Poincaré Sphere and a Unified Picture of Wigner's Little Groups

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Abstract. It is noted that the Poincaré sphere for polarization optics contains the symmetries of the Lorentz group. The sphere is thus capable of describing the internal space-time symmetries dictated by Wigner's little groups. For massive particles, the little group is like the three-dimensional rotation group, while it is like the two-dimensional Euclidean group for massless particles. It is shown that the Poincaré sphere, in addition, has a symmetry parameter corresponding to reducing the particle mass from a positive value to zero. The Poincaré sphere thus the gives one unified picture of Wigner's little groups for massive and massless particles.

1 Introduction

In 1939 [1], Wigner considered the subgroups of the Lorentz group whose transformations leave the four-momentum of a given particle invariant. These subgroups therefore dictate the internal space-time symmetries of the particles. They are called Wigner's little groups. The little groups for massive and massless particles are like $O(3)$ and $E(2)$ respectively.

The physics of $O(3)$-like little group is transparent. For a massive particle at rest, it can have the spin whose direction can be rotated according to the three-dimensional rotation group. On the other hand, the $E(2)$-like group for massless particles has a stormy history, and its physics had not been settled until 1990 [2]. It is not difficult to see its rotational degree of freedom corresponds to the helicity. The two translation-like variable collapse into that of one gauge-transformation variable.

The question is whether it is possible to derive this $E(2)$-like little group as a limiting case of the $O(3)$-like little group massive particles. It is not possible to reduce the value of mass continuously to zero within the framework of the Lorentz group because the mass is an invariant quantity. However, it is possible to make the momentum infinite where the mass-hyperbola becomes tangent to the light cone, and then reduce the energy to a finite value along this light cone [2].

It was noted that the Poincaré sphere is based on the two-by-two coherency matrix consisting of four stokes parameters constructed from the two-component Jones vector [3, 4]. The Jones vector consists of the two transverse components of a light beam. The transformation applicable to this vector is that of $SL(2, c)$ isomorphic to the Lorentz group. Thus, the Poincaré sphere contains the symmetries of the Lorentz group [5].

In addition, the coherency matrix measures the coherence or decoherence between the two transverse electric fields [6, 7]. The determinant of this two-by-two matrix gives the degree of coherence. It is shown in this note that the variation of this degree corresponds to variation of the mass variable

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not allowed in the Lorentz group. Thus, the Poincaré sphere allows to reduce the mass continuously to zero from a positive value.

In Sec. 2, we present the two-by-two representation of the Lorentz group where both the space-time and momentum-energy four-vectors are written in the form of the two-by-two matrix. The determinant of the momentum matrix is $(mass)^2$. In Sec. 3, Wigner's little group is formulated in the language of two-by-two matrices as in the case of the coherency matrix. In Sec. 4, it is noted that the coherency matrix for the Poincaré sphere contains the symmetries of the Lorentz group. In addition, it is shown the degree of coherence corresponds to the particle mass, which can be changed continuously from a positive value to zero.

2 Two-by-two representation of the Lorentz Group

Let us start with the two-by-two matrix

$$X = \begin{pmatrix} t + z & x - iy \\ x + iy & t - z \end{pmatrix}, \tag{1}$$

Then its determinant is $\left(t^2 - z^2 - x^2 - y^2\right)$. Thus, the Lorentz transformation is a determinant-preserving transformation. We can the consider a unimodular matrix of the form [8]

$$G = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \text{and} \quad G^\dagger = \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}, \tag{2}$$

with

$$\det(G) = 1, \tag{3}$$

and the transformation

$$X' = GXG^\dagger. \tag{4}$$

This can be explicitly written as

$$\begin{pmatrix} t' + z' & x' - iy' \\ x + iy & t' - z' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t + z & x - iy \\ x + iy & t - z \end{pmatrix} \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}. \tag{5}$$

Since G is not a unitary matrix, Eq.(4) not a unitary transformation. This two-by-two transformation can be translated into the language of the conventional four-by-four representation of the Lorentz group [6, 9].

The energy-momentum four-vector can also be written as a two-by-two matrix. It can be written as

$$P = \begin{pmatrix} p_0 + p_z & p_x - ip_y \\ p_x + ip_y & p_0 - p_z \end{pmatrix}, \tag{6}$$

with

$$\det(P) = p_0^2 - p_x^2 - p_y^2 - p_z^2, \tag{7}$$

which means

$$\det(P) = m^2, \tag{8}$$

where m is the particle mass.

The Lorentz transformation can be written explicitly as

$$P' = GPG^\dagger. \tag{9}$$

This is an unimodular transformation, and the mass is a Lorentz-invariant variable.

If the elements of the G matrix are complex, but it has six independent parameters due to the condition of Eq.(3), as in the case of the Lorentz group.

Furthermore, the elements of the X and P matrices have to be real. This means that the y components have to vanish. The transformation is applicable only to the t, z, and x coordinates.

In mathematics, the group represented by the two-by-two matrix of Eq.(2) is called $SL(2, c)$ locally isomorphic to the group of Lorentz transformation matrices applicable to the four-dimensional Minkowskian space. The three-parameter subgroup consisting only of real matrices is called $Sp(2)$ locally isomorphic to the group of three-by-three matrices performing Lorentz transformations on the space of t, z, and x.

For a physical system invariant under rotations around the z axis, it is enough to study the $Sp(2)$ subgroup. Here, for all practical purposes, it is sufficient to work with the following three matrices.

$$R(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \quad B(\eta) = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}, \quad S(\lambda) = \begin{pmatrix} \cosh(\lambda/2) & \sinh(\lambda/2) \\ \sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}, \quad (10)$$

corresponding to the rotation around the y axis, and the boost along the z and x directions, respectively. The rotation around the z axis corresponds to

$$Z(\phi) = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}. \tag{11}$$

3 Internal Space-time Symmetries

In 1939 [1], Wigner considered the subgroups of the Lorentz group whose transformations leave the four-momentum of a given particle invariant. We shall use the word "Wigner matrix" which leaves the four-momentum invariant. Then the Wigner matrix W is defined as

$$P = WPW^{\dagger}. \tag{12}$$

The four-momentum can be brought to the form proportional to

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{13}$$

if the mass of the particle is positive or imaginary respectively. If the mass is positive, it can be brought to its rest frame with zero momentum. If the mass is imaginary, it can be bought to the frame with zero energy and momentum along the z direction. The Wigner matrices which leave the these four-momenta are $R(\theta)$ and $S(\lambda)$ of Eq.(10) respectively.

If the particle is massless, the four-momentum becomes proportional to

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{14}$$

if the momentum is along the z direction. Its Wigner matrix takes the form

$$\begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix}. \tag{15}$$

While the physics of the Wigner matrix is transparent for massive and imaginary-mass particles, this triangular matrix is strange. If it is translated into the four-by-four matrix, the transformation matrix becomes

$$\Gamma(\gamma, \phi) = \begin{pmatrix} 1 + \gamma^2/2 & -\gamma^2/2 & \gamma & 0 \\ \gamma^2/2 & 1 - \gamma^2/2 & \gamma & 0 \\ -\gamma & \gamma & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{16}$$

applicable to the (t, z, x, y) coordinates. This matrix is in Wigner's original paper but has a stormy history. It was later shown that this matrix performs a gauge transformation when applied to the photon four-potential [2].

Our next question is whether it is possible to obtain the Wigner matrix of Eq.(15) starting from $R(\theta)$ of Eq.(10) for the massive particle. It is not possible to change the mass within the frame of the Lorentz group. However, let us look at the matrix $B(\eta)$ of Eq.(10). This matrix corresponds to a Lorentz boost which will transform the momentum of the massive particle from zero to p along the z direction, with the parameter η defined as

$$e^\eta = \left[\frac{p_0 + p}{p_0 - p} \right]^{1/2}, \tag{17}$$

which becomes $2p/m$, for large values of the momentum. This limit can also be obtained for small values of m, even though the variation of this parameter is not allowed in within the framework of the Lorentz group.

Let us apply this Lorentz boost to the Wigner matrix $B(\theta)$ for the massive particle. Then

$$B(\eta)R(\theta)B^{-1}(\eta) = \begin{pmatrix} \cos(\theta/2) & -e^\eta \sin(\theta/2) \\ e^{-\eta} \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}. \tag{18}$$

We can make η very large and make θ very small, so that $e^\eta \sin(\theta/2)$ remains as a finite number γ. Then the above expression becomes the Wigner matrix of Eq.(15). This process is illustrated in Fig. 1(b).

4 Symmetries derivable from the Poincaré Sphere

The Poincarś sphere was originally developed as an instrument for studying polarization of light waves [3, 4]. If the light wave propagates along the z direction, its electric field is perpendicular to the momentum. It can have its x and y component. Let us start with the Jones vector defined as

$$\begin{pmatrix} \psi_1(z, t) \\ \psi_2(z, t) \end{pmatrix} = \begin{pmatrix} \exp\left[i(kz - \omega t)\right] \\ \exp\left[i(kz - \omega t)\right] \end{pmatrix}, \tag{19}$$

where the upper and lower component of this column vector and x and y components of the electric field. We start here with the case with two identical components.

To this vector, we can apply the phase-shift matrix of the form

$$\begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}. \tag{20}$$

This matrix has the same mathematical expression as the rotation matrix given in Eq.(11), but performs a different physical transformation in polarization optics.
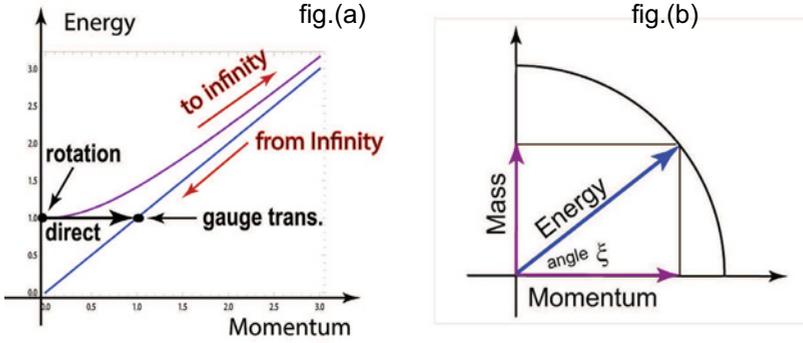
Figure 1. Transition from the $O(3)$-like little group for a massive particle to the $E(2)$-like little group. Within the framework of the Lorentz group, the particle mass is a Lorentz invariant quantity. However all mass hyperbolas collapse into the light cone when the particle momentum becomes infinity. Thus, it is possible to go to infinity first and come back after jumping to the light cone, as described in fig.(a). This figure also tells how to go to the massless case directly within the symmetry available from the Poincaré sphere. As is illustrated in fig.(b), it is possible have a symmetry where the mass and momentum obey the triangular rule for a fixed energy in the symmetry offered by the Poincaré sphere.

We can apply the attenuation matrix of the form of $B(\eta)$ given in Eq.(10), and the matrix of the form of $R(\theta)$ in order to rotate the polarization coordinate around the z direction. Then, repeated applications of these four matrices will lead to the most general form of $SL(2,c)$ matrix given in Eq.(2), and the application of this matrix will lead to the most general form of the Jones vector. These two-by-two matrices perform two different transformations in physics, as shown in Table 1.

However, the Jones vector cannot tell us whether the two components are coherent with each other. In order to address this important degree of freedom, we use the coherency matrix defined as [3, 7]

$$C = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}, \tag{21}$$

where

$$< \psi_i^* \psi_j > = \frac{1}{T} \int_0^T \psi_i^*(t + \tau) \psi_j(t) dt, \tag{22}$$

where T is a sufficiently long time interval. Then, those four elements become [6]

$$S_{11} = < \psi_1^* \psi_1 > = 1, \qquad S_{12} = < \psi_1^* \psi_2 > = (\cos \xi) e^{-i\phi},$$
$$S_{21} = < \psi_2^* \psi_1 > = (\cos \xi) e^{+i\phi}, \qquad S_{22} = < \psi_2^* \psi_2 > = 1. \tag{23}$$

The diagonal elements are the absolute values of ψ_1 and ψ_2 respectively. The angle ϕ could be different from the value of the phase-shift angle given in Eq.(20), but this difference does not play any role in our reasoning. The off-diagonal elements could be smaller than the product of ψ_1 and ψ_2, if the two polarizations are not completely coherent.

The angle ξ specifies the degree of coherency. If it is zero, the system is fully coherent, while the system is totally incoherent if ξ is 90°. This can therefore be called the "decoherence angle." While the most general form of the transformation applicable to the Jones vector is G of Eq.(2), the transformation applicable to the coherency matrix is

$$C' = G \, C \, G^\dagger. \tag{24}$$

Table 1. Polarization optics and special relativity sharing the same set of matrices. Each matrix has its well-defined role in optics and relativity. The determinant of the Stokes or the four-momentum matrix remains invariant under $SL(2,c)$ transformations. This determinant is the $(mass)^2$ in particle physics, while it corresponds to the decoherence parameter in optics. In particle physics, the determinant cannot be changed, while it is a variable that can be changed.

Polarization Optics	Transformation Matrix	Particle Symmetry
Phase shift by ϕ	$\begin{pmatrix} e^{\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}$	Rotation around z.
Rotation around z	$\begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$	Rotation around y.
Squeeze along x and y	$\begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}$	Boost along z.
Squeeze along 45°	$\begin{pmatrix} \cosh(\lambda/2) & \sinh(\lambda/2) \\ \sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}$	Boost along x.
$(\sin \xi)^2$	Determinant	$(mass)^2$

The determinant of the coherency matrix is invariant under this transformation, and it is

$$\det(C) = (\sin \xi)^2. \tag{25}$$

Thus, angle ξ remains invariant. In the language of the Lorentz transformation applicable to the four-vector, the determinant is equivalent to the $(mass)^2$ and is therefore a Lorentz-invariant quantity.

The coherency matrix of Eq.(21) can be diagonalized to

$$\begin{pmatrix} 1 + \cos \xi & 0 \\ 0 & 1 - \cos \xi \end{pmatrix} \tag{26}$$

by a rotation. Let us then go back to the four-momentum matrix of Eq.(6). If $p_x = p_y = 0$, and $p_z = p_0 \cos \xi$, we can write this matrix as

$$p_0 \begin{pmatrix} 1 + \cos \xi & 0 \\ 0 & 1 - \cos \xi \end{pmatrix}. \tag{27}$$

Thus, with this extra variable, it is possible to study the little groups for variable masses, including the small-mass limit and the zero-mass case. For a fixed value of P_0, the $(mass)^2$ becomes

$$(mass)^2 = (p_0 \sin \xi)^2, \quad \text{and} \quad (momentum)^2 = (p_0 \cos \xi)^2, \tag{28}$$
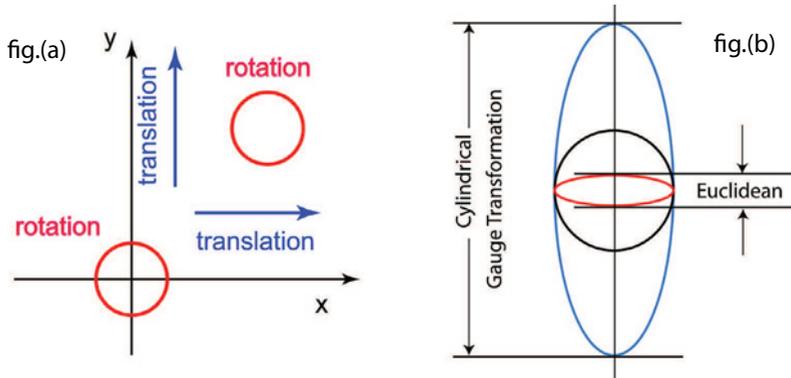
Figure 2. The geometry of the $E(2)$-like little group for massless particles. In terms of There are one rotational and two translations degrees of freedom in the two-dimensional Euclidean group, as shown in fig.(a). However, there is only one gauge degree of freedom. Is it possible to collapse these two translational degrees into one? The answer is that there is a cylindrical group isomorphic to $E(2)$ where both the two translations in the $E(2)$ plane collapse into one translation along the direction perpendicular to the plane, as illustrated in fig.(b).

resulting in

$$(energy)^2 = (mass)^2 + (momentum)^2. \tag{29}$$

This transition is illustrated in Fig. 1. We are interested in reaching a point on the light cone from a mass hyperbola while keeping to the energy fixed [5]. According to this figure, we do not have to make an excursion to infinite-momentum limit. If the energy is fixed during this process, Eq.(29) tells the mass and momentum relation, and Fig. 1b illustrates this relation. Indeed, this momentum-mass relation suggests a possibility of the $O(3, 2)$ space-time symmetry [10].

Within the framework of the Lorentz group, it is possible, by making an excursion to infinite momentum where the mass hyperbola coincides with the light cone, to then come back to the desired point. On the other hand, the mass formula of Eq.(28) allows us to go there directly. The decoherence mechanism of the coherency matrix makes this possible.

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I am very grateful to Professor Eugene Wigner. From 1985 to 1990, I went to Princeton frequently to do physics under his guidance. This is the reason why I am frequently introduced as Wigner's youngest student, as in the present conference.

When I was a graduate student at Princeton (1958-61), I was afraid of him. My thesis advisor was Professor Sam Treiman. I am not the first student of Treiman to be interested in Wigner's 1939 paper on the internal space-time symmetries of elementary particles. Steven Weinberg was Treiman's first student. In 1964, he published a series of papers to make his Wigner's paper useful to particle physics of that time. In one of his papers [11], he discussed massless particles and struggled with the four-by-four matrix given in Eq.(16). This matrix is contained Wigner's original paper [1].

Together with my younger colleagues, namely D. Han and D. Son, we were able to interpret what Weinberg wanted to say as summarized in Fig. 2a [12]. According to this figure, the photon

spin corresponds the rotational on a two-dimensional Euclidean space, whose Cartesian coordinates correspond to the gauge degrees of freedom.

I explained this aspect of his $E(2)$-like little group to Professor Wigner. He became very happy to hear that the translation-like variables correspond to gauge transformations. On the other hand, he noted that there is only one gauge degree of freedom, while there are two translational degrees of freedom.

After some hard work, we came to the conclusion that the $E(2)$ symmetry can be derived from a tangential plane on the north pole of the sphere. For the same sphere, we can consider a cylinder tangential to the equatorial belt of the same sphere. Equivalently, the sphere can be contracted or expanded along the z direction as shown in Fig. 2b [2]. The deformation takes place when the particle (*momentum/mass*) changes from zero to infinity.

The two translational degrees collapse into one translation along the z direction. This cylinder can rotate around the same axis. This rotational degree corresponds to the photon spin. I had a pleasure of publishing this paper with Wigner in the Journal of Mathematical Physics [2]. The editor of this journal was Lawrence Biedenharn at that time. He told me he was very happy to publish this paper in his journal.

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