

# The multiple-scattering series in few-nucleon systems

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**Abstract.** We discuss under which circumstances the resummation of the multiple-scattering series is justified from an EFT point of view. The application to  $\pi d$  and  $\bar{K}d$  scattering is briefly discussed.

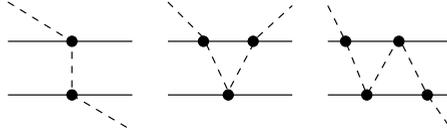
## 1. Introduction

The multiple-scattering series (MSS) is a particular class of diagrams where a meson (e.g. a pion) scatters many times between a pair of other particles such as nucleons, see Fig. 1 for illustration. This class of diagrams is especially relevant for an accurate description of meson–nucleus scattering, which is an important source of information about meson–nucleon (MN) scattering lengths [1–3]. With the recent advances of effective field theories, in particular chiral perturbation theory (ChPT), an accurate extraction of the MN scattering lengths with a controlled uncertainty became possible. In particular, the  $\pi N$  scattering lengths were extracted from a recent ChPT analysis of pionic hydrogen and deuterium atomic data with an unprecedented accuracy [1, 2]. Such an analysis calls for rigorous control over higher-order ChPT corrections to pion–nucleus scattering with higher-order terms of the MSS representing the most prominent ones.

In this paper we follow Ref. [4] to demonstrate that diagrams of the MSS topology are a factor of  $\pi^2$  larger compared to the standard ChPT counting predictions. This finding raise a question of the theoretical uncertainty estimate. The consequences of this enhancement for meson–deuteron scattering with a natural MN scattering length are discussed. In particular, we compare the results of the resummed MSS with a perturbative approach for  $\pi d$  scattering and discuss some insights into  $\bar{K}d$  scattering.

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**Figure 1.** Second, third, and fourth term in the MSS. Solid lines denote nucleons, dashed mesons.

## 2. MSS in perturbation theory

Already in 1953 Brückner pointed out that the MSS should provide a large contribution to pion–deuteron scattering [5]. Weinberg came to the same conclusion [7] within the ChPT classification of diagrams based on a systematic, model independent expansion of the amplitudes in terms of momenta and pion masses measured in units of  $\Lambda \sim 4\pi F_\pi \sim 1$  GeV. In particular, the first diagram of Fig. 1, the so-called double-scattering term, appears already at leading order (LO) in the chiral expansion of the 3-body ( $\pi NN$ ) operators [7], while the next diagram, the triple-scattering term, is suppressed by two orders in the expansion ( $N^2LO$ ).

Indeed, as long as nucleons are considered static and the incoming (outgoing) meson is at rest, the explicit evaluation of the double- and triple-scattering<sup>1</sup> terms yields [4, 8]

$$A^{(2)}(\mathbf{Q}) = \frac{a^2}{\mathbf{Q}^2}, \quad (1)$$

$$A^{(3)}(\mathbf{Q}) = -4\pi a^3 \int \frac{d^3l}{(2\pi)^3} \frac{1}{\mathbf{l}^2(\mathbf{l} - \mathbf{Q})^2} = -\frac{a^3}{2\pi|\mathbf{Q}|} J_0, \quad (2)$$

where  $\mathbf{Q} = \mathbf{p}' - \mathbf{p}$  denotes the three-momentum transfer between the incoming and outgoing nucleons, and  $a \simeq M_\pi/(8\pi F_\pi^2)$  is used as a measure of natural MN scattering lengths. Assuming for the dimensionless integral  $J_0$  to be of order 1, one finds the parametric suppression of the triple-scattering term predicted by Weinberg’s power counting

$$\frac{A^{(3)}}{A^{(2)}} = -\frac{aQ}{2\pi} J_0 \sim \left(\frac{M_\pi}{4\pi F_\pi}\right)^2 J_0. \quad (3)$$

However, an explicit evaluation of  $J_0$ ,

$$J_0 = \int_0^\infty \frac{dx}{x} \log\left(\frac{x+1}{x-1}\right)^2 = \pi^2, \quad (4)$$

leads to the numerical enhancement of  $A^{(3)}$ . To account for this effect the triple-scattering term was promoted to next-to-leading (NLO) order in Ref. [1, 2]. The origin of the enhancement was identified in Ref. [8] with the specific topology of the triple-scattering term consisting of two consecutive pion exchanges with Coulombic-type (no pion mass scale) pion propagators. This leads to concerns regarding the enhancement of the quadruple-scattering term, which up to higher-order terms reads

$$A^{(4)}(\mathbf{Q}) = (4\pi)^2 a^4 \int \frac{d^3l_1}{(2\pi)^3} \frac{d^3l_2}{(2\pi)^3} \frac{1}{\mathbf{l}_1^2(\mathbf{l}_1 - \mathbf{l}_2)^2(\mathbf{l}_2 - \mathbf{Q})^2} = -a^4 \log \frac{Q}{\mu} + \frac{f_0(\mu)}{32\pi^2}. \quad (5)$$

The divergent integral in Eq. (5) was first regularized and then renormalized by adding the regulator-dependent contact term  $f_0(\mu)$ . The comparison of Eqs. (1) and (5) reveals also the  $\pi^2$ -enhancement of

<sup>1</sup> In the evaluation of the triple-scattering term only the relevant for our work (enhanced) contribution is kept. The full result is given in Ref. [8], where it is also shown that the residual terms are of natural size.

the quadruple-scattering term

$$\frac{A^{(4)}}{A^{(2)}} = a^2 Q^2 \sim \pi^2 \left( \frac{M_\pi}{4\pi F_\pi} \right)^4. \quad (6)$$

This kind of enhancement takes place also at higher MSS orders, as discussed in Ref. [4]. The observed numerical enhancement of the MSS diagrams is potentially troublesome for the theoretical uncertainty estimate and predictive power of the theory, since it may suggest the existence of larger-than-expected contact operators.

### 3. MSS resummation

A solution to this problem was proposed in Ref. [4], where it was shown that the UV divergences of the individual MSS terms cancel once the whole series is resummed. Hence, no enhanced short-range operators are expected to contribute in the resummed MSS.

The multiple scattering amplitude for a meson scattering from the static  $NN$  pair can be written as

$$A(\mathbf{Q}) = t_{\pi N}(0) \frac{\pi}{2} \delta^{(3)}(\mathbf{Q}) + \int \frac{d^3 p''}{(2\pi)^3} t_{\pi N}(\mathbf{p} - \mathbf{p}'') \frac{1}{(\mathbf{p} - \mathbf{p}'')^2} A(\mathbf{p}' - \mathbf{p}''), \quad (7)$$

where the elementary MN amplitude reads  $t_{\pi N}(\mathbf{p}) = -4\pi a \hat{g}\left(\frac{|\mathbf{p}|}{\Lambda_{\pi N}}\right)$ , and  $\hat{g}(x)$  takes into account a non-zero range of MN interaction ( $\hat{g}(x) \rightarrow 1$  when  $x \rightarrow 0$ ). From an EFT point of view  $\Lambda_{\pi N}$  plays the role of a regulator, and Eq. (7) is valid when  $Q < \Lambda_{\pi N} \simeq \Lambda$ . Taking the Fourier transform of Eq. (7), one finds the solution for the resummed amplitude in coordinate space

$$A(r) = -\frac{ar}{4\pi(r + ag(r))}. \quad (8)$$

From this one can draw the following conclusions

- The perturbative treatment of Eq. (7) is justified as long as the MN scattering length is natural

$$a \lesssim \Lambda^{-1} \simeq (4\pi F_\pi)^{-1} \implies \frac{a}{r} \sim \frac{Q}{\Lambda} \ll 1. \quad (9)$$

- The expansion of Eq. (8) in  $a/r$  with the subsequent Fourier transform to momentum space reproduces the individual MSS terms discussed in Sect. 2.
- The resummed MSS (Eq. (8)) is much less regulator dependent than the individual contributions.
- For  $\pi d$  scattering taking the limit  $\Lambda_{\pi N} \rightarrow \infty$  in Eq. (8) results in the correction to the scattering length

$$\langle \Delta A \rangle \equiv \langle A_{\Lambda_{\pi N}}(r) - A_{\Lambda_{\pi N} \rightarrow \infty}(r) \rangle \leq 3\% a_{\pi d}. \quad (10)$$

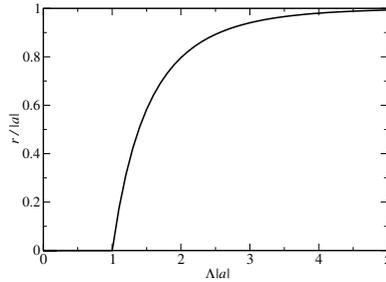
This is fully in line with the estimate of the (natural) contact term at N<sup>2</sup>LO in chiral counting [1, 2, 4].

### 4. A problem with the pole

The resummed MSS result exhibits an unphysical pole for attractive MN interaction ( $a < 0$ ) since the denominator of Eq. (8) may go to zero in this case,<sup>2</sup> see Fig. 2. This can be naively interpreted as a

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<sup>2</sup> A similar effect is discussed in Ref. [9] as the result of the solution of the momentum-space integral equation.



**Figure 2.** Position of the pole at  $r = -ag(r)$  for  $a < 0$ .

failure of the resummed results. However, for a natural scattering length the pole occurs very close to the origin for  $r \sim a < \Lambda^{-1}$  or  $Q > \Lambda$ , i.e. it is already beyond the region of applicability of the EFT. Thus we believe that the MSS delivers reliable results as long as  $a$  is natural. On the contrary, for an unnaturally large and negative scattering length ( $a\Lambda > 1$ ) the pole indeed appears in the physical region  $r \sim Q^{-1} \sim a > \Lambda^{-1}$ , but as long as  $Q^{-1} \sim a$  a shallow MN state can be formed. Therefore, the 2-body input to the integral equation (7) should be modified to include at least the unitarity corrections

$$t_{\pi N}(\mathbf{p}) = -\frac{4\pi}{1/a - ip} \hat{g} \left( \frac{|\mathbf{p}|}{\Lambda_{\pi N}} \right). \quad (11)$$

In addition, nucleon recoil effects should also be considered [10]. We expect that keeping these effects would shift the pole towards the origin, i.e. outside the physical region.

## 5. Some applications

Due to the natural size of the  $\pi N$  scattering length both perturbative and non-perturbative treatments are applicable to  $\pi d$  scattering: in the perturbative case the numerical enhancement of the short-range operator turns out to be not sufficient to overcome the strong parametric suppression of the quadruple-scattering term, see Eq. (6), while the non-perturbative scheme is applicable since the pole does not affect the results. Given the larger magnitude of the kaon–nucleon scattering lengths ( $\sim 1$  fm), one may wonder if the resummed approach is applicable to kaonic deuterium [11, 12]. We believe that it does apply for the following reasons: first, in the case of isovector dominated and/or absorptive interactions—both true for  $\bar{K}N$  interactions—no pole appears on the real axis. Second, the natural hard scale in the EFT for the  $\bar{K}d$  system is  $\Lambda \sim 2M_\pi$  [13], which means that  $\Lambda a \sim 1$  and the pole should lie not far from the origin, see Fig. 2.

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