frac{\partial u}{\partial y}$. Then we get equation for ω, which was derived by Wilcox:

$$0 = \nu\frac{\partial^2\omega}{\partial y} - \beta\omega^2$$

We use ansatz $\omega = Cy^{-\alpha}$ and it yields to:

$$\omega = \frac{6\nu}{\beta y^2} \tag{6}$$

Formula (6) is used for evaluating values of ω until the third line of cells.

We know, that turbulent kinetic energy decrease to 0 on a wall. We use previous assumptions for deriving equation for k and ansatz $k = Cy^{\alpha}$:

$$0 = \frac{u_\tau^4}{\nu} + \nu\frac{\partial^2 k}{\partial y^2} - \beta^* k\omega$$

$$k = \frac{u_\tau^4}{\nu^2}\frac{1}{6\frac{\beta^*}{\beta} - 2}y^2 \tag{7}$$

Equation for ϵ is from formula$\epsilon = k\omega$.

$$\epsilon = \frac{3u_\tau^4}{\nu(3\beta^* - \beta)}$$

Suitable boundary conditions are: $k = 0$ or $\frac{\partial k}{\partial n} = 0$. If we use $\epsilon = k\omega$, we see $\epsilon = O(1)$ and there is only one suitable boundary condition $\frac{\partial \epsilon}{\partial n} = 0$. Any Dirichlet boundary condition is unsuitable due to friction velocity. We will have to know values of friction velocity during all times. In case of boundary conditions is "k−ω" better then "k−ϵ" model. We can see from approximations for k and ω, that $\nu_T = O(y^4)$.

4 Wall functions

Wall functions are approximations of k, ω or ϵ in so-called "log - layer". Velocity profile is described by " the law of the wall" $u(y) = \frac{u_\tau}{\kappa}(\ln y + C)$. Where κ is von Karmar constant and constant C depends on roughness of the wall and friction velocity. In this layer $1 \ll Re_T$. Then equations are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{8}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\varrho}\frac{\partial p_w}{\partial x} + \frac{\partial}{\partial y}\left(\frac{k}{\omega}\frac{\partial u}{\partial y}\right) \tag{9}$$

$$u\frac{\partial k}{\partial x} + v\frac{\partial k}{\partial y} = \frac{k}{\omega}\left(\frac{\partial u}{\partial y}\right)^2 + \sigma^*\frac{\partial}{\partial y}\left(\frac{k}{\omega}\frac{\partial k}{\partial y}\right) - \beta^* k\omega \tag{10}$$

$$u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \alpha\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left(\frac{k}{\omega}\frac{\partial \omega}{\partial y}\right) - \beta\omega^2 \tag{11}$$

We assume an infinite plate parallel to the flow : $p_w =$ const., $\frac{\partial}{\partial x} \ll \frac{\partial}{\partial y}$, "log-layer": $u = \frac{u_\tau}{\kappa}(\ln y + C)$, then the system of equations is over-setted. It means, that we have 4 equations and 3 unknown functions. We will see, the system has point solution only on one line.

Equations (8) - (10) under the assumed "log-layer" approximation have solution:

$$u = \frac{u_\tau}{\kappa}(\ln y + C_1) \quad v = C_2 \tag{12}$$

$$k = \frac{u_\tau}{\kappa}\frac{C_2}{\sqrt{\beta^*}}\ln(C_3 y) \tag{13}$$

$$\omega = \frac{u_\tau}{\kappa\sqrt{\beta^*}y} \tag{14}$$

$$\sigma^* = 1 \tag{15}$$

Equation (11) yields to point solution on the critical line:

$$\ln(C_3 y_{crit}) = 1 - \frac{1}{\sigma} + \frac{u_\tau}{\kappa\sigma}\frac{\sqrt{\beta^*}}{C_2}\left(\frac{\beta}{\beta^*} - \alpha\right) \tag{16}$$

k on the critical line is:

$$k_{crit} = \left(1 - \frac{1}{\sigma}\right)\frac{u_\tau C_2}{\kappa\sqrt{\beta^*}} + \frac{u_\tau^2}{\kappa^2\sigma}\left(\frac{\beta}{\beta^*} - \alpha\right) \tag{17}$$

We use well known formula for turbulent viscosity in "log-layer"

$$\nu_T = u_\tau\kappa y$$

Then we can evaluate:

$$C_2 = \frac{u_\tau\kappa}{\ln(C_3 y_{crit})}$$
$$= u_\tau\kappa\frac{\sigma}{\sigma - 1}\left(1 - \frac{\sqrt{\beta*}}{\sigma\kappa^2}\left(\frac{\beta}{\beta^*} - \alpha\right)\right)$$
$$\ln(C_3 y_{crit}) = \frac{\sigma - 1}{\sigma}\left(1 - \frac{\sqrt{\beta*}}{\sigma\kappa^2}\left(\frac{\beta}{\beta^*} - \alpha\right)\right)^{-1}$$

In light of the last formulated formulas and formula for k_{crit} we get

$$k_{crit} = \frac{u_\tau^2}{\sqrt{\beta^*}}$$

.

The right one wall functions on the critical line ($y = y_{crit}$) are:

$$u = \frac{u_\tau}{\kappa}(\ln y + C_1)$$
$$v = u_\tau\kappa\frac{\sigma}{\sigma - 1}\left(1 - \frac{\sqrt{\beta*}}{\sigma\kappa^2}\left(\frac{\beta}{\beta^*} - \alpha\right)\right)$$
$$k = \frac{u_\tau^2}{\sqrt{\beta^*}}$$
$$\omega = \frac{u_\tau}{\kappa\sqrt{\beta^*}y}$$

Important is, we don't need to know value of constant C_3 for prescribing boundary conditions on the critical line. Standard wall functions for $k - \omega$ model. See Wilcox

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$
$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\varrho}\frac{\partial p}{\partial x}$$

$$u\frac{\partial k}{\partial x}+v\frac{\partial k}{\partial y}=\frac{k}{\omega}\left(\frac{\partial u}{\partial y}\right)^2+\sigma^*\frac{\partial}{\partial y}\left(\frac{k}{\omega}\frac{\partial k}{\partial y}\right)-\beta^* k\omega$$
$$u\frac{\partial \omega}{\partial x}+v\frac{\partial \omega}{\partial y}=\alpha\left(\frac{\partial u}{\partial y}\right)^2+\frac{\partial}{\partial y}\left(\frac{k}{\omega}\frac{\partial \omega}{\partial y}\right)-\beta\omega^2$$

With the same "log-layer" assumptions.Solution (standard wall functions):

$$u=\frac{u_\tau}{\kappa}(\ln y+C_1)\qquad v=0$$
$$k=\frac{u_\tau^2}{\kappa^2\sigma}\left(\frac{\beta}{\beta^*}-\alpha\right)$$
$$\omega=\frac{u_\tau}{\kappa\sqrt{\beta^*}y}$$

If we use $\lim_{C_2\to 0+}$ on solutions (12) - (14) and concept of critical line (16), we get "standard wall functions". But $y_{crit}\to+\infty$ The limiting process shows, that presented theory is more complex and explain why "standard wall functions" yields to poor results. Another shape of this theory is impossibility to find extension under the same set up.

We have to figure out by experiment value of constant C_1, we don't need to know value of constant C_3, but we assume it's dependence on u_τ. Value of y_{crit} has to evaluated from empirical formula. Following chapter shows limits Of "simple" methods in boundary layer theory.

5 Impossibility theorem

We saw in previous section approximation of vertical part of velocity by a constant. In this section we formulate and prove theorem, that we can't use any method of self-similar solutions, which are well known from laminar boundary layer theory

Theorem 1 *Self-similar solutions can occur only in laminar boundary layer.*

Proof We create proof by contradiction. Let exist a self-similar solution of an incompressible turbulent boundary layer equations described by this system of equations with a turbulent model.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$
$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\varrho}\frac{\partial p_w}{\partial x}+\frac{\partial}{\partial y}\left((\nu+\nu_T)\frac{\partial u}{\partial y}\right)\tag{18}$$

With a turbulent model, we don't need equations from the turbulent model. We have these boundary conditions:

$$y=0:\quad u=v=0,\quad \nu_T=0\quad p_w(x)=f(x)$$
$$x=0:\quad u=U(0,y),\quad v=0\quad \nu_T=\nu_T(0,y)$$
$$y\to\infty:\quad u=U(x),\quad v=0,\quad \nu_T=\nu_T(\infty)$$

We assume a self-similar solution, we need to find proper boundary conditions.. We can find parts of velocity in this form:

$$u=U(x)f\left(\frac{y}{l(x)}\right)\tag{19}$$
$$v=U(x)l'(x)\frac{y}{l(x)}f'-\frac{d}{dx}(U(x)l(x))f\tag{20}$$

We put formulas (19) and (20) into the equation (18) and we seek for self-similar solutions. It yields to this condition:

$$\frac{l(x)}{\nu+\nu_T}\frac{\partial(\nu+\nu_T)}{\partial y}=C$$

Solution is:

$$\nu+\nu_T=\nu\exp\left(\frac{Cy}{l(x)}\right)$$

Constant $C\leq 0$, or we get infinite viscosity. It yields to this system of inequalities:

$$0\leq\nu_T=\nu(\exp\left(\frac{Cy}{l(x)}\right)-1)\leq 0$$

We get contradiction - $\nu_T=0$ everywhere in turbulent boundary layer. We can see sharp border between methods of laminar and turbulent boundary layer.

6 Comparison and figures

We compare theoretical results with numerical solutions of compressible flow in this section. Main goal is check out how incompressible flow is good approximation for compressible flow. Numerical solution was done with $k-\omega$ model with these constants: $\alpha=\frac{5}{9}$, $\beta=\frac{3}{40}$, $\beta^*=\frac{9}{100}$, $\sigma=\sigma^*=\frac{1}{2}$. Computation mesh: 245 x 64. Outer velocity $30m/s$.The first level of knots of the mesh 2.10^{-6}, quotient if increasing high $q=1.2$. Coordinate axes are in logaritmic scaling fpor better view

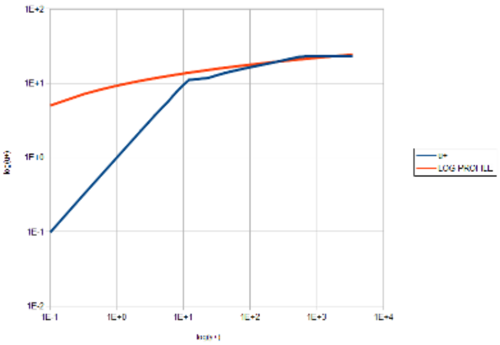

Fig. 1. velocity profile

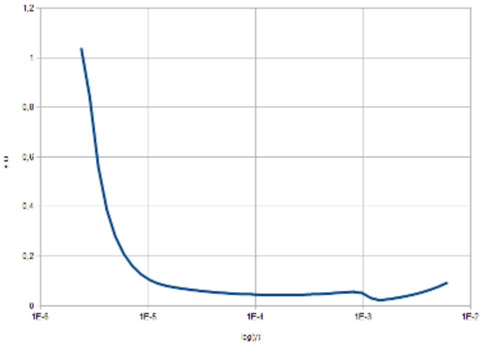

Fig. 2. $y\omega$

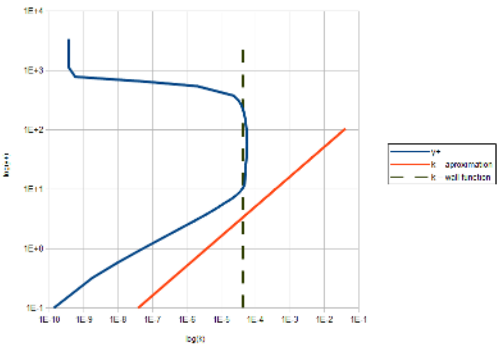

Fig. 3. turbulent kinetic energy

We can see velocity profile of $u^+ = \frac{u}{u_\tau}$ (blue curve) compared with "log- profile" (red curve)in figure 1.$\kappa = 0.54$, the value is much higher, than in literature.We can see one point intersection.

We can see evolution of $y\omega$ in figure 2. We can see constant part. We compared the value of constant part with evaluated value for ω:

$$\left|\omega - \frac{u_\tau}{\kappa\sqrt{\beta^*}y}\right| \approx 0.011$$

We can see evolution of the turbulent kinetic energy (blue curve) and comparison with standard wall function (bared line) for k in figure 3. The comparison is good due to $\kappa =$ 0.54 - constant line. Wall behaviour (7) (orange curve) fails.

7 Conclusion

We saw in section 4 the most complex wall functions for $k-\omega$ model based on fully developed "log-layer" and their link to "standard wall functions". These new wall functions explain problems in using standard ones and shows, that logarithmic approximation isn't the best option for $k-\omega$ model. Futher research will be focused on extending presented theory on whole log-layer by using special perturbation functions. Setting boundary conditions on the critical line is complicated for numerical solutions and preparing commutation grids. The greatest disadvantage of all wall functions based on log- layer is strong assumption in stationary flow with fully developed log-layer. Near to point of separation or in non-stationary flows are standard boundary conditions better choice.

Compartment showed poor results and hight κ. We explain it as comparison of compressible and incompressible flow, not enough dense mesh, $\sigma^* = 0.5$ instead $\sigma^* = 1$ as was evaluated. More tests are required and work extension of new wall functions and change them by perturbation functions and constants into useful form in compressible flow. Then wall function become power tool in modelling of atmosphere or other cases, where is problem to set boundary conditions on walls (surfaces)

References

1. Wikcix C. David, *Turbulence Modeling for CFD - second edition* (DCW industries Inc., La Canada, California 1998) page numbers