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# Jacobi map and geodesic light-cone gauge: an exact solution[⋆]

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**Abstract.** An exact expression for the Jacobi map is furnished. This result has been achieved thanks to the properties of the recently proposed geodesic light-cone gauge. From here, a non-perturbative expression for the area distance is given. It is remarkable that the expression we get nicely appears as the product of a pure source term times a pure observer one. This is true only in the particular gauge we adopt, as shown by transforming the area distance in the synchronous gauge and in the longitudinal one. The consistency among the two expressions is finally checked.

## 1 Introduction

In this paper, we are going to show and comment the main results achieved in [1]. Our main purpose was finding an exact expression for the the Jacobi map (JM) in the recently proposed [2] geodesic light-cone (GLC) gauge. As already known, this gauge is well adapted to the description of most of physical observables involving light-like signals because it is built on the past light-cone of the observer; in particular, the luminosity/redshift relation in presence of inhomogeneities has been studied a lot [3–6]. On the other hand, the JM concerns the trajectory of the bundle of photons from the source we are interested in to the observer: it is natural indeed to find how to express the JM in the GLC gauge. Nevertheless its theoretical importance is evident because it allows us to link directly the evolution of light-like signals from the source tho the observer. Furthermore, it is also connected to observations: in particular to the so-called area distance $d_A$ defined as [7]:

$$d_A^2 \equiv \frac{dS_s}{d\Omega_o}, \tag{1}$$

where $dS_s$ is the cross-sectional area element perpendicular to the light-ray at the source position and $d\Omega_o$ is the infinitesimal solid angle subtending the source at the observer position. Moreover, it is well-known that the area distance is related to the luminosity distance $d_L$ by the distance duality relation[1] [8] $d_L = (1+z)^2 d_A$, where $z$ is the source's redshift.

The link with the luminosity distance and the possibility to consider an exact non-perturbative scenario are the main motivations for this work because it allows to consider the entire effects of

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[1]It is based on the invariance of some geometric properties, when source and observer are exchanged. Moreover, it is valid in any metric space-time, if the photons number is conserved during the bundle's travel.

inhomogeneities and their afflictions to the physical observable concerning the acceleration of the universe.

Our discussion is organized as follows. Sect. 2 will recall the main ideas in the general definition of the JM. In Sec. 3 we will explain how the GLC gauge can furnish a simple, but exact, JM. Sect. 4 and Sect. 5 are dedicated to expressing the GLC gauge result in two well-know ones: respectively the synchronous gauge (SG) and the longitudinal gauge (LG); physical consequences of these transformations at the observer position are discussed. The consistency among the two previous results is explained in Sect. 6. In Sect. 7 we will summarize our results, drawing a few conclusions.

## 2 The Jacobi Map: general definition

Following the notation of [9], let us consider two trajectories on the light-cone $x_1^\mu$ and $x_2^\mu$ that depend by the affine parameter $\lambda$; in such a way, we define the displacement $\xi^\mu = x_1^\mu - x_2^\mu$ which is orthogonal to $k^\mu$ and evolves according to the geodesic deviation equation:

$$\nabla_\lambda^2 \xi^\mu = R_{\alpha\beta\nu}{}^{\mu} k^\alpha k^\nu \xi^\beta \,, \tag{2}$$

where $\nabla_\lambda \equiv k^\alpha \nabla_\alpha$ and $R_{\alpha\beta\mu}{}^{\nu}$ is the Riemman tensor. Because of the orthogonality among $\xi^\mu$ and $k^\mu$, we are left with only two non-trivial components of this equation; therefore let us define a two dimensional subspace using the so-called Sachs basis $\{s_A^\mu\}_{A=1,2}$[2] [10, 11] defined as follows:

$$g_{\mu\nu} s_A^\mu s_B^\nu = \delta_{AB} \,, \tag{3}$$

$$s_A^\mu u_\mu = 0 \qquad , \qquad s_A^\mu k_\mu = 0, \tag{4}$$

$$\Pi_\nu^\mu \nabla_\lambda s_A^\nu = 0 \qquad \text{with} \qquad \Pi_\nu^\mu = \delta_\nu^\mu - \frac{k^\mu k_\nu}{(u^\alpha k_\alpha)^2} - \frac{k^\mu u_\nu + u^\mu k_\nu}{u^\alpha k_\alpha} \tag{5}$$

where $\Pi_\nu^\mu$ projects on the two dimensional space orthogonal to the quadri-velocity field $u_\mu$ and $n_\mu = u_\mu + (u^\alpha k_\alpha)^{-1} k_\mu$; this allows us to express the displacement as $\xi^\mu = \xi^A s_A^\mu$ and $\xi^A = \xi^\mu s_\mu^A = g_{\mu\nu} \xi^\mu s_A^\nu$. With the aim of these definitions, we can project eq.(2) along the Sachs basis; so provided $d/d\lambda \equiv k^\mu \partial_\mu$ and $R_B^A \equiv R_{\alpha\beta\nu\mu} k^\alpha k^\nu s_B^\beta s_A^\mu$ eq.(2) becomes:

$$\frac{d^2 \xi^A}{d\lambda^2} = R_B^A \xi^B. \tag{6}$$

Without any lack of generality, once we express the solution of eq.(6) as[3] $\xi^A(\lambda_s) = J_B^A(\lambda_s, \lambda_o) \left(\frac{k^\mu \partial_\mu \xi^B}{k^\nu u_\nu}\right)_o$, we obtain a second order differential equation for $J_B^A$:

$$\frac{d^2}{d\lambda^2} J_B^A(\lambda, \lambda_o) = R_C^A J_B^C \,, \tag{7}$$

with the initial condition $J_B^A(\lambda_o, \lambda_o) = 0$ and $\frac{d}{d\lambda} J_B^A(\lambda_o, \lambda_o) = \delta_B^A \, (k^\nu u_\nu)_o$.

$J_B^A(\lambda_s, \lambda_o)$ is the so-called Jacobi map and its knowledge is important because it is related to some relevant physical observables; in particular, as we said in Sect. 1, the one we are interested in is the area distance $d_A^2 \equiv \frac{dS_s}{d\Omega_o}$. The link between $J_B^A$ and $d_A$ is given by the relation [12]:

$$d_A^2 = \det J_B^A(\lambda_s, \lambda_o). \tag{8}$$

[2]Capital latin indices refer to a flat space, so their position is irrelevant.
[3]Suffices $o$ and $s$ respectively indicate the observer and source position.

Solving eq.(7) immediately allows to connect with the observations. Unfortunately this is not so simple in general; in fact, until now only perturbed solution of an highly symmetric background are given in literature. However, in the next section, we will furnish an exact non-perturbative solution for the Jacobi map by performing its evaluation in a specific coordinates system: the so-called GLC gauge.

## 3 The Jacobi map and the GLC gauge

The GLC coordinates derive from entirely fixing the gauge degrees of freedom of the metric on the observer [2]. In particular the timelike coordinate $\tau$ is always identified with the proper time of the SG [3] while the three spatial-like ones consist of a null coordinate $w$ and two angular ones $\tilde{\theta}^a$. In such a way, the line element can be written as:

$$ds^2 = \Upsilon^2 dw^2 - 2\Upsilon dw d\tau + \gamma_{ab}(d\tilde{\theta}^a - U^a dw)(d\tilde{\theta}^b - U^b dw) \ , \quad a, b = 1, 2 \ . \tag{9}$$

This gauge is qualitatively different by the so-called observational coordinates [13–15], where the timelike coordinate $\tau$ is replaced by the spacelike $y$. Let us stress that, in such a choice, the metric has no symmetry i.e. all of the effects due to the presence of inhomogeneities are exactly taken in account. Moreover this set of coordinates is well-adapted to the observer description[4] because: i) the observer past light-cone is exactly given by the hyper-surface $w = const$ ii) static observers are also geodesic, just like happens in the SG iii) photons travel at constant $\tilde{\theta}^a$ and $w$, with[5] $k^\mu = \Upsilon^{-1}\delta^\mu_\tau$ iv) as a consequence of this last point we can say that the spatial part of the displacement vector $\xi^\mu$ is constant for every couple of photons; in particular, if they travel on the same light-cone $\xi^w = 0$[6].

For all of these reasons, using the GLC coordinates is crucial in order to easily find the exact expression of the JM; in fact in this gauge, from conditions (3) and (4), Sachs basis looks like $s^\mu_A = (s^\tau_A, 0, s^a_A)$, or equivalently $s^A_\mu = g_{\mu\nu} s^\nu_A = (0, s^A_w, s^A_a)$ and the condition $s^A_a s^A_b = \gamma_{ab}$ must hold, hence eq.(6) becomes[7]:

$$\frac{d^2 s^A_a}{d\lambda^2} = R_{\alpha a\beta}{}^j k^\alpha k^\beta s^A_j = R_{\alpha a\beta}{}^b k^\alpha k^\beta s^A_b . \tag{10}$$

Last equation allows us to construct the Jacobi map with the following ansatz:

$$J^A_B(\lambda, \lambda_o) = s^A_a(\lambda) \left\{ \left[ \left( \frac{k^\mu \partial_\mu s}{k^\mu u_\mu} \right)^{-1} \right]^a_B \right\}_{\lambda=\lambda_o} = s^A_a(\lambda) \left\{ \left[ \left( u_\tau^{-1} \partial_\tau s \right)^{-1} \right]^a_B \right\}_{\lambda=\lambda_o} , \tag{11}$$

where the matrix in brackets is given by the initial conditions of eq.(7). In such a way, the area distance is immediately given by:

$$d_A^2 = \det\left( J^A_B(\lambda_s, \lambda_o) \right) = \frac{\sqrt{\gamma(\lambda_s)}}{\frac{1}{4}\left[\det\left(u_\tau^{-1}\partial_\tau \gamma^{ab}\right)\gamma^{3/2}\right]_o} , \qquad \gamma \equiv \det \gamma_{ab} \ . \tag{12}$$

For the full rigorous derivation of eq.(12), we refer to [1]. Last expression is the main result we are showing here. First of all, it is the first exact non-perturbative expression for the area distance and once again we stress that it has been possible to find it thanks to the properties of the GLC gauge.

[4] GLC gauge also provides a gauge invariant averaging procedure for spacelike domains [16, 17] and on the light-cone [3, 4, 6].

[5] More generally, $k^\mu = \omega\Upsilon^{-1}\delta^\mu_\tau$ where $\omega$ is an arbitrarily normalization constant.

[6] This result has been rigorously proved in [1] where $\xi^w$ was left different to 0.

[7] Constancy of $\xi^a$ is crucial for such equation.

Moreover, the result appears as the product of two terms evaluated in two different regions of the space-time: numerator is related to the metric at the source while the derivatives with respect to $\tau$ at the observer appear in the denominator. This derivatives take in account the aberration effect due to the peculiar velocity of the observer once we express the GLC quantities with other sets of coordinates (this effect must appear if the geodesic observer is not static anymore; in particular, from the definition of $d_A^2$ the solid angle $d\Omega_o$ changes due to relativistic effects). In fact for a static observer we expect that $\frac{1}{4}\left[\det\left(u_\tau^{-1}\partial_\tau\gamma^{ab}\right)\gamma^{3/2}\right]_o = \sin\tilde{\theta}_o^1$, which means that no aberration are present just like happens in the FRW limit where observer is static. Because of this, in the following sections, all of the evaluations at the observer will be compared to the pure $\sin\tilde{\theta}_o^1$, in order to correctly interpret the results we will get. Before concluding this section, it is important to underline that the GLC gauge too admits the pure $\sin\tilde{\theta}_o^1$: in particular, it can be achieved by a time independent transformation which fixes a residual gauge freedom still present.

Next sections are dedicated to the expression of our result in two well-know gauges: the SG and the longitudinal gauge (LG), focusing on the observer effects coming out.

## 4 First-order Jacobi map in the synchronous gauge

Now let us express our result in terms of the SG coordinates. Studying what happens in this gauge is useful because it shares the same proper time of the GLC gauge [3] ($\tau = t$); moreover static observers are geodesic in both gauges, so any peculiar velocity correction is expected at the observer position. Instead of the usual carthesian coordinates:

$$ds^2_{SG} = -dt^2 + a^2(t)\left[(1-2\bar{\psi})\delta_{ij} + D_{ij}\bar{E}\right]dx^i dx^j \quad ; \qquad D_{ij} = \partial_i\partial_j - \frac{1}{3}\delta_{ij}\Delta_3 \tag{13}$$

where $\Delta_3$ is the 3-dimensional laplacian, we adopt the spatial polar ones; therefore the line element appears as follows:

$$ds^2_{SG} = -dt^2 + a^2(t)\left[(1-2Z)dr^2 - 2S_a dr d\theta^a + h_{ab}d\theta^a d\theta^b\right] \tag{14}$$

with

$$Z = \bar{\psi} - \frac{1}{2}\left(\frac{\partial^2}{\partial r^2} - \frac{1}{3}\Delta_3\right)\bar{E} \quad ; \qquad S_a = -\left(\partial_r - \frac{1}{r}\right)\partial_a\bar{E},$$
$$h_{ab} = \gamma^0_{ab}\left[1 - 2\bar{\psi} - \left(\frac{1}{3}\Delta_3 - \frac{1}{r}\partial_r\right)\bar{E}\right] + \nabla_a\partial_b\bar{E}. \tag{15}$$

What we want to do is expressing the GLC quantities with the polar synchronous coordinates. At first-order in perturbation theory, the transformation is:

$$\tau = t, \tag{16}$$
$$w = \eta_+ + \frac{1}{2}\int_{\eta_+}^{\eta_-} dx\, Z(\eta_+, x, \theta^a), \tag{17}$$
$$\tilde{\theta}^a = \theta^a + \frac{1}{2}\int_{\eta_+}^{\eta_-} dx\, \chi^a(\eta_+, x, \theta^a)\,, \tag{18}$$

where $\eta_\pm = \eta \pm r$ are the light-cone variables, $\partial_\pm = (\partial_\eta \pm \partial_r)/2$ and $\chi^a = S^a + \frac{1}{2}\gamma_0^{ac}\int_{\eta_+}^{\eta_-} dx\,\partial_c Z(\eta_+, x, \theta^a)$. Thanks to the equivalence $\tau = t$ at every space-time position (and at every order in perturbation theory), two gauges can share the same static observer as geodesic one. In such a way, once we have required that two sets of coordinates become equal as $r \to 0$, we expect that $\frac{1}{4}\left[\det\left(u_\tau^{-1}\partial_\tau\gamma^{ab}\right)\gamma^{3/2}\right]_o =$ $\sin\theta_o$. However, the explicit first-order calculation gives the following result:

$$\frac{\sin\theta_o}{\det\left(\partial_\tau s_b^B(\lambda_o)\right)} = 1 - \frac{1}{2}\left(\Delta_3 - 3\partial_r^2\right)\bar{E}_o = 1 - \frac{1}{2}\left(\frac{\Delta_2}{r^2} - 2\partial_r^2 + 2\frac{1}{r}\partial_r\right)\bar{E}_o\,. \tag{19}$$

The extra terms appearing here cannot be interpretated as the observer peculiar velocity's aberration because it is null with our choice. By the way, thay can be erased by fixing the residual gauge freedom still present in the synchronous gauge, as we said for the GLC gauge: to do this, let us expand $\bar{E}$ with respect to the carthesian coordinates around the observer position, so[8] $\bar{E} = E_0 + E'\eta + E_i x^i + \frac{1}{2}E''\eta^2 + \frac{1}{2}E_{ij}x^i x^j + E_i' x^i \eta + \ldots$ where all coefficients are constant. Eq.(19) becomes:

$$\det\left(\partial_\tau s_b^B(\lambda_o)\right) = \sin\theta_o\left[1 + \frac{1}{2}\left(\delta^{ij}E_{ij} - \frac{3}{r^2}E_{ij}x^i x^j\right)\right]_o \quad ; \tag{20}$$

therefore the spatial coordinates transformation $x^i \to \bar{x}^i = x^i + \frac{1}{2}L^i_j x^j$ erases extra terms in eq.(20) at the present time with the choice $E_{ij} = 0$, i.e. the right physical interpretation is recovered.

A part the matter of the observer, the full first-order expression of $d_A$ in the synchronous gauge looks like:

$$\begin{aligned}\left(d_A^2\right)_{SG} = \frac{a_s^2 r_s^2 \sin\theta_s}{\sin\theta_o}\Bigg\{1 &+ \left(\frac{3}{2}\partial_r^2 - \frac{1}{2}\Delta_3\right)\bar{E}_o - 2\bar{\psi}_s + \left(\frac{1}{6}\Delta_3 - \frac{1}{2}\partial_r^2\right)\bar{E}_s \\ &+ \frac{1}{2}\int_{\eta_s^+}^{\eta_s^-} dx\gamma_0^{ab}\partial_a\partial_b\left(\partial_r - \frac{1}{r}\right)\bar{E} - \frac{1}{4}\int_{\eta_s^+}^{\eta_s^-} dx\gamma_0^{ab}\int_{\eta_s^+}^{x} dy\partial_a\partial_b\left[\bar{\psi} + \left(\frac{1}{6}\Delta_3 - \frac{1}{2}\partial_r^2\right)\bar{E}\right]\Bigg\}.\end{aligned} \tag{21}$$

Before concluding, let us stress that the perturbed result appears more complicated than the exact one of the GLC; some multiple integrals on the light-cone are involved and any factorization among source and observer are present anymore. Previous calculations of $d_A$ in the SG are available in the literature; for instance, in [18], the area distance is given up to the second order only for a dust-dominated while eq.(21) is independent by the sources and the dynamics. So far, we were not able to prove the agreement of the two expressions up to a residual gauge fixing.

In the next section, we will discuss the result obtained once we transform the GLC gauge result in the LG.

## 5 First-order Jacobi map in the longitudinal gauge

In this presentation, we limit ourselves to discuss the first-order LG. A full treatment of the second-order perturbation (the so-called Poisson gauge [19]) is given in Appendix B of [1]. Here we will discuss only the second-order peculiar velocity effect. Expressing the GLC area distance with the coordinates of LG is very interesting for a qualitative matter: in the LG, static observer is no longer geodesic, so the aberration effect due to his/her relativistic velocity can be considered. This different physical situation generates some technical difficulties in the coordinates transformation from GLC to LG: in the previous section we have imposed that two sets of angles are equal at the observer position at every time and it has been possible because in both gauges static observer remains static. This degeneracy is broken in the LG therefore we can impose the equality among GLC and LG angles only at a given time (we choose today).

[8] We choose $\eta_o = 0$ in order to simplify the notation.

Having this in mind, the LG element line we start from is:

$$ds^2_{LG} = a^2(\eta)\left[-(1+2\phi)d\eta^2 + (1-2\psi)(dr^2 + r^2 d^2\Omega)\right]. \tag{22}$$

As before, we adopt the polar spatial coordinates; moreover let us underline that $\phi \neq \psi$ which means that any dynamical constrain from Einstein equations has been used. Therefore the coordinates transformation we need for is given by:

$$\tau = \int_{\eta_{in}}^{\eta} d\eta' a(\eta') + a(\eta) P(\eta, r, \theta^a)\,, \tag{23}$$

$$w = \eta_+ + Q(\eta_+, \eta_-, \theta^a)\ \,, \tag{24}$$

$$\tilde{\theta}^a = \theta^a + \frac{1}{2}\int_{\eta_o}^{\eta_-} dx\ \left[\gamma_0^{ab}\partial_b Q\right](\eta_+, x, \theta^a)\ \,, \tag{25}$$

where $P(\eta, r, \theta^a) = \int_{\eta_{in}}^{\eta} d\eta' \frac{a(\eta')}{a(\eta)}\phi(\eta', r, \theta^a)$, $Q(\eta_+, \eta_-, \theta^a) = \int_{\eta_o}^{\eta_-} dx\, \frac{1}{2}(\psi + \phi)(\eta_+, x, \theta^a)$ and $\eta_{in}$ is a early enough time when the integrand was negligible. In such a way, as did before, we find that the area distance is:

$$\begin{aligned}\left(d_A^2\right)_{LG} = \frac{a_s^2 r_s^2 \sin\theta_s}{\sin\theta_o}\Bigg\{1 - 2\int_{\eta_{in}}^{\eta_o} d\eta' \frac{a(\eta')}{a(\eta_o)}\partial_r\phi\,(\eta', 0, \theta^a) - 2\psi_s -\\ -\frac{1}{2}\int_{\eta_o}^{\eta_s^-} dx\gamma_0^{ab}\int_{\eta_o}^{x} dy \frac{1}{2}\partial_a\partial_b\left[\psi(\eta_+, y, \theta^a) + \phi(\eta_+, y, \theta^a)\right]\Bigg\}\,,\end{aligned} \tag{26}$$

while the observer term becomes:

$$\frac{\sin\theta_o}{\det\left(\partial_\tau s_b^B(\lambda_o)\right)} = 1 - 2\partial_r P_o\,. \tag{27}$$

Once again, we get that the $d_A$ is no longer separable and the pure observer term differs from $\sin\theta_o$; however we cannot remove the extra term in eq.(27) as we did before because LG has no residual degrees of freedom (this is true also in PG [20]). This correction can be correctly view as the effect of a Lorentz boost with peculiar velocity $\vec{v}$ on the observer solid angle $d\Omega$ appearing in the definition of $d_A^2$: in fact, in general $d\Omega = \frac{1-v^2}{(1-\vec{v}\cdot\vec{n})^2}\,d\widetilde{\Omega}$, where $\vec{n}$ is the unit vector connecting observer and source, so we can expect that $d_A^2 = \left[\frac{(1-\vec{v}\cdot\vec{n})^2}{1-v^2}\right]_o\left(\frac{\sqrt{\gamma_s}}{\sin\theta_o}\right)$. In such a way, first-order expansion of the aberration can reproduce the extra term in eq.(27) if $v_i = -a(\eta)\,\partial_i P_o$ and[9] $n^i = -a^{-1}(\eta)\,\delta_r^i$. Let us notice that the velocity we are interested in is deeply related to the transformation law among $\tau$ and $\eta$. In fact, eq.(23) can be seen as $\tau = \tau^{(0)} + \tau^{(1)}$ so, at the same way, $v_i = -\partial_i\tau^{(1)}$.

In order to convince ourselves of this, let us report[10] the result of a second-order expansion in this gauge. We get that the second-order correction is given by:

$$\frac{\sin\theta_o}{\det\left(\partial_\tau s_b^B(\lambda_o)\right)} = \left[(1-\partial_r P)^2 + \nabla_i P\nabla^i P - 2\psi_o\partial_r P\right]_o - \int_{\eta_{in}}^{\eta_o} d\eta' \frac{a(\eta')}{a(\eta_o)}\partial_r\left(\phi^{(2)} - \psi^2 + \nabla_i P\nabla^i P\right), \tag{28}$$

where $\phi^{(2)}$ is the second-order perturbation of $g_{\eta\eta}$. Eq.(28) must be equal to the expansion (up to the second order) of the Lorentz boost: $1 + v^2 - 2\vec{v}\cdot\vec{n} + (\vec{v}\cdot\vec{n})^2$. If we identify $v_i = -\partial_i\tau^{(1)} - \partial_i\tau^{(2)}$ and

[9] $\vec{n}$ is a zero-th order quantity because it multiplies a $\vec{v}$, which is a first-order one.

[10] Technical details are given in [1].

$n^i = -a^{-1}(\eta)(1+\psi)\,\delta^i_r$ the interpretation still perfectly works. Once again, the peculiar velocity we need is related to the (second-order) perturbation of the coordinates transformation among $\tau$ and $\eta$. In the SG, there aren't perturbations at any order of expansion: this is another way to understand the different kind of correction we got.

The last check we left to do is the consistency of the results in the two gauges. This will be discussed in the next section.

## 6 Comparison of the results in two gauges

The comparison among two results must be studied in order to show the physical equivalence of the expressions. In order to do that we have to consider that $d_A^2$ is evaluated in two different point of space-time; therefore, under a first-order infinitesimal gauge transformation, it changes as a bi-scalar object does, so following equality must hold:

$$\left(d_A^2\right)_{LG} = \left(d_A^2\right)_{SG} - \left[\epsilon^\mu_{(1)}\partial_\mu\left(d_A^2\right)^{(0)}\right]_o - \left[\epsilon^\mu_{(1)}\partial_\mu\left(d_A^2\right)^{(0)}\right]_s , \tag{29}$$

where $\epsilon^\mu_{(1)} = \frac{1}{2}\left(a\bar{E}', \partial^i\bar{E}\right)$ is the infinitesimal generator and the superscript (0) indicates the zero-th order quantities. The relations among $\phi$, $\psi$ and $\bar{E}$ are [21] $\phi = -\frac{a'\bar{E}'}{2a} - \frac{\bar{E}''}{2}$ and $\psi = \bar{\psi} + \frac{a'\bar{E}'}{2a} + \frac{1}{6}\Delta_3\bar{E}$. Starting from here and after some technical passages, eq.(29) gives the following condition

$$\left(\partial_r\bar{E}'\right)_o = \left(\frac{3}{2}\partial_r^2 - \frac{1}{2}\Delta_3\right)\bar{E}_o + \frac{1}{2}\left(\frac{\Delta_2}{r^2}\bar{E}\right)_o + \left[\left(r\gamma_0^{ab}\right)_s - \left(r\gamma_0^{ab}\right)_o\right]\left(\partial_a\partial_b\partial_+\bar{E}\right)_o , \tag{30}$$

where second and third terms approach to the observer along the light-cone. At this point, let us expand again $\bar{E}$ around the observer position, just as we have done in section 4, so eq.(30) becomes[11] side by side:

$$\left(\frac{E_i'x^i}{r}\right)_o = \left(\frac{E_i'x^i}{r}\right)_o - \frac{1}{2}\left[\left(\partial_\theta^2 + \frac{1}{\sin^2\theta}\partial_\phi^2\right)\left(E_{ij}\frac{x^ix^j}{r^2}\right)\right]_o \tag{31}$$

which means that $\left(\partial_\theta^2 + \frac{1}{\sin^2\theta}\partial_\phi^2\right)\left(E_{ij}\frac{x^ix^j}{r^2}\right)_o = 0$. This condition can be satisfied by partially fixing the residual gauge freedom of the SG. The amazing result is that here we need the same gauge fixing that we already discussed in order to recover the pure $\sin\theta_o$ in the SG. We can understand this by noticing that in LG all of the equal time hypersurfaces are shear-free, so it is not surprising that the same condition have to be required in the SG, even if only at the observing time. On the contrary, the peculiar velocity term appearing in the LG expression automatically disappears in the SG.

## 7 Summary and conclusions

We have briefly shown and commented the results of [1]. We have exactly solved the geodesic deviation equation only by using the GLC gauge. This furnishes us a fully non-perturbative expression for the Jacobi map, given by eq.(11), and for the area distance, eq.(12). Remembering that also redshift is exactly given in the GLC gauge, we are able to express the exact non-perturbative luminosity distance by the reciprocity relation among $d_A$ and $d_L$.

This solution for $d_A$ appears as a nice factorization between a pure source term and a pure observer one; this is true only in the GLC gauge. In fact, once we have expressed the GLC quantities with the SG coordinates and the LG ones, this separation is no longer true. Furthermore, $d_A$ correctly takes

[11] Along the light-cone, $\eta$ can be replaced by $-r$. This is crucial here.

in account the observer corrections due to his/her peculiar velocity, just as expected from the general definition of the area distance: they are null in the first-order SG, where the static observer is geodesic, and they are equals to the Lorentz boost prediction up to second-order in the LG, when the geodesic observer can't be static anymore. The consistency of two results has been provided too: they perfectly agree if the anisotropy around the SG observer is removed by a residual gauge fixing.

The expression we gave takes care of all of the inhomogeneities and anisotropies that in principle could be in the Universe; the knowledge of an exact Jacobi map can be used to study non-perturbatively other physical quantities (for instance weak lensing). Moreover it is a really interesting result because it can allow for a non-perturbative approach to the back reaction problem. To do this, the main purpose we can reach is finding a convenient formulation of Einstein's equations in the GLC gauge which is still missing.

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