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Solitonic Charged Pion Crystal in Dense QCD

– from a generalized Ginzburg-Landau approach –

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Abstract. We present a systematic study of the phase structure of QCD near the critical point within a general Ginzburg-Landau framework. In particular, we are interested in clarifying the effects of isospin mismatch on the critical point and inhomogeneous phases expected to show up in its neighborhood. To this end, we first derive the generalized Ginzburg-Landau potential expanded up to the sixth order in the order parameter, derivative and the isospin chemical potential. We then show that, going down in density from high density quark matter, the system might go through a particular kind of inhomogeneous charged pion crystal phase before forming a soliton lattice of chiral condensate.

1 Introduction

The possibility of spatially inhomogeneous realizations of chiral symmetry breaking in quark matter has recently been the subject of extensive research [1–7]. Such inhomogeneities are known to be driven by the competition and compromise between quark-antiquark pairing and the pair breaking effect provided by quark chemical potential μ; the latter stresses the former trying to generate an imbalance via populating a net excess of quarks over antiquarks.

Most of study concentrate on the ideal situation where equal numbers of up and down quarks are populated in the system. However, in realistic bulk systems such as those realized in compact stars, down quark density is almost twice as large as up quark density due to the charge neutrality constraint. The isospin density is also known to be a driving force to another interesting condensation of charged pions in quark matter [8].

We here present for the first time a systematic study on the impact of isospin asymmetry on the critical point (CP) and phases in its vicinity. The method we use is the generalized Ginzburg-Landau (gGL) approach which is based on the gradient expansion of thermodynamic potential. Starting from the sixth order Ginzburg-Landau potential designed to minimally describe the tricritical point at the chiral limit, we construct the most general gGL potential at the same order by adding all possible gradient terms as well as new terms which are allowed in the presence of isospin chemical potential $\mu_{\rm I}$. Analysing the obtained gGL potential, we will show that the phase structure in the neighborhood of CP is dramatically modified in the presence of $\mu_{\rm I}$. In particular, we find that, going down in density from high density side, quark matter may go through a solitonic charged pion crystal (SPC) phase before forming a soliton lattice of chiral condensate (solitonic chiral crystal phase; SCC hereafter).

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2 Evaluating gGL phase diagram for vanishing isospin density

We start from the gGL potential at $\mu_{\rm I} = 0$ for spatially varying condensates in the scalar and pseudoscalar isovector channels: $\sigma(\mathbf{x}) \sim \langle \bar{q}q \rangle$ and $\pi_a(\mathbf{x}) = \langle \bar{q} i \gamma_5 \tau_a q \rangle$ (a = 1, 2, 3). Using the real representation with chiral four-vector notation $\phi(\mathbf{x}) = (\sigma(\mathbf{x}), \pi_1(\mathbf{x}), \pi_2(\mathbf{x}), \pi_3(\mathbf{x}))$, we find, up to the sixth order in the order parameter ϕ and derivatives acting on ϕ [5]

$$\begin{aligned} \omega(\mathbf{x}) &= -h\sigma + \frac{\alpha_2}{2}\phi^2 + \frac{\alpha_4}{4}(\phi^4 + (\nabla\phi)^2) \\ &\quad + \frac{\alpha_6}{6}\left(\phi^6 + 3[\phi^2(\nabla\phi)^2 - (\phi, \nabla\phi)^2] + 5(\phi, \nabla\phi)^2 + \frac{1}{2}(\Delta\phi)^2\right). \end{aligned} \tag{1}$$

$(\phi, \nabla\phi) = \sigma\nabla\sigma + \boldsymbol{\pi} \cdot \nabla\boldsymbol{\pi}$, and h is an external field which is responsible for current quark mass; the h-term tries to make the chiral four-vector align in the σ-direction. If we assume the chiral limit $h = 0$ with $\pi_a = 0$, replacing the symbol σ with M, Eq. (1) reduces to the gGL potential derived in [2]:

$$\omega(\mathbf{x}) = \frac{\alpha_2}{2}M^2 + \frac{\alpha_4}{4}(M^4 + (\nabla M)^2) + \frac{\alpha_6}{6}\left(M^6 + 5M^2(\nabla M)^2 + \frac{1}{2}(\Delta M)^2\right). \tag{2}$$

On the other hand, assuming $h = 0$, $\pi_1 = \pi_2 = 0$ and one-dimensional structure for σ and π_3, we find, with the complex notation $\Delta(z) = \sigma(z) + i\pi_3(z)$,

$$\omega(z) = \frac{\alpha_2}{2}|\Delta|^2 + \frac{\alpha_4}{4}(|\Delta|^4 + |\Delta'|^2) + \frac{\alpha_6}{6}\left(|\Delta|^6 + 3|\Delta|^2|\Delta'|^2 + (\mathrm{Re}\,[\Delta^*\Delta'])^2 + \frac{1}{2}|\Delta''|^2\right), \tag{3}$$

which is formally equivalent to the gGL expression derived for two-dimensional chiral model with a continuous chiral symmetry [9].

We assume $\alpha_6 > 0$ for stability of the system, and replace α_6 with 1 following the convention that every quantity with an energy dimension is to be measured in the unit $\alpha_6^{-1/2}$. Then via scaling $\phi \to h^{1/5}\phi$, $\mathbf{x} \to \mathbf{x}^{-1/5}$, $\alpha_2 \to \alpha_2 h^{4/5}$, $\alpha_4 \to \alpha_4 h^{2/5}$, we can get rid of h in ω apart from an overall factor $h^{6/5}$. Then what we need to do is to evaluate the phase structure in the (α_2, α_4)-plane. The result is displayed in Fig. 1. A particular kind of inhomogeneous phase is realized in the shaded area which we call the inhomogeneous island following Ref. [10]. The phase is the one-dimensional soliton lattice of chiral condensate characterized by three parameters, k, b and ν, as

$$\sigma(z) = k\nu^2 \mathrm{sn}(b, \nu)\mathrm{sn}(kz - b/2, \nu)\mathrm{sn}(kz + b/2, \nu) + k\frac{\mathrm{cn}(b, \nu)\mathrm{dn}(b, \nu)}{\mathrm{sn}(b, \nu)}. \tag{4}$$

This is the mathematical expression of the SCC condensate. It is explicitly shown in [5] that this actually provides a one-parameter family of solution to the Euler-Lagrange equation $\delta\Omega/\delta\sigma(z) = 0$, that is a forth order nonlinear differential equation. The SCC condensate $\sigma(z)$ smoothly interpolates between the chiral symmetry broken (χSB) phase and nearly symmetric phase: The spatial profile of $\sigma(z)$ looks like a lattice of soliton-antisoliton pairs near the phase boundary to the χSB phase, while near the other phase boundary it is more like a vanishingly small ripple of sinusoidal wave on the homogeneous sea of small but nonvanishing chiral condensate. The solid curve depicted in the shaded area is the first-order chiral phase transition which would have been realized if we did not take care the possibility of inhomogeneity discarding gradient terms in the gGL potential Eq. (1). The solid curve ends at the point "LCP" which stands for the Lifshitz critical point. At this point the solitonic chiral condensate ends together, and this fact constitutes the reason for the term "Lifshitz" in front of "CP". The dotted line above the LCP represents the chiral crossover transition. When the chiral limit is approached as $h \to 0$, the LCP smoothly continues to the Lifshitz tricritical point at $(\alpha_2, \alpha_4) = (0, 0)$, and the crossover transition turns into a second-order phase transition at $\alpha_2 = 0$ for $\alpha_4 > 0$ [2].

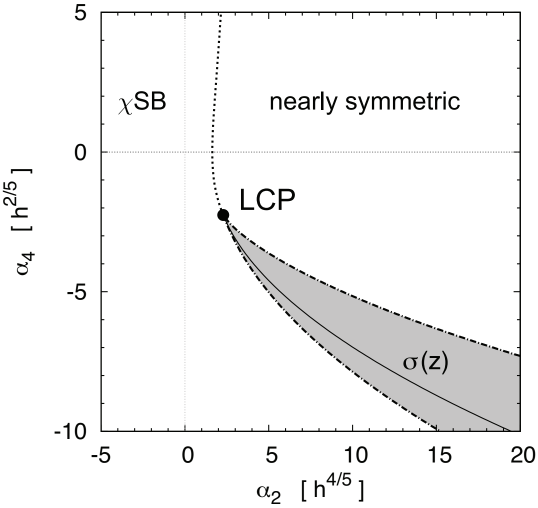

Figure 1. The gGL phase diagram for vanishing isospin density. The SCC phase is realized in the shaded area which we call the inhomogeneous island. The island is surrounded by the dash-dot line representing continuous second-order phase transitions. The χSB phase and the nearly symmetric phase are separated by a crossover (dotted line) or the inhomogeneous island.

3 Evaluating gGL phase diagram for nonzero isospin density

We now take into account the isospin density which can be accommodated by the isospin chemical potential $\mu_{\rm I}$ [4, 5]. The isospin chemical potential brings about two effects into gGL potential: These are (i) simple shifts in gGL couplings according to $\mu_{\rm I}^2$, and (ii) new terms responding to the explicit symmetry breaking from isospin SU(2) to its diagonal subgroup U(1). For (i), we find $\alpha_4 \to \alpha_4 + \mu_{\rm I}^2\alpha_6 + O(\mu_{\rm I}^4)$. The shifts in α_2 and α_6 only appear at higher orders. On the other hand, possible new terms for (ii), which are not be forbidden by symmetry, can be written as

$$\delta\omega(\mathbf{x}) = \frac{\beta_2}{2}\boldsymbol{\pi}_\perp^2 + \frac{\beta_4}{4}\boldsymbol{\pi}_\perp^4 + \frac{\beta_{4b}}{4}(\phi^2 - \boldsymbol{\pi}_\perp^2)\boldsymbol{\pi}_\perp^2 + \frac{\beta_{4c}}{4}(\nabla\boldsymbol{\pi}_\perp)^2. \tag{5}$$

We have defined the charged pion doublet $\boldsymbol{\pi}_\perp = (\pi_1, \pi_2)$. When the system is in the charged pion condensate (PC) characterized by a nonvanishing condensate $|\boldsymbol{\pi}_\perp| \neq 0$, the U(1) symmetry is spontaneously broken. Within the series expansion in $\mu_{\rm I}$ up to appropriate orders, we find

$$\beta_2 = -\frac{1}{2}\mu_{\rm I}^2\alpha_4, \quad \beta_4 = -2\mu_{\rm I}^2\alpha_6, \quad \beta_{4b} = -2\mu_{\rm I}^2\alpha_6, \quad \beta_{4c} = -\frac{4}{3}\mu_{\rm I}^2\alpha_6. \tag{6}$$

The gGL potential $\omega(\mathbf{x}) + \delta\omega(\mathbf{x})$ is now characterized by five parameters, α_2, α_4, α_6, h, and $\mu_{\rm I}^2$. In the same way as before, we can get rid of α_6 and h, and accordingly there are three parameters α_2, α_4 and $\mu_{\rm I}^2$ left in the problem, which should be measured in the units $h^{4/5}$, $h^{2/5}$ and $h^{2/5}$ respectively.

The remaining task is to evaluate the phase diagram in (α_2, α_4)-plane for a given value of $\mu_{\rm I}^2$. To do this in full generality is technically involved, so we here examine several ansatz states listed in Table. 1. DCDW is a simple extension of the dual chiral density wave defined in the chiral limit [1], in which the scalar condensate and the pseudoscalar neutral pion condensate vary alternatingly

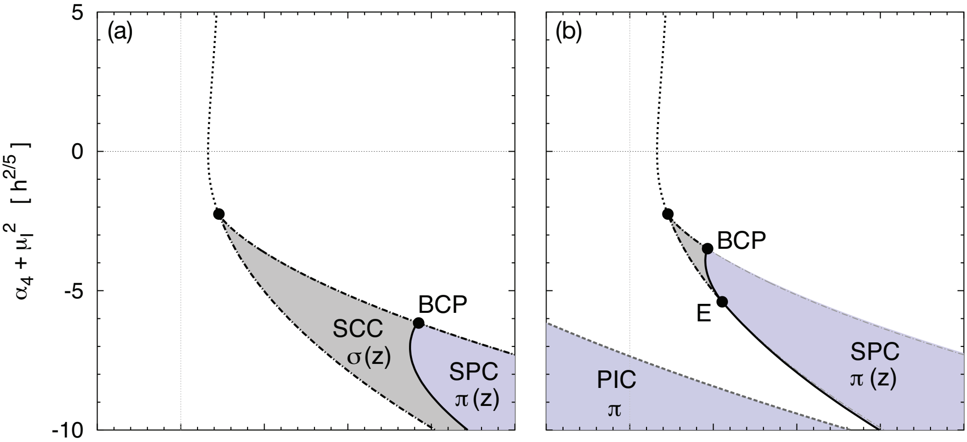

Figure 2. The gGL phase diagram for nonvanishing isospin densities: (a) for $\mu_{\mathrm{I}}^2 = 0.01$, and (b) for $\mu_{\mathrm{I}}^2 = 0.1$.

in space. In the chiral limit $h \to 0$, $c = 0$ and $a = b$ are expected, and there remains a particular combination of translation and chiral transformation intact. $\mathrm{DCDW_c}$ is the charged pion version of DCDW phase. "SPC" is an abbreviation of the solitonic charged pion crystal condensate defined by the ansatz; $\sigma \neq 0$, and

$$\pi_1(z) = k\nu \mathrm{sn}(kz, \nu). \tag{7}$$

This is a charged pion analog to the SCC state in the chiral limit. In this phase, the charged pion component is oscillating in the homogeneous see of scalar condensate. It can be shown that the SPC condensate actually constitutes a one-parameter family of solutions to the Euler-Lagrange equation.

The numerical results for the gGL phase diagram for two representative values for μ_{I}^2 are displayed in Fig. 2; (a) is for $\mu_{\mathrm{I}}^2 = 0.01$, and (b) is for $\mu_{\mathrm{I}}^2 = 0.1$. The former roughly corresponds to $\mu_{\mathrm{I}} \sim 50\,\mathrm{MeV}$, and the latter to $\mu_{\mathrm{I}} \sim 100\,\mathrm{MeV}$ according to a crude order estimate in [5]. In the figure, the scale of vertical axis is redefined to $\alpha_4 + \mu_{\mathrm{I}}^2$, so that the trivial shift of location of LCP $(\alpha_2, \alpha_4) \cong (2.28, -2.25) \to (2.28, -2.25 - \mu_{\mathrm{I}}^2)$ is invisible. Using a typical mapping between the (α_2, α_4)- and (μ, T)-planes within an NJL-type model, the shift is in the direction of higher μ and lower T [5].

Apart from the shift of LCP, we notice in Fig. 2(a) that a part of SCC is replaced by SPC. The SPC phase and the nearly symmetric phase are separated by a continuous second-order phase transition, while the phase transition between SPC and SCC is of first-order. As a consequence, there is a bicritical point denoted by "BCP" where two second-order phase transitions and a first-order phase

Table 1. Candidate phases at finite isospin density $\mu_{\mathrm{I}} \neq 0$.

	σ	$\boldsymbol{\pi}$	Internal symmetry	Translation
χSB	$\sigma \neq 0$	$\pi_\perp = 0$	$\mathrm{U(1)_B \times U(1)_Q}$	unbroken
PC	$\sigma \sim 0$	$\pi_1 \neq 0$	$\mathrm{U(1)_B}$	unbroken
SCC	$\sigma = \sigma(z)$	$\pi_\perp = 0$	$\mathrm{U(1)_B \times U(1)_Q}$	broken
SPC	$\sigma \sim 0$	$\pi_1 = \pi(z)$	$\mathrm{U(1)_B}$	broken
DCDW	$\sigma = c + a\cos(kz)$	$\pi_3 = b\sin(kz)$	$\mathrm{U(1)_B \times U(1)_Q}$	broken (modified when $h = 0$)
$\mathrm{DCDW_c}$	$\sigma = c + a\cos(kz)$	$\pi_1 = b\sin(kz)$	$\mathrm{U(1)_B}$	broken even at $h = 0$

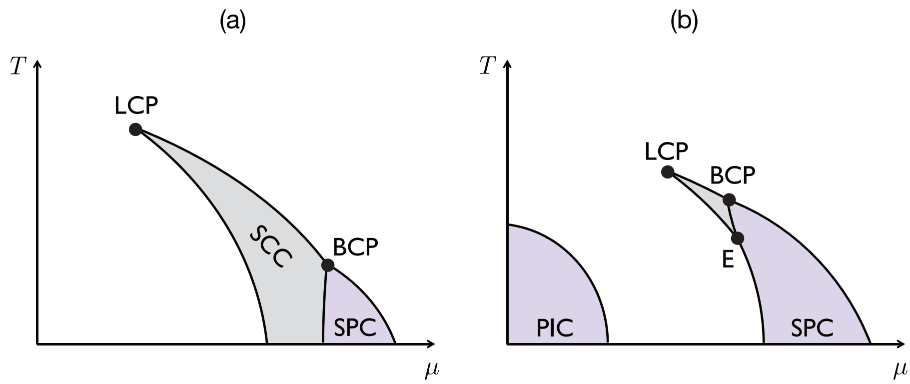

Figure 3. Schematic illustration of possible QCD phase diagram for $\mu_{\rm I} \neq 0$, speculated from gGL analysis which is formally valid only in the vicinity of LCP. (a) Phase diagram for small $\mu_{\rm I}^2$ which shares the same topology with Fig. 2(a). (b) Phase diagram for large $\mu_I^2 > m_\pi^2$, which possesses the same topology as Fig. 2(b).

transition meet up all at once. At the second-order phase transition from the nearly symmetric phase to the SPC phase, a charged pion mode at finite wavevector becomes tachyonic before the σ-mode gets unstable. This is because finite $\mu_{\rm I}^2$ gives a negative contribution to the gradient term $(\nabla \boldsymbol{\pi}_\perp)^2$ according to $\beta_{4c} = -\frac{4}{3}\mu_{\rm I}^2\alpha_6 < 0$.

Microscopically, the SPC phase is driven by the competition and compromise between the u-$\bar{d}$ pairing and the pair breaking effect; the former is promoted by a large $\mu_{\rm I}$ where the Fermi surfaces of up and anti-down quarks are equal and large, while the latter is provided by $\mu \neq 0$ which favors the situation $n_q \neq 0$, that is actually the population imbalance between quark and antiquark.

In Fig. 2(b), we see that the region for SPC phase gets enlarged. SPC now takes over a major part of SCC. Actually for large negative α_4, SCC disappears completely, and the SPC and χSB phases are directly separated by a single first-order phase transition which is accompanied by an abrupt change in the magnitude of σ. Accordingly there shows up an interesting point denoted by "E" at which a second order phase transition comes across two first-order phase transitions.

Another notable change in the topology of phase diagram (b) from (a) is the appearance of continent of PC in the deep inside the χSB phase [4]. This is triggered by the cross-term $\alpha_4|\sigma|^2\boldsymbol{\pi}_\perp^2$ in the gGL potential Eq. (5) which promotes strongly the instability in the charged pion mode at vanishing wavevector if $\alpha_4 < 0$ and $|\sigma|$ is large.

4 Conclusion

We have investigated systematically the phase diagram of two-flavor quark matter near the CP using the gGL approach. In particular, we focused on clarifying what are the effects of isospin density on the CP and inhomogeneous chiral phases in its neighborhood. To this end, we derived the gGL potential expanded up to sixth order in the condensate, derivative, and the isospin chemical potential $\mu_{\rm I}$. Based on the gGL potential, we computed numerically the phase diagram in the (α_2, α_4)-plane which in principle has a unique map on to (μ, T)-plane at least in the vicinity of the QCD critical point, if any. We have confirmed that nonvanishing $\mu_{\rm I}$ not only brings about the PC phase (as expected) but also stabilizes the SPC phase replacing a part of SCC island in the phase diagram.

Let us finally draw schematic QCD phase diagram in (μ, T)-plane, even if it stays at the speculative and conceptual level. This is done in Fig. 3, where we simply assumed the topologies found in gGL phase diagram in (α_2, α_4)-plane remain unchanged. Fig. 3(a) corresponds to Fig. 2(a) where a small value is set for $\mu_{\rm I}$. We see that, going down in density from high density quark matter, the system goes through the SPC phase before making a soliton lattice of chiral condensate, i.e., the SCC condensate. Fig. 3(b) corresponds to the Fig. 2(b) where a relatively large value is taken for $\mu_{\rm I}$. LCP moves towards higher chemical potential and lower temperature because of the shift $\alpha_4 \to \alpha_4 + \mu_{\rm I}^2$. Assuming that the value of $\mu_{\rm I}$ is above the mass of charged pion m_π, we know that the pion condensate is realized at vacuum [8]. For this reason the PC continent in (α_2, α_4)-plane is mapped on to the area including the origin of (μ, T)-plane.

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