EPJ Web of Conferences **58**, 01013 (2013)
DOI: 10.1051/epjconf/20135801013

Discrete mechanics, "time machines" and hybrid systems

Hans-Thomas Elze[1,a]

[1] *Dipartimento di Fisica "Enrico Fermi", Università di Pisa, Largo Pontecorvo 3, I-56127 Pisa, Italia*

Abstract. Modifying the *discrete mechanics* proposed by T.D. Lee, we construct a class of discrete classical Hamiltonian systems, in which time is one of the dynamical variables. This includes a toy model of "time machines" which can travel forward and backward in time and which differ from models based on closed timelike curves (CTCs). In the continuum limit, we explore the interaction between such time reversing machines and quantum mechanical objects, employing a recent description of quantum-classical hybrids.

1 Introduction

Time travel and time machines have been the stuff of science fiction for a while and possibly excited human minds much earlier than that. – However, they have become a topic of active scientific enquiry since the realization that certain cosmological solutions of Einstein's equations of general relativity allow for *closed timelike curves* (CTCs). Here an object can travel in an unusual geometry of spacetime, such that it encounters the past and, in particular, its own past. It is obvious that - with the link between quantum mechanics and general relativity still little understood - this provides an arena for producing paradoxes (*e.g.*, grandfather paradox, unproved theorem paradox) and testing new ideas how to resolve them, besides availing surprising computational resources. – Discussion of background, an overview of existing literature, and the state of the art of constructing *quantum mechanical time machines* can be found in Refs. [1, 2].

Our aim here is threefold. – In Sect. 2, we recall T.D. Lee's proposal of *time as a fundamentally discrete dynamical variable* [3, 4]. Limiting the number of events or measurements in a given spacetime region, this can have surprising consequences in the continuum limit. We modify his action principle in such a way that a Hamiltonian formulation for such discrete systems becomes available. Furthermore, we show that these systems allow for a particular kind of time machines, namely *time reversing machines*. – In Sect. 3, we review a recent attempt to construct a theory that describes *quantum-classical hybrids*, consisting of quantum mechanical and classical objects that interact directly with each other [5–7]. Hybrids might exist as a fundamentally different species of composite objects "out there", with consequences for the range of applicability of quantum mechanics, or they may serve as approximate description for certain complex quantum systems.

We employ the concept of quantum-classical hybrids, in order to explore a hypothetical *direct coupling of classical time machines to quantum objects*. In Sect. 4, we introduce a specific model, obtain its equations of motion, and discuss consequences for the study of time machines, followed by conclusions in Sect. 5.

[a] e-mail: elze@df.unipi.it

2 Discrete Hamiltonian mechanics

Discrete dynamical systems arise in many contexts in physics or mathematics, for example, in discrete approximations or maps facilitating numerical studies of complex systems, as regularized versions of quantum field theories on spacetime lattices, or describing intrinsically discrete processes. Here, the usual preponderance of differential equations over finite difference equations is given up for reasons which may be more or less fundamental.

In a series of articles T.D. Lee and collaborators have proposed to incorporate discreteness as a fundamental aspect of dynamics, see Refs. [3, 4] and further references therein, and have elaborated various classical and quantum models in this vein, which share desirable symmetries with the corresponding continuum theories while presenting finite degrees of freedom.

For our purposes, it will be sufficient to consider classical discrete mechanics which derives from the basic assumption that *time is a discrete dynamical variable*. This naturally invokes a *fundamental length or time* (in natural units), l. Which can be rephrased as the assumption that in a fixed $(d+1)$-dimensional spacetime volume Ω maximally $N = \Omega/l^{d+1}$ measurements can be performed or this number of events take place.

In Refs. [3, 4] a variational principle was presented, based on a Lagrangian formulation of the postulated action. Various forms and (dis)advantages of such an approach have subsequently been discussed, *e.g.*, in Refs. [8–10]. Presently, we present a Hamiltonian formulation which differs from all previous ones in that it leads to particularly transparent and symmetric equations of motion. This allows us to introduce a suitable Poisson bracket and a phase space description of the dynamics. The latter is an essential ingredient when constructing quantum-classical hybrids, as we shall see in the following Sect. 3.

We describe the state n of a discrete mechanical object by its positions in spacetime in terms of the real *dynamical variables* x_n, τ_n and corresponding *conjugated momenta* $p_n, \mathcal{P}_n$, with $n = 0, 1, 2, \dots$.[1] We postulate that its dynamics is governed by the stationarity of this *action*:

$$A := \sum_{n>0} \Big[(p_n + p_{n-1})\Delta x_n + (\mathcal{P}_n + \mathcal{P}_{n-1})\Delta\tau_n - \mathcal{H}_n \Big] \ , \tag{1}$$

under independent variations of all variables and momenta; the finite differences are defined by:

$$\Delta x_n := x_n - x_{n-1} \ , \quad \Delta\tau_n := \tau_n - \tau_{n-1} \ , \tag{2}$$

and the Hamiltonian function by $\mathcal{H} := \sum_n \mathcal{H}_n$, with:

$$\mathcal{H}_n := \Delta\tau_n \Big[\frac{p_n^2 + p_{n-1}^2}{2} + V(x_n) + V(x_{n-1}) \Big] + \mathcal{K}_n \ , \tag{3}$$

where V is a sufficiently smooth potential and $\mathcal{K}_n$ will be specified in due course.

The variations of the action amount to differentiations here and lead to the equations of motion:

$$\dot{x}_n = \dot{\tau}_n p_n + \partial_{p_n} \sum_{n'} \mathcal{K}_{n'} \ = \ \partial_{p_n}\mathcal{H} \ \equiv \ \{x_n, \mathcal{H}\} \ , \tag{4}$$

$$[1ex]\dot{p}_n = -\dot{\tau}_n \partial_{x_n} V(x_n) - \partial_{x_n} \sum_{n'} \mathcal{K}_{n'} \ = \ -\partial_{x_n}\mathcal{H} \ \equiv \ \{p_n, \mathcal{H}\} \ , \tag{5}$$

$$[1ex]\dot{\tau}_n = \partial_{\mathcal{P}_n} \sum_{n'} \mathcal{K}_{n'} \ = \ \partial_{\mathcal{P}_n}\mathcal{H} \ \equiv \ \{\tau_n, \mathcal{H}\} \ , \tag{6}$$

$$[1ex]\dot{\mathcal{P}}_n = E_{n+1} - E_n - \partial_{\tau_n} \sum_{n'} \mathcal{K}_{n'} \ \equiv \ \{\mathcal{P}_n, \mathcal{H}\} \ , \tag{7}$$

[1] x_n and p_n might be vectors, depending on the dimensionality of space, while τ_n and $\mathcal{P}_n$ are assumed onedimensional.

introducing the discrete "time derivative", $\dot{O}_n := O_{n+1} - O_{n-1}$, on the left-hand and Poisson brackets (cf. below) on the right-hand sides, respectively; furthermore, $E_n := \frac{1}{2}(p_n^2 + p_{n-1}^2) + V(x_n) + V(x_{n-1})$.

Several remarks are in order here. – Assuming that $\mathcal{K}_n$ depends only on the state n, it can be shown that the set of Eqs. (4)–(7) is *time reversal invariant*; the state $n+1$ can be calculated from knowledge of the earlier states n and $n-1$ and the state $n-1$ from the later ones $n+1$ and n.

Furthermore, stationarity of the action under independent variations of τ_n and x_n for every state n implies *invariance under translations in time and space*, respectively, and the conservation of energy and momentum (modulo the effect of the external force deriving from V). This holds under the further assumption that $\mathcal{K}_n$ does *not* enter through its derivatives in Eqs. (4)–(5).

Explicit solutions of the equations of motion can be easily found in the case that $\mathcal{K}_n := 0$ and the potential V is constant or a linear function, *i.e.*, for zero or constant external force . This recovers the behaviour discussed in Refs. [3, 8]. However, in the present formulation, we also have the possibility to study more exotic models, in which the dynamics of the time variable τ_n itself plays an important role when $\mathcal{K}_n \neq 0$, cf. Sect. 2.2.

Considering the dynamical variables and canonically conjugated momenta as canonical coordinates for the phase space spanned by $\{x_n, \tau_n; p_n, \mathcal{P}_n\}$, we introduce the *Poisson bracket* of any two regular functions f and g on this space:

$$\{f, g\} := \sum_n \Big(\frac{\partial f}{\partial_{x_n}} \frac{\partial g}{\partial_{p_n}} - \frac{\partial f}{\partial_{p_n}} \frac{\partial g}{\partial_{x_n}} + \frac{\partial f}{\partial_{\tau_n}} \frac{\partial g}{\partial_{\mathcal{P}_n}} - \frac{\partial f}{\partial_{\mathcal{P}_n}} \frac{\partial g}{\partial_{\tau_n}} \Big) \ , \tag{8}$$

which has been indicated already on the right-hand sides of Eqs. (4)–(5),

This allows a convenient description also of ensembles of discrete mechanical objects, which individually follow the above equations of motion. Let us collectively denote variables and momenta as Q_n and P_n, respectively, such that $\{f, g\} = \sum_n \Big(\frac{\partial f}{\partial_{Q_n}} \frac{\partial g}{\partial_{P_n}} - \frac{\partial f}{\partial_{P_n}} \frac{\partial g}{\partial_{Q_n}} \Big)$. Then, we postulate, in analogy to continuum mechanics (see Subsect. 2.1), a *continuity equation* to determine the flow of the probability density $\rho_n \equiv \rho_n(Q_n; P_n)$ of the ensemble in phase space:

$$0 = \dot{\rho}_n + \partial_{Q_n}(\rho_n \dot{Q}_n) + \partial_{P_n}(\rho_n \dot{P}_n) \ , \tag{9}$$

with $\dot{O}_n := O_{n+1} - O_{n-1}$, as before. Employing the equations of motion (4)–(7), this continuity equation can be rewritten as the discrete mechanics analogue of the *Liouville equation*:

$$\dot{\rho}_n = \{\mathcal{H}, \rho_n\} \ . \tag{10}$$

In the following, we study the continuum limit of the equations of motion, in which time remains one of the dynamical variables.

2.1 The continuum limit

In order to discuss the continuum limit, we let the fundamental time (or length) constant become arbitrarily small, $l \to 0$, such that the density of events or measurements becomes correspondingly large, $N \to \infty$. Furthermore, we introduce the *external time*, $t := nl$, with $n = 0, 1, 2, \ldots$, and define $x(t) := x_n$, $\tau(t) := \tau_n$, $p(t) := p_n$, $\mathcal{P}(t) := \mathcal{P}_n$, *i.e.*, in terms of the discrete dynamical variables and conjugated momenta. Thus, for example, $\tau_{n+1} - \tau_n = \tau(t + l) - \tau(t) = \dot{\tau}(t)l + \mathrm{O}(l^2)$, where $\dot{\tau} := \mathrm{d}\tau/\mathrm{d}t$, *etc.* – In this way, we obtain the equations of motion in the continuum limit:

$$\dot{x} = \dot{\tau} p + \frac{1}{2l}\partial_{p_n} \sum_{n'} \mathcal{K}_{n'} \quad , \tag{11}$$

$$\dot{p} = -\dot{\tau}\nabla V(x) - \frac{1}{2l}\partial_{x_n} \sum_{n'} \mathcal{K}_{n'} \quad , \tag{12}$$

$$\dot{\tau} = \frac{1}{2l}\partial_{\mathcal{P}_n} \sum_{n'} \mathcal{K}_{n'} \quad , \tag{13}$$

$$\dot{\mathcal{P}} = \frac{\mathrm{d}}{\mathrm{d}t}\Big[\frac{p^2}{2} + V(x)\Big] - \frac{1}{2l}\partial_{\tau_n} \sum_{n'} \mathcal{K}_{n'} \quad , \tag{14}$$

where terms containing $\sum \mathcal{K}_{n'}$ will be defined and evaluated shortly.

It suffices here to assume that all $\mathcal{K}_{n'}$ are independent of $\{x_n, p_n\}$. This simplifies Eqs. (11)–(12):

$$\dot{x} = \dot{\tau} p \quad , \quad \dot{p} = -\dot{\tau}\nabla V(x) \quad , \tag{15}$$

implies $\mathrm{d}/\mathrm{d}t[p^2/2 + V(x)] = 0$, and, consequently, simplifies also Eq. (14). In this case, $\dot{\tau}$ plays the role of a given *"lapse" function* for the subsystem described by x and p, which can be separately determined (cf. Sect. 2.2). I.e., if Eqs. (13)–(14) are integrated explicitly, the remaining Eqs. (15) follow from the time dependent effective Hamiltonian:

$$\mathcal{H}_c(x, p; t) := \dot{\tau}(t)[\frac{p^2}{2} + V(x)] \quad , \tag{16}$$

with $\dot{\tau}$ as a time dependent parameter.

The existence of a simple continuum Hamiltonian, such as $\mathcal{H}_c$, is not obvious, in general, since $\Delta\tau_n$ on the right-hand side of Eq. (3) becomes proportional to $\dot{\tau}$, if one performs the continuum limit directly on the discrete dynamics Hamiltonian; the presence of this factor can spoil the Hamiltonian picture of the resulting dynamics.

2.2 Time machines

Here we illustrate the continuum limit of the discrete mechanics that we obtained. We choose $\mathcal{K}_n := l[\mathcal{P}_n^2 + \mathcal{V}(\tau_n)]$. Then, the continuum limit applied to Eqs. (13)–(14) gives simply:

$$\dot{\tau} = \mathcal{P} \quad , \quad \dot{\mathcal{P}} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{V}(\tau) \quad . \tag{17}$$

with $\dot{\tau} := \mathrm{d}\tau/\mathrm{d}t$, *etc.*

We observe that for suitable potentials $\mathcal{V}(\tau)$ and initial conditions the *internal time* τ will perform a *bounded periodic motion* as function of the *external time* t. For example, for an oscillator potential, $\mathcal{V}(\tau) := \omega^2\tau^2$, we obtain solutions $\tau(t) = \bar{\tau}\sin(\omega t)$, with amplitude $\bar{\tau}$ and phase determined by the initial conditions, such that $\dot{\tau}(t) = \dot{\tau}(-t)$ is time reversal invariant.

Furthermore, the Eqs. (15) can be rewritten as a single second order equation:

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}x = -\nabla V(x) \quad , \quad \text{with } \tau \equiv \tau(t) \quad , \tag{18}$$

i.e., as an ordinary equation of motion with respect to the internal time, which is considered as a function of the external time, to be obtained from Eqs. (17).

This situation describes a toy model of *time machines*: the x, p-subsystem moves forward in time on a particular trajectory in phase space, as long as $\tau(t)$ increases; when, due to its periodicity, this function decreases, this trajectory is traced identically backwards! Thus, the behaviour in the external time t is cyclic, alternating between forward and backward evolution.

We remark that this dynamical implementation of "time travel" differs from a frequently considered one, which is based on modifying the background spacetime structure. In particular, Politzer's spacetime, which allows closed timelike curves (CTCs), is obtained by identifying a certain spatial region at one time with the same region at a later time [11]; thus, an object may *transit* instantaneously from a final state to the corresponding (identical) initial state of its evolution. In our model, it *evolves* identically backwards from a final state to its initial state; it is conceivable that this can be realized in physical analogue models.

In Sect. 4, we explore the coupling of such a classical time machine to a quantum object in a particular framework describing quantum-classical hybrids.

3 Quantum-classical hybrids

The direct coupling of quantum mechanical (QM) and classical (CL) degrees of freedom – *"hybrid dynamics"* – departs from quantum mechanics. We summarize here briefly the theory presented in Refs. [5–7], where also additional references and discussion of related works can be found.

Hybrid dynamics has been researched extensively for various reasons. – For example, the Copenhagen interpretation of quantum mechanics entails the measurement problem which, together with the fact that quantum mechanics needs interpretation, in order to be operationally well defined, may indicate that it needs amendments. Such as a theory of the *dynamical* coexistence of QM and CL objects. This should have impact on the measurement problem [12] as well as on the description of the interaction between quantum matter and (possibly) classical spacetime [13].

Furthermore, it is of great practical interest to better understand QM-CL hybrids appearing in QM approximation schemes addressing many-body systems or interacting fields, which are naturally separable into QM and CL subsystems; for example, representing fast and slow degrees of freedom, mean fields and fluctuations, *etc.*

Concerning the hypothetical emergence of quantum mechanics from some coarse-grained deterministic dynamics (see Refs. [14–16] with numerous references to related work), the quantum-classical backreaction problem might appear in new form, namely regarding the interplay of fluctuations among underlying deterministic and emergent QM degrees of freedom. Which can be rephrased succinctly as: *"Can quantum mechanics be seeded?"*

Thus, there is ample motivation for the numerous attempts to formulate a satisfactory hybrid dynamics. Generally, they are deficient in one or another respect. Which has led to various no-go theorems, in view of the lengthy list of desirable properties or consistency requirements that "*the*" hybrid theory should fulfil, see, for example, Refs. [17, 18]:

- Conservation of energy.
- Conservation and positivity of probability.
- Separability of QM and CL subsystems in the absence of their interaction, recovering the correct QM and CL equations of motion, respectively.
- Consistent definitions of states and observables; existence of a Lie bracket structure on the algebra of observables that suitably generalizes Poisson and commutator brackets.
- Existence of canonical transformations generated by the observables; invariance of the classical sector under canonical transformations performed on the quantum sector only and *vice versa*.

- Existence of generalized Ehrenfest relations (*i.e.* the correspondence limit) which, for bilinearly coupled CL and QM oscillators, are to assume the form of the CL equations of motion ("Peres-Terno benchmark" test [19]).
- 'Free Will' [20].
- Locality.
- No-signalling.
- QM / CL symmetries and ensuing separability carry over to hybrids.

These issues have also been discussed for the hybrid ensemble theory of Hall and Reginatto, which does conform with the first six points listed but is in conflict with the last two [21, 22].

We have proposed an alternative theory of hybrid dynamics based on notions of phase space [5]. This extends work by Heslot, demonstrating that quantum mechanics can entirely be rephrased in the language and formalism of classical analytical mechanics [23]. Introducing unified notions of states on phase space, observables, canonical transformations, and a generalized quantum-classical Poisson bracket, this has led to an intrinsically linear hybrid theory, which allows to fulfil *all* of the above consistency requirements.

Recently Burić and collaborators have shown that the dynamical aspects of our proposal can indeed be derived for an all-quantum mechanical composite system by imposing constraints on fluctuations in one subsystem, followed by suitable coarse-graining [24, 25].

Besides constructing the QM-CL hybrid formalism and showing how it conforms with the above consistency requirements, we earlier discussed the possibility to have classical-environment induced decoherence, quantum-classical backreaction, a deviation from the Hall-Reginatto proposal in presence of translation symmetry, and closure of the algebra of hybrid observables [5, 7]. Questions of locality, symmetry vs. separability, incorporation of superposition, separable, and entangled QM states, and 'Free Will' were considered in Ref. [6].

3.1 Quantum mechanics – rewritten in classical terms

We recall that evolution of a *classical* object can be described in relation to its $2n$-dimensional phase space, its *state space*. A real-valued regular function on this space defines an *observable*, *i.e.*, a differentiable function on this smooth manifold.

There always exist (local) systems of *canonical coordinates*, commonly denoted by $(x_k, p_k),\ k = 1, \ldots, n$, such that the *Poisson bracket* of any pair of observables f, g assumes the standard form:

$$\{f, g\} \ = \ \sum_k \Big(\frac{\partial f}{\partial_{x_k}} \frac{\partial g}{\partial_{p_k}} - \frac{\partial f}{\partial_{p_k}} \frac{\partial g}{\partial_{x_k}} \Big) \ . \tag{19}$$

This is consistent with $\{x_k, p_l\} = \delta_{kl}$, $\{x_k, x_l\} = \{p_k, p_l\} = 0,\ k, l = 1, \ldots, n$, and has the properties defining a Lie bracket operation, mapping a pair of observables to an observable.

General transformations $\mathcal{G}$ of the state space are restricted by compatibility with the Poisson bracket structure to so-called *canonical transformations*, which do not change physical properties of an object. They form a Lie group and it is sufficient to consider infinitesimal transformations. An *infinitesimal transformation* $\mathcal{G}$ is *canonical*, if and only if for any observable f the map $f \rightarrow \mathcal{G}(f)$ is given by $f \rightarrow f' = f + \{f, g\}\delta\alpha$, with some observable g, the so-called *generator* of $\mathcal{G}$, and $\delta\alpha$ an infinitesimal real parameter. – Thus, for example, the canonical coordinates transform as follows:

$$x_k \ \rightarrow \ x'_k = x_k + \frac{\partial g}{\partial p_k}\delta\alpha \quad , \quad p_k \ \rightarrow \ p'_k = p_k - \frac{\partial g}{\partial x_k}\delta\alpha \ . \tag{20}$$

This illustrates the fundamental relation between observables and generators of infinitesimal canonical transformations in classical Hamiltonian mechanics.

Following Heslot's work, we learn that the previous analysis can be generalized and applied to quantum mechanics; this concerns the dynamical aspects as well as the notions of states, canonical transformations, and observables [23].

The *Schrödinger equation* and its adjoint can be derived as Hamiltonian equations from an action principle [5]. We must add the *normalization condition*, $C := \langle\Psi(t)|\Psi(t)\rangle \stackrel{!}{=} 1$, for all state vectors $|\Psi\rangle$, which is essential for the probability interpretation of amplitudes; state vectors that differ by an unphysical constant phase are to be identified. Thus, the *QM state space* is formed by the rays of the underlying Hilbert space.

3.1.1 Oscillator representation

A unitary transformation describes QM evolution, $|\Psi(t)\rangle = \hat{U}(t-t_0)|\Psi(t_0)\rangle$, with $U(t-t_0) = \exp[-i\hat{H}(t-t_0)/\hbar]$, solving the Schrödinger equation. Thus, a stationary state, characterized by $\hat{H}|\phi_i\rangle = E_i|\phi_i\rangle$, with real energy eigenvalue E_i, performs a harmonic motion, *i.e.*, $|\psi_i(t)\rangle = \exp[-iE_i(t-t_0)/\hbar]|\psi_i(t_0)\rangle \equiv \exp[-iE_i(t-t_0)/\hbar]|\phi_i\rangle$. We assume a denumerable set of such states. Following these observations, it is quite natural to introduce the following *oscillator representation*.

We expand state vectors with respect to a complete orthonormal basis, $\{|\Phi_i\rangle\}$:

$$|\Psi\rangle = \sum_i |\Phi_i\rangle(X_i + iP_i)/\sqrt{2\hbar} \ , \tag{21}$$

where the time dependent coefficients are separated into real and imaginary parts, X_i, P_i [23]. This expansion allows to evaluate what we *define* as *Hamiltonian function*, *i.e.*, $\mathcal{H} := \langle\Psi|\hat{H}|\Psi\rangle$:

$$\mathcal{H} = \frac{1}{2\hbar}\sum_{i,j}\langle\Phi_i|\hat{H}|\Phi_j\rangle(X_i - iP_i)(X_j + iP_j) =: \mathcal{H}(X_i, P_i) \ . \tag{22}$$

Choosing the set of energy eigenstates, $\{|\phi_i\rangle\}$, as basis of the expansion, we obtain:

$$\mathcal{H}(X_i, P_i) = \sum_i \frac{E_i}{2\hbar}(P_i^2 + X_i^2) \ , \tag{23}$$

hence the name *oscillator representation*. – Evaluating $|\dot{\Psi}\rangle = \sum_i |\Phi_i\rangle(\dot{X}_i + i\dot{P}_i)/\sqrt{2\hbar}$ according to Hamilton's equations with $\mathcal{H}$ of Eq. (22) or (23), gives back the Schrödinger equation. – Furthermore, the *normalization condition* becomes:

$$C(X_i, P_i) = \frac{1}{2\hbar}\sum_i(X_i^2 + P_i^2) \stackrel{!}{=} 1 \ . \tag{24}$$

Thus, the vector with components given by (X_i, P_i), $i = 1, \dots, N$, is confined to the surface of a $2N$-dimensional sphere with radius $\sqrt{2\hbar}$, which presents a major difference to CL Hamiltonian mechanics.

The (X_i, P_i) may be considered as *canonical coordinates* for the state space of a QM object. Correspondingly, we introduce a *Poisson bracket*, cf. Eq.(19), for any two *observables* on the *spherically compactified state space*, *i.e.* real-valued regular functions F, G of the coordinates (X_i, P_i):

$$\{F, G\} = \sum_i\Big(\frac{\partial F}{\partial_{X_i}}\frac{\partial G}{\partial_{P_i}} - \frac{\partial F}{\partial_{P_i}}\frac{\partial G}{\partial_{X_i}}\Big) \ . \tag{25}$$

As usual, time evolution of an observable O is generated by the Hamiltonian: $\mathrm{d}O/\mathrm{d}t = \partial_t O + \{O, \mathcal{H}\}$. In particular, we find that the constraint of Eq. (24) is conserved: $\mathrm{d}C/\mathrm{d}t = \{C, \mathcal{H}\} = 0$.

3.1.2 Canonical transformations and quantum observables

In the following, we recall briefly the compatibility of the notion of observable introduced in passing above – as in classical mechanics – with the usual QM one. This can be demonstrated rigourously by the implementation of canonical transformations and analysis of the role of observables as their generators. For details, see Refs. [5–7, 23].

The Hamiltonian function has been defined as observable in Eq. (22), which relates it directly to the corresponding QM observable, namely the expectation of the self-adjoint Hamilton operator. This is indicative of the general structure with the following most important features:

• A) *Compatibility of unitary transformations and Poisson structure.* – Classical canonical transformations are automorphisms of the state space which are compatible with the Poisson bracket. Automorphisms of the QM Hilbert space are implemented by unitary transformations. This implies a transformation of the canonical coordinates (X_i, P_i) here. From this, one derives the invariance of the Poisson bracket defined in Eq. (25) under unitary transformations. Consequently, the *unitary transformations on Hilbert space are canonical transformations on the* (X, P) *state space*.

• B) *Self-adjoint operators as observables.* – Any infinitesimal unitary transformation $\hat{U}$ can be generated by a self-adjoint operator $\hat{G}$, such that: $\hat{U} = 1 - (i/\hbar)\hat{G}\delta\alpha$, which leads to the QM relation between an observable and a self-adjoint operator. By a simple calculation, one obtains:

$$X_i \;\rightarrow\; X_i' = X_i + \frac{\partial\langle\Psi|\hat{G}|\Psi\rangle}{\partial P_i}\delta\alpha \;\;,\quad P_i \;\rightarrow\; P_i' = P_i - \frac{\partial\langle\Psi|\hat{G}|\Psi\rangle}{\partial X_i}\delta\alpha \;\;. \tag{26}$$

From these equations, the relation between an observable G, defined in analogy to classical mechanics (as above), and a self-adjoint operator $\hat{G}$ can be inferred:

$$G(X_i, P_i) = \langle\Psi|\hat{G}|\Psi\rangle \;\;, \tag{27}$$

i.e., by comparison with the classical result. Hence, a *real-valued regular function G of the state is an observable, if and only if there exists a self-adjoint operator $\hat{G}$ such that Eq. (27) holds*. This implies that *all QM observables are quadratic forms* in the X_i's and P_i's, which are essentially fewer than in the corresponding CL case; interacting QM-CL hybrids require additional discussion, see Ref. [7].

• C) *Commutators as Poisson brackets.* – From the relation (27) between observables and self-adjoint operators and the Poisson bracket (25) one derives:

$$\{F, G\} = \langle\Psi|\frac{1}{i\hbar}[\hat{F}, \hat{G}]|\Psi\rangle \;\;, \tag{28}$$

with both sides of the equality considered as functions of the variables X_i, P_i and with the commutator defined as usual. Hence, the *QM commutator is a Poisson bracket with respect to the* (X, P) *state space* and relates the algebra of its observables to the algebra of self-adjoint operators.

In conclusion, quantum mechanics shares with classical mechanics an even dimensional state space, a Poisson structure, and a related algebra of observables. It differs essentially by a restricted set of observables and the requirements of phase invariance and normalization, which compactify the underlying Hilbert space to the complex projective space formed by its rays.

3.2 Quantum-classical Poisson bracket, hybrid states and their evolution

The far-reaching parallel of classical and quantum mechanics, as we have seen, suggests to introduce a *generalized Poisson bracket* for QM-CL hybrids:

$$\{A, B\}_\times := \{A, B\}_{\rm CL} + \{A, B\}_{\rm QM} \tag{29}$$

$$:= \sum_k \Big(\frac{\partial A}{\partial_{x_k}}\frac{\partial B}{\partial_{p_k}} - \frac{\partial A}{\partial_{p_k}}\frac{\partial B}{\partial_{x_k}}\Big) + \sum_i \Big(\frac{\partial A}{\partial_{X_i}}\frac{\partial B}{\partial_{P_i}} - \frac{\partial A}{\partial_{P_i}}\frac{\partial B}{\partial_{X_i}}\Big) \ , \tag{30}$$

of any two observables A, B defined on the Cartesian product of CL *and* QM state spaces. It shares the usual properties of a Poisson bracket. – Note that due to the convention introduced by Heslot [23], to which we adhered in Sect. 3.1, the QM variables X_i, P_i have dimensions of $(\text{action})^{1/2}$ and, consequently, no $\hbar$ appears in Eqs. (29)–(30). At the expense of introducing appropriate rescalings, these variables could be made to have their usual dimensions and $\hbar$ to appear explicitly here. – For the remainder of this article, instead we choose units such that $\hbar \equiv 1$.

Let an *observable "belong" to the CL (QM) sector, if it is constant with respect to the canonical coordinates of the QM (CL) sector*. Then, the $\{\ ,\ \}_\times$-bracket has the important properties:
• D) It reduces to the Poisson brackets introduced in Eqs. (19) and (25), respectively, for pairs of observables that belong *either* to the CL *or* the QM sector. • E) It reduces to the appropriate one of the former brackets, if one of the observables belongs only to either one of the two sectors. • F) It reflects the *separability* of CL and QM sectors, since $\{A, B\}_\times = 0$, if A and B belong to different sectors. Hence, *if a canonical tranformation is performed on the QM (CL) sector only, then observables that belong to the CL (QM) sector remain invariant.*

The hybrid density ρ for a self-adjoint density operator $\hat\rho$ in a given state $|\Psi\rangle$ is defined by [5] :

$$\rho(x_k, p_k; X_i, P_i) := \langle\Psi|\hat\rho(x_k, p_k)|\Psi\rangle = \frac{1}{2}\sum_{i,j} \rho_{ij}(x_k, p_k)(X_i - iP_i)(X_j + iP_j) \ , \tag{31}$$

using Eq. (21) and $\rho_{ij}(x_k, p_k) := \langle\Phi_i|\hat\rho(x_k, p_k)|\Phi_j\rangle = \rho^*_{ji}(x_k, p_k)$. It describes a *QM-CL hybrid ensemble* by a real-valued, positive semi-definite, normalized, and possibly time dependent regular function on the Cartesian product state space canonically coordinated by $2(n + N)$-tuples $(x_k, p_k; X_i, P_i)$; the variables $x_k, p_k,\ k = 1, \ldots, n$ and $X_i, P_i,\ i = 1, \ldots, N$ are reserved for CL and QM sectors, respectively.

It can be shown that $\rho(x_k, p_k; X_i, P_i)$ is the *probability density to find in the hybrid ensemble the QM state* $|\Psi\rangle$, parametrized by X_i, P_i through Eq. (21), *together with the CL state* given by a point in phase space, specified by coordinates x_k, p_k. – Further remarks, concerning superposition, pure/mixed, or separable/entangled QM states that may enter the hybrid density can be found in Ref. [6].

Furthermore, the simple form of ρ as bilinear function of QM "phase space" variables X_i, P_i, stemming from the expectation of a density operator $\hat\rho$, has to be generalized for interacting hybrids, allowing for so-called *almost-classical observables*; see Sect. 5.4 of Ref. [5] and a related study [7].

We are now in the position to introduce the appropriate *Liouville equation* for the dynamical evolution of hybrid ensembles [5]. Based on Liouville's theorem and the generalized Poisson bracket defined in Eqs. (29)–(30), we are led to:

$$-\partial_t\rho = \{\rho, \mathcal{H}_\Sigma\}_\times \ , \tag{32}$$

with $\mathcal{H}_\Sigma \equiv \mathcal{H}_\Sigma(x_k, p_k; X_i, P_i)$ and:

$$\mathcal{H}_\Sigma := \mathcal{H}_{\rm CL}(x_k, p_k) + \mathcal{H}_{\rm QM}(X_i, P_i) + \mathcal{I}(x_k, p_k; X_i, P_i) \ , \tag{33}$$

which defines the relevant Hamiltonian function, including a hybrid interaction; $\mathcal{H}_\Sigma$ is required to be an *observable*, in order to have a meaningful notion of energy. Note that *energy conservation* follows from $\{\mathcal{H}_\Sigma, \mathcal{H}_\Sigma\}_\times = 0$.

An important advantage of Hamiltonian flow and a general property of the Liouville equation is:
• G) The normalization and positivity of the probability density ρ are conserved in presence of a hybrid interaction; hence, its interpretation remains valid.

4 Quantum control by a classical time machine

Our aim here is to combine the results on discrete mechanics (Sect. 2), where time is one of the dynamical variables and which consequently allows to model a particular kind of time machines (Sect. 2.2), with those on QM-CL hybrids (Sect. 3). We explore in this framework, how such a classical time machine interacts with a quantum object.

As a concrete example, we consider an *oscillator-like time machine coupled to a q-bit*. The former is represented by the Hamiltonian function:

$$\mathcal{H}_{\rm CL}(x,p;t) := \frac{\zeta}{2}\cos(\omega t)[p^2+\Omega^2x^2] \ , \tag{34}$$

cf. Eq. (16), where Ω denotes the proper oscillator frequency, while ω is the frequency of the change of time direction, cf. Sect. 2.2, and the dimensionless constant ζ parametrizes its amplitude. For $\Omega \gg \omega$, the oscillator performs many oscillations (circles in phase space), before the time direction changes; conversely, for $\Omega \ll \omega$, the oscillator moves only little before beginning to trace its trajectory in phase space in the opposite direction. Qualitatively similar behaviour of the time machine is expected for other than oscillator potentials.

The q-bit is described, in the oscillator representation, cf. Sect. 3.1.1, by the Hamiltonian function:

$$\mathcal{H}_{\rm QM}(X_1,X_2,P_1,P_2) := \frac{E_0}{2}\sum_{i=1,2}(-1)^i(P_i^2+X_i^2) \ , \tag{35}$$

with E_0 an energy scale, cf. Eq. (23). Wave function normalization, Eq. (24), is required by:

$$2C \equiv X_1^2+X_2^2+P_1^2+P_2^2 \stackrel{!}{=} 2 \ . \tag{36}$$

The model is completed by choosing a hybrid interaction, for example:

$$\mathcal{I}(x,;X_i,P_i) := \lambda x\cos(\omega t)\langle\Psi|\hat{O}|\Psi\rangle = \lambda x\cos(\omega t)\sum_{i,j=1,2}O_{ij}(X_i-iP_i)(X_j+iP_j) \ , \tag{37}$$

using the oscillator expansion of a generic state $|\Psi\rangle$; $O_{ij} := \langle\phi_i|\hat{O}|\phi_j\rangle$ denotes a matrix element of the q-bit observable $\hat{O}$ $(=\hat{O}^\dagger)$ in the basis of energy eigenstates corresponding to $\mathcal{H}_{\rm QM}$ and λ is a coupling constant. Naturally, other and more general interactions may be considered.

Then, the following Hamilton equations are obtained in the usual way from the hybrid Hamiltonian $\mathcal{H}_\Sigma := \mathcal{H}_{\rm CL} + \mathcal{H}_{\rm QM} + \mathcal{I}$:

$$\dot{x} = p\cos(\omega t) \ , \tag{38}$$

$$\dot{p} = -\Big(\Omega^2x + \lambda\langle\Psi|\hat{O}|\Psi\rangle\Big)\cos(\omega t) \ , \tag{39}$$

$$\dot{X}_i = (-1)^iE_0P_i + \lambda x\partial_{P_i}\langle\Psi|\hat{O}|\Psi\rangle\cos(\omega t) \ , \tag{40}$$

$$\dot{P}_i = -(-1)^iE_0X_i - \lambda x\partial_{X_i}\langle\Psi|\hat{O}|\Psi\rangle\cos(\omega t) \ , \tag{41}$$

where we set $\zeta \equiv 1$, which can always be implemented by rescaling time, E_0, and λ. In agreement with the general result in Eqs. (27)–(28) of Ref. [5], the constraint of Eq. (36) is conserved under this Hamiltonian flow, $\mathrm{d}C/\mathrm{d}t = \{C, \mathcal{H}_\Sigma\}_\times = 0$, and, therefore, it is sufficient to impose the constraint on the initial conditions of the equations of motion (38)–(41).

In order to uncover some characteristic features of this hybrid model, we introduce the *internal time* variable $\tau(t) := \omega^{-1}\sin(\omega t)$ into Eqs. (38)–(39). The resulting second order equation (for $x(\tau)$) of a driven harmonic oscillator can be solved with the help of its retarded Green's function:

$$x(t) = x_1\cos[\Omega\tau(t)+\phi] - \lambda\Omega^{-1}\int_{-\infty}^{\tau(t)}\mathrm{d}s\;\sin[\Omega(\tau(t)-s)]\tilde{O}(s) \;\;, \tag{42}$$

where the first term solves the homogeneous equation, incorporating integration constants x_1 and ϕ, and where the inhomogeneity is given by:

$$\tilde{O}(s) := (X_1P_2 - X_2P_1)_{t(s)} \;\;, \tag{43}$$

with X's and P's evaluated at $t(s)$, determined (modulo π/ω) by $t = \omega^{-1}\arcsin(\omega s)$; for simplicity, the q-bit obervable has been assumed to be proportional to the spin-1/2 Pauli matrix σ_2, such that $-O_{12} = O_{21} = i/2$ and $O_{11} = O_{22} = 0$. Using solution (42) and Eq. (38), we obtain:

$$p(t) = \mathrm{d}x/\mathrm{d}\tau = -x_1\Omega\sin[\Omega\tau(t)+\phi] - \lambda\int_{-\infty}^{\tau(t)}\mathrm{d}s\;\cos[\Omega(\tau(t)-s)]\tilde{O}(s) \;\;. \tag{44}$$

We see explicitly that the time machine travels periodically forwards and backwards in time, due to the periodicity of its *internal time* τ with respect to the *external time* t governing the chronology respecting q-bit (described by the X, P-variables). Most notably, $\dot{x}$ and p are not always aligned, *i.e.*, of same sign.

However, since from one period of forward (or backward) evolution to the next the external time increases by $2\pi/\omega$, generally, the value of the function $\tilde{O}$ in Eqs. (42)–(44) will change accordingly. This implies that the classical time machine that interacts with the q-bit, will go backwards in (x, p) phase space in a different way than it came! Which can entail known paradoxes of time travel, such as the *grandfather paradox* or the *unproved theorem paradox* [1, 2].

Summarizing, the interaction of a classical time machine with a chronology respecting system, the q-bit here, introduces an aspect of "ageing" into its dynamics: despite going forwards and backwards in time, in general, its state does evolve and depends on the external time t.

Unlike solutions to such paradoxes proposed in the literature for *quantum systems* in the presence of closed timelike curves (CTCs) by Deutsch [26], Lloyd and collaborators in the form of post-selected teleportation (P-CTCs) [1, 2], or the consideration of open timelike curves (OTCs) by Ralph and collaborators [27], our model of a *classical* time machine does not provide enough freedom to eliminate paradoxical situations (Novikov principle) by imposing additional constraints on its dynamics. Apparently, *quantum-classical hybrids* do not work here. – Considering a suitably constrained *ensemble* of classical time machines might help. However, its physical relevance remains to be seen.

Of course, having a QM time machine consistently interacting with a classical object is not ruled out by the present model. In fact, previously considered CTC scenarios should reduce to such a hybrid situation under suitable circumstances.

We consider the effect of the time machine on the q-bit next. In this case, we conveniently rewrite Eqs. (40)–(41) by undoing the oscillator expansion, cf. Eq. (21):

$$i\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} X_1+iP_1 \\ X_2+iP_2 \end{pmatrix} = (-E_0\hat{\sigma}_3 + \lambda x(t)\cos(\omega t)\hat{\sigma}_2)\begin{pmatrix} X_1+iP_1 \\ X_2+iP_2 \end{pmatrix} \;\;, \tag{45}$$

where $\hat{\sigma}_{2,3}$ are the imaginary and diagonal spin-1/2 Pauli matrices, respectively. This is a Schrödinger equation representing a spin-1/2 in a *magnetic field*. In particular, here its 2-component is time dependent. With the time dependence arising from $x(t)\cos(\omega t)$, cf. Eqs. (42)–(43), this *effective Schrödinger equation is nonlinear and non-Markovian*.

The nonlinear and non-Markovian behaviour can be neglected for sufficiently small coupling λ, in which case the time dependence of the effective magnetic field is given by the following factor:

$$B(t) := \lambda x_1 \cos(\omega t) \cos\left(\Omega\omega^{-1}\sin(\omega t)\right) \quad , \tag{46}$$

incorporating only the first term from the right-hand side of Eq. (42). Concerning the q-bit, the corresponding instantaneous eigenvalues of the effective Hamiltonian are shifted in this approximation (lowest nonvanishing order in λ) and are simply given by:

$$E_{\pm} = \pm E_0\left(1 + B^2(t)/2E_0^{\,2}\right) \quad . \tag{47}$$

This result would, in principle, allow to constrain parameters defining the present toy model of a quantum-classical hybrid, consisting of a classical time machine interacting with a q-bit, given the manifold laboratory realizations of q-bits.

Furthermore, among others, there are generalizations of the hybrid interaction, Eq. (37), which could give rise to a *rotating magnetic field* instead of the oscillating one in Eqs. (45)–(46). This, in turn, produces effects like a *Berry phase*, or its generalizations (see, *e.g.*, the recent Ref. [28] and references therein), which could serve as well to constrain such models.

However, as we have discussed, the classical time machines addressed here are likely bound to reproduce the paradoxes of time travel. If they are not directly observable, for some reason, their indirect effects on quantum systems may be worth further study.

5 Conclusions

Our purpose here has been to explore the possibility that classical "time machines" couple directly to quantum mechanical objects.

We invoked the discrete mechanics proposed by T.D. Lee, in which time belongs to the set of dynamical variables [3, 4]. Suitably modifying the underlying action, we have developed a Hamiltonian theory of such *discrete classical dynamical systems*. Choosing the dynamics of the time variable appropriately, we are led to systems which evolve forward and backward in time, *time machines* or, more precisely, time reversing machines.

In the continuum limit and for particular choices of the dynamics of time, the motion is periodic. Thus, such an object evolves forward in time, forming a trajectory in phase space, until it comes to a halt, then traces this trajectory backwards in time, comes to halt, evolves forward again, and so on. A clock carried on board would be seen running alternatingly forwards and backwards. – These time machines are distinct from the closed timelike curves (CTCs) on which an object travels, which have been frequently discussed in the literature, see Refs. [1, 2], for example, and works referred to there. They might be realizable in physical analogue models.

In order to describe the direct coupling of such a time machine with a quantum object, *e.g.* a q-bit, we reviewed our recent proposal for a consistent quantum-classical hybrid dynamics, which is based on a phase space formalism for classical as well as for quantum mechanics [5–7, 23, 24].

We have defined a toy model of such a QM-CL hybrid, consisting of an oscillator like classical time machine coupled to a q-bit and discussed its Hamiltonian equations of motion. While this could lead to observe the action of a time reversing machine through its effects on a quantum object, we have also pointed out that common time travel paradoxes would affect the classical time machine.

In retrospect, the latter is understandable, since the outcome of the evolution of a classical object is deterministic and fixed, *e.g.*, by initial conditions, to the extend that no additional (nonlinear) constraints can be imposed, as in the quantum case. The QM-CL hybrids that we described do *not* alter this circumstance. For quantum mechanical objects travelling on CTCs, however, such constraints serve to suppress the paradoxes by selecting well-behaved ones from the ensemble of all possible histories [1, 2, 26]. Which poses the question whether a *quantum mechanical time reversing machine*, based on the Hamiltonian discrete meachanics presented here, can similarly avoid time travel paradoxes?

Acknowledgements: It is a pleasure to thank M. Crosta, M. Gramegna and M. Ruggiero for the invitation to give a talk in the inspiring atmosphere of the conference *Time machine Factory* (Torino, October 2012) and to thank N. Burić, L. Maccone, and C. Stoica for discussions and correspondence.

References

[1] S. Lloyd, L. Maccone, R. Garcia-Patron, V. Giovannetti, Y. Shikano, S. Pirandola, L.A. Rozema, A. Darabi, Y. Soudagar, L.K. Shalm, A.M. Steinberg, Phys. Rev. Lett. **106**, 040403 (2011)

[2] S. Lloyd, L. Maccone, R. Garcia-Patron, V. Giovannetti and Y. Shikano, Phys. Rev. D **84**, 025007 (2011)

[3] T.D. Lee, Phys. Lett. **122**B, 217 (1983)

[4] T.D. Lee, J. Statist. Phys. **46**, 843 (1987)

[5] H.-T. Elze, Phys. Rev. A **85**, 052109 (2012)

[6] H.-T. Elze, Journ. Phys.: Conf. Ser. **361**, 012004 (2012)

[7] H.-T. Elze, Int. J. Qu. Inf. (IJQI) **10**, No. 8, 1241012 (2012)

[8] A. D'Innocenzo, L. Renna and P. Rotelli, Eur. J. Phys. **8**, 245 (1987)

[9] Z. Hong-Bin, C. Li-Qun and L. Rong-Wan, Chinese Phys. **14**, 1063 (2005)

[10] S. Capobianco and T. Toffoli, arXiv:1103.4785v2

[11] H.D. Politzer, Phys. Rev. D **46**, 4470 (1992)

[12] T.N. Sherry and E.C.G. Sudarshan, Phys. Rev. D **18**, 4580 (1978); do., *II.* Phys. Rev. D **20**, 857 (1979)

[13] W. Boucher and J. Traschen, Phys. Rev. D **37**, 3522 (1988)

[14] G. 't Hooft, Int. J. Mod. Phys. A **25**, 4385 (2010); *do.*, arXiv:0908.3408 (2009)

[15] H.-T. Elze, J. Phys.: Conf. Ser. **174**, 012009 (2009); *do.*, J. Phys.: Conf. Ser. **33**, 399 (2006)

[16] S.L. Adler, *Quantum Mechanics as an Emergent Phenomenon* (Cambridge, UK: Cambridge U. Press, 2005)

[17] J. Caro and L.L. Salcedo, Phys. Rev. A **60**, 842 (1999)

[18] L. Diósi, N. Gisin and W.T. Strunz, Phys. Rev. A **61**, 022108 (2000)

[19] A. Peres and D.R. Terno, Phys. Rev. A **63**, 022101 (2001)

[20] L. Diósi, Journ. Phys.: Conf. Ser. **361**, 012028 (2012)

[21] M.J.W. Hall and M. Reginatto, Phys. Rev. A **72**, 062109 (2005)

[22] M.J.W. Hall, Phys. Rev. A **78** , 042104 (2008)

[23] A. Heslot, Phys. Rev. D **31**, 1341 (1985)

[24] M. Radonjić, S. Prvanović and N. Burić, Phys. Rev. A **85**, 064101 (2012)

[25] N. Burić, I. Mendas, D. B. Popović, M. Radonjić and S. Prvanović, Phys. Rev. A **86**, 034104 (2012)

[26] D. Deutsch, Phys.Rev. D **44**, 3197 (1991)
[27] J.L. Pienaar, C.R. Myers and T.C. Ralph, preprint arXiv:1206.5485 (2012)
[28] S.S. Gousheh, A. Mohammadi and L. Shahkarami, preprint arXiv:1212.2104 (2012)