Balitsky-JIMWLK evolution equation at NLO

Giovanni Antonio Chirilli[1,a]

[1]*Department of Physics, The Ohio State University, Columbus, OH 43210, USA*

Abstract. Wilson line operators are infinite gauge factors ordered along the straight lines of the fast moving particles. Scattering amplitudes of proton-Nucleus or Nucleus-Nucleus collisions at high-energy are written in terms of matrix elements of these operators and the energy dependence of such amplitudes is obtained by the evolution equation with respect to the rapidity parameter: the Balitsky-JIMWLK evolution equation. A brief description of the derivation of the Balitsky-JIMWLK evolution equation at leading order and next-to-leading order will be presented.

1 Introduction

The relevant operators for the description of high energy scattering amplitudes are Wilson lines: infinite gauge factors ordered along the straight lines of the fast moving particles. In quantum mechanics, if one considers a particle propagating at very high energy through an external potential, the wave function of the particle is modified by multiplying the wave function of the particle in free space by a phase factor given by the exponential of the external potential ordered along the line parallel to the particle's velocity. In quantum electrodynamics, instead, is the gauge field $A^\mu(x)$ that enters in the exponential phase factor.

In QCD, since the gauge field is a matrix valued function, the phase factor is a path-ordered exponential along the trajectory of the particle

$$U(x_\perp) = 1 + ig \int_{-\infty}^{\infty} du\, A_\bullet(u\, p_1 + x_\perp) + (ig)^2 \int_{-\infty}^{+\infty} du \int_{-\infty}^{u} dv\, A_\bullet(up_1 + x_\perp) A_\bullet(v\, p_1 + x_\perp) + \dots \quad (1)$$

Here, we used the notation $x_\bullet = \sqrt{\frac{s}{2}} x^-$ and $x_* = \sqrt{\frac{s}{2}} x^+$ with x^+ and x^- light-cone coordinates defined as $x^\pm = \frac{x^0 \pm x^3}{\sqrt{2}}$; p_1^μ and p_2^μ are ligh-cone vector such that $p_1^2 = p_2^2 = 0$ and $p_1 \cdot p_2 = \frac{s}{2}$ with s the Mandelstam variable for the center-of-mass energy.

The Operator Product Expansion (OPE) in Wilson lines [1, 2] allows one to write the scattering amplitude as an expansion in terms of coefficient functions convoluted with matrix elements of Wilson lines ordered along the particles' velocities. The energy dependence of the amplitudes is given by the evolution equations in rapidity of the Wilson line operators.

In high-energy Deep Inelastic Scattering (DIS) the virtual photon emitted by the lepton long before scattering off the hadronic target, splits into a quark and anti-quark pair. The propagation of the color dipole through the hadronic target, which is made mainly by gluon degrees of freedom, is given by

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[a]e-mail: chirilli.1@osu.edu

a scattering amplitude proportional to two Wilson lines. The amplitude can then be written as a convolution of the a coefficient function (the photon impact factor now known at NLO [3, 4]) and a matrix element of a color dipole. The evolution equation of the color dipole represent the first of the Balitsky hierarchy of evolution equations which, in the large N_c approximation, reduces to the Balitsky-Kovchegov equation [1, 5] (for a review see Ref. [6]).

The scattering amplitudes of proton-Nucleus or Nucleus-Nucleus collisions are made of several Wilson lines and the corresponding evolution equation is known as the Balitsky-JIMWLK evolution equation [1, 7].

In the next section we will provide a brief pedagogical introduction to the background field technique and derive as an example the evolution of one Wilson line operator. In section 3 we will present the NLO correction to the evolution equation of one Wilson line. The complete result of the Balitsky-JIMWLK evolution equation can be found in Ref. [8].

2 Leading Order Evolution Equation

In this section we give a brief introduction to the background field method which is used to derive the evolution equation of the matrix elements made of the relevant operators to describe the scattering amplitude.

At high-energy (high-parton density) the energy dependence of the scattering amplitude is encoded in the evolution of matrix elements made of Wilson lines.

Let us indicate with O^{η_1} an operator made of several Wilson lines with rapidity dependence η_1 and if we indicate with $|B\rangle$ the target state, then the scattering amplitude is proportional to

$$\langle B|O^{\eta_1}|B\rangle \tag{2}$$

The Wilson lines may depend on the rapidity parameter in at least two different ways. One way is the dependence by slope: if the particle propagate at infinite energy then its trajectory is on the light cone. If we assume that the energy is very large but not infinite, then, the trajectory of the particle is slightly off the light cone *i.e* it is along $n^\mu = p_1^\mu + e^{-2\eta} p_2^\mu$ direction. The energy of the particle is given by its rapidity η. It is easy to see that if η is infinite, the vector n^μ becomes the light-cone vector p_1^μ. The Wilson line with rapidity dependence by slope is then written as

$$U^\eta(x_\perp) = \text{Pexp}\Big\{ig \int_{-\infty}^{\infty} du\, n_\mu\, A^\mu(un + x_\perp)\Big\} \tag{3}$$

Another way to include the energy (rapidity) dependence into the Wilson line operator si thorough rigid cut-off *i.e.* by cutting off the longitudinal component of the momentum of the gluon in the following way

$$U_x^\eta = \text{Pexp}\Big[ig \int_{-\infty}^{\infty} du\, p_1^\mu A_\mu^\eta(up_1 + x_\perp)\Big] \tag{4}$$

with

$$A_\mu^\eta(x) = \int \frac{d^4k}{(2\pi)^4} \theta(e^\eta - |\alpha_k|) e^{-ik\cdot x} A_\mu(k) \tag{5}$$

and with $k^\mu = \alpha_k\, p_1^\mu + \beta_k\, p_2^\mu + k_\perp^\mu$. As it has been shown in Ref. [8, 11], at NLO it is more convenient to use the rapidity dependence by rigid cut-off.

Since the main degree of freedom at high energy scattering are gluons, we may assume, in the first approximation, that the target is made of gluon field. Thus, we evaluate the operator O^{η_1} in the background of an external gluonic field.

We now introduce a rapidity divide η_2 which separates the classical fields having rapidity up to η_2 from the quantum fields having rapidity $\eta_1 > \eta > \eta_2$ that will be integrated out to form Feynman diagrams. Formally, the separation of the correlation function in classical and quantum fields may be written as follows

$$\langle O^{\eta_1} \rangle_A \rightarrow \langle O'^{\eta_2} \otimes O'^{\eta_1} \rangle_A \tag{6}$$

The subscript A indicates that the matrix elements are evaluated in the background of the gluonic external field. In principle, after the separation of the fields in classical and quantum components, the operator may be different from the one we started with. We have indicated this with a prime on the operator O in Eq. (6). However, since particles with different rapidity perceive each other as Wilson lines, the operators obtained after the splitting of the fields in classical and quantum are still Wilson lines.

The result of the integration of the matrix element on the right-hand-side (RHS) of Eq. (6) over the quantum fields is the kernel of the one loop evolution equation times the matrix element of the operator made of the classical fields *i.e.* with rapidity parallel to η_2, and times the infinitesimal step in rapidity $\Delta \eta = \eta_2 - \eta_1$. The one-loop evolution equation of the O operator with respect to rapidity is

$$\langle O^{\eta_1} \rangle_A = \alpha_s(\eta_1 - \eta_2) K_{\text{evol}} \otimes \langle O'^{\eta_2} \rangle_A \tag{7}$$

The resulting evolution equation, obtained following the semi-classical approach just described, can be linear or non-linear:

- Linear case $\quad O^{\eta_1} = \alpha_s \Delta \eta \, K_{\text{evol}} \otimes O^{\eta_2}$
- Non-linear case $\quad O^{\eta_1} = \alpha_s \Delta \eta \, K_{\text{evol}} \otimes \{O^{\eta_2} O^{\eta_2}\}$

For the evolution of a matrix element made of one Wilson line we obtain the Feynman diagrams given in Fig. (2). The red strip in the figure represents the background field in the spectator frame: the external field is highly boosted, and it gets contracted in the direction of the boost and time-dilated. Thus, the propagation of the particle is in the background of a shock-wave external field. The
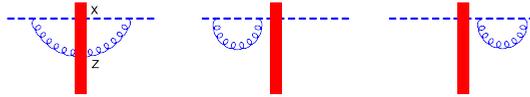
Figure 1. Feynman diagrams for one loop evolution of one Wilson lines.

evolution equation for one Wilson line is

$$\langle \{U_x^{\eta_1}\}_{ij} \rangle_A = \frac{\alpha_s}{2\pi^2} \Delta \eta \int \frac{d^2 z}{(x-z)_\perp^2} \left[\langle \text{tr}\{U_x^{\eta_2} U_z^{\eta_2 \dagger}\} \{U_z^{\eta_2}\}_{ij} \rangle_A - \langle \frac{1}{N_c} \{U_x^{\eta_2}\}_{ij} \rangle_A \right] \tag{8}$$

where we have used the short-hand notation $U_x \equiv U(x_\perp)$, and the color index $i, j = 1, 2, 3$. The kernel of the evolution in this case is $K(x, z) = \frac{1}{(x-z)_\perp^2}$, and the equation is clearly non-linear. Before the one-loop evolution, we have one Wilson line with rapidity η_1 corresponding to the propagation of one quark in the background of a shock-wave. At one loop order, instead, we have a quark and a gluon propagating in the shock-wave.

The scattering amplitude of DIS is proportional to

$$\mathcal{U}(x_\perp, y_\perp) = 1 - \frac{1}{N_c} \text{tr}\{U(x_\perp) U^\dagger(y_\perp)\} \tag{9}$$

To get the evolution equation of the color-dipole we need the evolution equation of the operator U_x and $U_y^\dagger$, and also the evolution equation of pairwise interaction. Thus, the operator $\frac{d}{d\eta}$ of the evolution equation does not follow the Leibniz rule for derivative of product of function. Indeed, this is represented by the Feynman diagrams in Fig. (2),
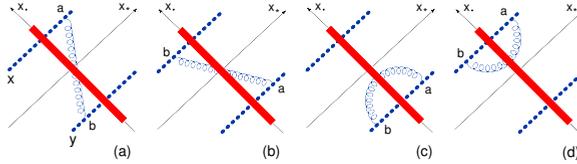
Figure 2. Feynamn diagram for the LO evolution equation of a color-dipole. We omitted the virtual correction diagrams.

and the corresponding evolution equation is

$$
\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) \;=\; \frac{\alpha_s N_c}{2\pi^2}\int \frac{d^2z\,(x-y)^2}{(x-z)^2(y-z)^2}\Big\{\hat{\mathcal{U}}(x,z)+\hat{\mathcal{U}}(z,y)-\hat{\mathcal{U}}(x,y)-\hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\Big\} \tag{10}
$$

Equation (10) is the LO Balitsky evolution equation [1] for color dipole. When the non-linear term operator $\langle\hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\rangle$ factorizes at large N_c as $\langle\hat{\mathcal{U}}(x,z)\rangle\langle\hat{\mathcal{U}}(z,y)\rangle$, Eq. (10) becomes the Balitsky-Kovchegov equation [1, 5]. The linear terms in Eq. (10) correspond to the BFKL evolution equation obtained in perturbative QCD in the leading-log resummation $\alpha \ll 1$ and $\alpha_s\eta \sim 1$; while the non-linear term appears because of the semi-classical approach where the new resummation parameter is $\alpha_s^2 A^{1/3}$ with A being the atomic number in the case of DIS off a nuclear target. The BFKL equation is known to violate unitarity, but the non linear term in Eq. (10) preserves unitarity.

In order to get the LO evolution equation for trace of any number of Wilson lines or product of any number of Wilson lines, which would correspond to the Balitsky-JIMWLK evolution equation [1, 7], one needs to obtain the evolution equation for the following operators as well: $\{U_x^{\dagger\eta_1}\}_{ij}$, $\{U_x^{\eta_1}U_y^{\eta_1}\}_{ij}$, $\{U_x^{\eta_1}U_y^{\dagger\eta_1}\}_{ij}$, $\{U_x^{\dagger\eta_1}U_y^{\dagger\eta_1}\}_{ij}$. Thus, one obtains a set of five evolution equations that can be used to calculate the evolution equation of operators with any trace of Wilson lines. As an example, let us calculate the evolution equation of a four Wilson lines operator $\mathrm{tr}\{U_x U_y^\dagger U_w U_z^\dagger\}$. To this end one has to sum the evolution of each single Wilson line using the evolution equation like Eq. (2), and similar evolution equations for each paring.

In order to obtain the evolution equations of matrix elements of operators with any trace of Wilson lines at NLO we have to extend these rules at the next-to-leading order (NLO).

3 Next-to-leading order evolution equation

In order to obtain the evolution equation of operators with any trace of Wilson lines (or also product of Wilson lines) at the next to leading order, one has to calculate a similar set of evolution equations at NLO. At NLO, however, we may have not only the evolution of single Wilson lines and of two connected Wilson lines but also evolution of triple Wilson lines interaction (see Fig. 3 e) and f)). The diagrams contributing to the NLO B-JIMWLK with three Wilson lines were calculated in Ref. [9]. In Ref. [8] we have calculated the full Balitsky hierarchy at NLO, confirming also the result obtained in Ref. [9]. The NLO JIMWLK Hamiltonian [10], on the other hand, was obtained using the NLO BK equation calculated in Ref. [11, 12] and the evolution with three connected Wilson lines of Ref. [9].
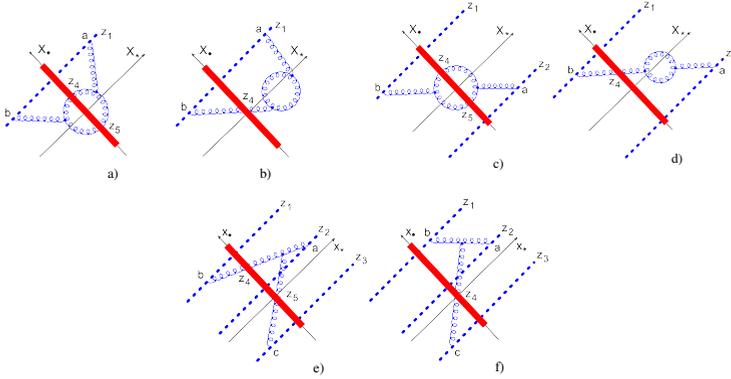
Figure 3. Sample of Feynman diagrams contributing to the NLO B-JIMWLK evolution equation.

For simplicity, we present here only the one-particle interaction ("gluon reggeization" term) at NLO

$$\frac{d}{d\eta}(U_1)_{ij} = \frac{\alpha_s^2}{8\pi^4} \int \frac{d^2z_4 d^2z_5}{z_{45}^2} \left\{ U_4^{dd'}(U_5^{ee'} - U_4^{ee'}) \left(\left[2I_1 - \frac{4}{z_{45}^2} \right] f^{ade} f^{bd'e'} (t^a U_1 t^b)_{ij} + \frac{(z_{14}, z_{15})}{z_{14}^2 z_{15}^2} \ln \frac{z_{14}^2}{z_{15}^2} \right. \right.$$

$$\times \left. \left. [i f^{ad'e'}(\{t^d, t^e\} U_1 t^a)_{ij} - i f^{ade}(t^a U_1 \{t^{d'}, t^{e'}\})_{ij}] \right) \right\}$$

$$+ \frac{\alpha_s^2 N_c}{4\pi^3} \int d^2z_4 z_{14}^2 \, (U_4^{ab} - U_1^{ab})(t^a U_1 t^b)_{ij} \left\{ \left[\frac{11}{3} \ln z_{14}^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right] \right\} \tag{11}$$

where

$$I_1 \equiv I(z_1, z_4, z_5) = \frac{\ln z_{14}^2/z_{15}^2}{z_{14}^2 - z_{15}^2} \left[\frac{z_{14}^2 + z_{15}^2}{z_{45}^2} - \frac{(z_{14}, z_{15})}{z_{14}^2} - \frac{(z_{14}, z_{15})}{z_{15}^2} - 2 \right]$$

and we have used the short-hand notation $U_1 \equiv U(z_1)$ etc. In Eq. (11) we have omitted the quark-loop contributions. The pairwise interaction and the triple interaction at NLO can be found in Ref. [8].

4 Conclusions

We presented a brief pedagogical introduction to the background field method and presented the LO and NLO evolution equation of one Wilson line operator which is part of the NLO Balitsky-JIMWLK equation whose complete NLO result can be found in Ref. [8]. The B-JIMWLK is relevant for the description of scattering amplitude of proton-nucleus or nucleus-nucleus collision where the scattering amplitude is proportional to matrix elements of Wilson lines.

The factorization in rapidity of scattering amplitudes using the OPE at high-energy in Wilson lines is a successful tool for the study of processes at high energy. Its validity has been proven at NLO accuracy [3, 4]. The BK equation is a large N_c limit of the first of the Balitsky hierarchy equations. In order to generalize NLO BK equation to the NLO evolution of any number of Wilson lines, we considered the triple interaction besides the self-energy interaction and the pairwise interaction. Here, we have presented only the evolution equation for self-energy interaction at NLO, while the pairwise interaction at NLO and the triple interaction can be found in Ref. [8].

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