The spectrum of atomic levels of hydrogen-like ions originating from the lowest Landau level in an external homogeneous superstrong magnetic field is obtained. The influence of the screening of the Coulomb potential on the values of critical nuclear charges is studied.

1 Introduction

We will discuss the modification of the Coulomb law and atomic spectra in superstrong magnetic field. The talk is based on papers [1–3], see also [4].

2 $D = 2$ QED

Let us consider two dimensional QED with massive charged fermions. The electric potential of the external point-like charge equals:

$$
\Phi(k) = -\frac{4\pi g}{k^2 + \Pi(k^2)},
$$

where $\Pi(k^2)$ is the one-loop expression for the photon polarization operator:

$$
\Pi(k^2) = 4g^2 \left[ \frac{1}{\sqrt{1 + t}} \ln(\sqrt{1 + t} + \sqrt{t} - 1) \right] \equiv -4g^2 P(t),
$$

and $t \equiv -k^2/4m^2$, $[g]$ = mass.

In the coordinate representation for $k = (0, k_\parallel)$ we obtain:

$$
\Phi(z) = 4\pi g \int_{-\infty}^{\infty} \frac{e^{ik_\parallel z}dk_\parallel/2\pi}{k_\parallel^2 + 4g^2 P(k_\parallel^2/4m^2)}.
$$

With the help of the interpolating formula

$$
\overline{P}(t) = \frac{2t}{3 + 2t}
$$

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the accuracy of which is better than 10% for $0 < t < \infty$ we obtain:

$$\Phi = 4\pi g \int_{-\infty}^{\infty} \frac{e^{ikz}dk/2\pi}{k^2 + 4g^2(k^2/2m^2)/(3 + k^2/2m^2)} =$$

$$= \frac{4\pi g}{1 + 2g^2/3m^2} \left[ \frac{-1/2|z| + \frac{g^2/3m^2}{\sqrt{6m^2 + 4g^2}} \exp(-\sqrt{6m^2 + 4g^2|z|})}{\sqrt{6m^2 + 4g^2}} \right]. \quad (5)$$

In the case of heavy fermions ($m \gg g$) the potential is given by the tree level expression; the corrections are suppressed as $g^2/m^2$.

In the case of light fermions ($m \ll g$):

$$\Phi(z) \bigg| m \ll g = \left\{ \begin{array}{ll}
\pi e^{-2g|z|}, & z \ll \frac{1}{g} \ln \left( \frac{g}{m} \right), \\
-2\pi g \left( \frac{3m^2}{2g^2} \right)|z|, & z \gg \frac{1}{g} \ln \left( \frac{g}{m} \right). 
\end{array} \right. \quad (6)$$

$m = 0$ corresponds to the Schwinger model; photon gets a mass due to a photon polarization operator with massless fermions.

### 3 Electric potential of the point-like charge in $D = 4$ in superstrong $B$

We need an expression for the polarization operator in the external magnetic field $B$. It simplifies greatly for $B \gg B_0 = m_e^2/e$, where $m_e$ is the electron mass and we use Gauss units, $e^2 = \alpha = 1/137...$

The following results were obtained in [2]:

$$\Phi(k) = \frac{4\pi e}{k^2 + k^2_{\perp} + \frac{2e^3B}{\pi} \exp \left( -\frac{k^2_{\perp}}{2eB} \right) \left( \frac{k^2}{4m_e^2} \right)} \, . \quad (7)$$

$$\Phi(z) = 4\pi e \int k^2_{\perp} + k^2_{\perp} + \frac{2e^3B}{\pi} \exp \left( -\frac{k^2_{\perp}}{(2eB)}(k^2_{\perp}/2m_e^2)/(3 + k^2_{\perp}/2m_e^2) \right) =$$

$$= \exp \left( -\frac{6m_e^2}{2eB} \right) + \exp \left[ -\left( \sqrt{2/\pi} e^3 B + 6m_e^2 \right) |z| \right]. \quad (8)$$

For $B \ll 3\pi m^2/e^3$ the potential is Coulomb up to small corrections:

$$\Phi(z) \bigg| e^3B \ll m_e^2 = \frac{e}{|z|} \left[ 1 + O \left( \frac{e^3B}{m_e^2} \right) \right], \quad (9)$$

analogously to $D = 2$ case with substitution $e^3B \rightarrow g^2$.

For $B \gg 3\pi m^2/e^3$ we obtain:

$$\Phi(z) = \left\{ \begin{array}{ll}
\frac{e}{|z|} e^{-\frac{2\sqrt{2/\pi} e^3 B |z|}} \, , & \frac{1}{\sqrt{2/\pi} e^3 B} \ln \left( \frac{e^3 B}{3\pi m_e^2} \right) > |z| > \frac{1}{\sqrt{eB}} \\
\frac{e}{|z|} \left( 1 - e^{-\frac{2\sqrt{2/\pi} e^3 B |z|}} \right) \, , & \frac{1}{m_e} > |z| > \frac{1}{\sqrt{2/\pi} e^3 B} \ln \left( \frac{e^3 B}{3\pi m_e^2} \right), \\
\frac{e}{|z|} \, , & |z| > \frac{1}{m_e}. 
\end{array} \right. \quad (10)$$

$$V(z) = -e\Phi(z) \, . \quad (11)$$

The close relation of the radiative corrections at $B \gg B_0$ in $D = 4$ to the radiative corrections in $D = 2$ QED allows to prove that just like in $D = 2$ case higher loops are not essential (see, for example, [5]).
4氢原子在磁场中

对于$B > B_0 = m_e^2/e$，Dirac方程的谱由超相对论电子组成，唯一的例外是：最低Landau能级(LLL, $n = 0$，$\sigma_z = -1$)的电子是非相对论的。因此我们将会找到LLL中电子在屏蔽库仑场中的谱。

电子在LLL中的波函数为：

$$R_{0m}(\rho) = \left[\pi(2a_H^2)^{1+|m|}|m|!!\right]^{-1/2} \rho^{|m|} e^{i(m\rho - \rho^2/(4a_H^2))}, \tag{12}$$

其中$m = 0, -1, -2$是电子轨道角动量在磁场方向上的投影。

对于$a_H \equiv 1/\sqrt{eB} \ll a_B = 1/(m_e e^2)$，阿达布型近似适用，波函数形式为：

$$\Psi_{n0m-1} = R_{0m}(\rho)\chi_n(z), \tag{13}$$

其中$\chi_n(z)$满足一维Schrödinger方程：

$$\left[-\frac{1}{2m_e}\frac{d^2}{dz^2} + U_{eff}(z)\right]\chi_n(z) = E_n\chi_n(z). \tag{14}$$

由于屏蔽在非常短的距离发生，对于奇数状态，有效势是不重要的，有效势为：

$$U_{eff}(z) = -e^2 \int |R_{0m}(\rho)|^2 d^2\rho \left[1 - e^{-\sqrt{6m_e^2}z} + e^{-\sqrt{(2/\pi)e^3B+6m_e^2}z}\right]. \tag{15}$$

它等同于库仑势，当$|z| \gg a_H$时，并且当$z = 0$时是正则的。

因此，奇数状态的能量是：

$$E_{odd} = -\frac{m_e^4}{2n^2} + O\left(\frac{m_e^2 e^3}{B}\right), \quad n = 1, 2, \ldots, \tag{16}$$

而对于超强磁场$B > m_e^2/e^3$，它们与巴尔梅系列一致，准确度很高。

对于偶数状态的有效势看起来像：

$$\tilde{U}_{eff}(z) = -e^2 \int \frac{|R_{0m}(\rho)|^2}{\sqrt{\rho^2 + z^2}} d^2\rho \left[1 - e^{-\sqrt{6m_e^2}z} + e^{-\sqrt{(2/\pi)e^3B+6m_e^2}z}\right]. \tag{17}$$

积分Schrödinger方程与有效势的方程从$x = 0$到$x = z$，其中$a_H \ll z \ll a_B$，并计算获得的表达式$\chi'(z)$与Whittaker函数的对数导数的方程一致。Schrödinger方程和库仑势的方程，我们得到偶数状态的能量方程：

$$\ln\left(\frac{H}{1 + \frac{e^6}{3\pi}H}\right) = \lambda + 2\ln\lambda + 2\psi\left(1 - \frac{1}{\lambda}\right) + \ln 2 + 4\gamma + \psi(1 + |m|), \tag{18}$$

其中$H \equiv B/(m_e^2 e^3)$，$\psi(x)$是伽马函数的对数导数，

$$E = -(m_e^4 / 2)\lambda^2. \tag{19}$$

氢原子的谱在极限$B \gg m_e^2/e^3$在图1中显示。
Figure 1. Spectrum of the hydrogen atom in the limit of the infinite magnetic field. Energies are given in Rydberg units, $R_y \equiv 13.6 \, eV$. 
5 Screening versus critical nucleus charge

Hydrogen-like ion becomes critical at \( Z \approx 170 \): the ground level reaches lower continuum, \( \epsilon_0 = -m_e \), and two \( e^+e^- \) pairs are produced from vacuum. Electrons with the opposite spins occupy the ground level, while positrons are emitted to infinity [6]. According to [7] in the strong magnetic field \( Z_{ct} \) diminishes: it equals approximately 90 at \( B = 100B_0 \); at \( B = 3 \cdot 10^4 B_0 \) it equals approximately 40. Screening of the Coulomb potential by the magnetic field acts in the opposite direction and with account of it larger magnetic fields are needed for a nucleus to become critical.

Let us parametrize bispinor which describes electron wave function in the following way:

\[
\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}, \quad \varphi = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad \chi = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.
\] (20)

Substituting \( \Psi \) in the Dirac equation for the electron in an external electromagnetic field we obtain:

\[
\begin{pmatrix} (\epsilon - m - e\varphi) & (c_1) \\ (-i\sigma \frac{\partial}{\partial \rho} + e\tilde{A}\tilde{\sigma}) & (b_1) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0
\]

\( (\epsilon + m - e\varphi) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + (\epsilon + m - e\varphi) \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0 \] (21)

Taking vector potential which describes constant magnetic field \( B \) directed along \( z \) axis in the form \( \tilde{A} = (-\frac{1}{2}By, \frac{1}{2}Bx, 0) \), we get:

\[
e\tilde{A}\tilde{\sigma} = -\frac{e}{2}B \begin{pmatrix} 0 & y + ix \\ y - ix & 0 \end{pmatrix} = -\frac{i}{2}eB\rho \begin{pmatrix} 0 & e^{-i\theta} \\ -e^{i\theta} & 0 \end{pmatrix},
\] (22)

where \( \rho = \sqrt{x^2 + y^2}, \theta = \arctan(y/x) \). Analogously we obtain:

\[
-i\sigma \frac{\partial}{\partial \rho} = -i \left( e^{i\theta} \frac{\partial}{\partial \rho} + ie^{i\theta} \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \varphi} \right).
\] (23)

Substituting two last expressions in the Dirac equation we get:

\[
\begin{pmatrix} (\epsilon - m - e\varphi) & (c_1) \\ (-i\sigma \frac{\partial}{\partial \rho} + e\tilde{A}\tilde{\sigma}) & (b_1) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0
\]

\( (\epsilon + m - e\varphi) \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + (\epsilon + m - e\varphi) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0 \] (24)

Axial symmetry of electromagnetic field allows to determine \( \theta \) dependence of the functions \( c_i \) and \( b_i \):

\[
\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1(\rho, z) \\ c_2(\rho, z) \end{pmatrix} e^{i(M-1/2)\theta}, \quad \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} b_1(\rho, z) \\ b_2(\rho, z) \end{pmatrix} e^{i(M+1/2)\theta},
\] (25)

where \( M = \pm1/2, \pm3/2, \ldots \) is the projection of electron angular momentum on \( z \) axis. Substituting (25) in (24) we get four linear equations for four unknown functions \( c_i \) and \( b_i \) (here and below \( c_1 \equiv c_1(\rho, z), \))
\[ \begin{align*}
\epsilon - m - e\varphi c_1 + i(b_1 z - b_2) & = \frac{M + 1/2}{\rho} b_2 - \frac{eB\rho}{2} b_2 = 0 \\
\epsilon - m - e\varphi c_2 + i(b_1 e + \frac{M - 1/2}{\rho} b_1 + eB\rho b_2) & = 0 \\
(\epsilon + m - e\varphi) b_1 + i(-c_1 z - c_2) & = \frac{M + 1/2}{\rho} c_2 - \frac{eB\rho}{2} c_2 = 0 \\
(\epsilon + m - e\varphi) b_2 + i(-c_1 e + \frac{M - 1/2}{\rho} c_1 + eB\rho b_2) & = 0 ,
\end{align*} \]

where \( b_{1z} \equiv \partial b_1 / \partial z, b_{1\rho} \equiv \partial b_1 / \partial \rho, \ldots \). Ground energy state has \( s_z = -1/2, l_z = 0 \). Taking \( M = -1/2 \) we should look for solution of (26) with \( c_1 = b_1 = 0 \):

\[ \begin{align*}
\begin{cases}
 b_2 + \frac{eB\rho}{2} b_2 = 0 \\
 c_2 + \frac{eB\rho}{2} c_2 = 0 ,
\end{cases} \quad (27)
\end{align*} \]

The dependence on \( \rho \) is determined by (27):

\[ \begin{align*}
\begin{cases}
 b_2(\rho, z) = e^{-eB\rho^2/4} (-i)f(z) \\
c_2(\rho, z) = e^{-eB\rho^2/4} g(z) .
\end{cases} \quad (29)
\end{align*} \]

Substituting the last expressions in (28) and averaging over fast motion in transverse to the magnetic field plane we obtain two first order differential equation which describes electron motion along magnetic field [7]:

\[ \begin{align*}
g_z - (\epsilon + m_e - \bar{V}) f = 0 , \\
f_z + (\epsilon - m_e - \bar{V}) g = 0 ,
\end{align*} \]

where \( g_z \equiv dg/dz, f_z \equiv df/dz \). They describe the electron motion in the effective potential \( \bar{V}(z) \):

\[ \begin{align*}
\bar{V}(z) & = -\frac{Z e^2}{\alpha \hbar^2} \left[ 1 - e^{-\sqrt{6m_e^2} |z|} + e^{-\sqrt{(2/\pi)^3 B + 6m_e^2} |z|} \right] \times \\
& \times \int_0^{\infty} \frac{e^{-\rho^2/2\alpha^2}}{\sqrt{\rho^2 + z^2}} \rho d\rho .
\end{align*} \]

Intergrating (30) numerically we find the dependence of \( Z_{cr} \) on the magnetic field with the account of screening. The results are shown in Fig. 2. For the given nucleus to become critical larger magnetic fields are needed and the nuclei with \( Z < 52 \) do not become critical.

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Figure 2. The values of $B^2_{z}$: a) without screening according to [7], dashed (green) line; b) numerical results with screening, solid (blue) line. The dotted (black) line corresponds to the field at which $a_H$ becomes smaller than the size of the nucleus.

References