EPJ Web of Conferences **81**, 05010 (2014)
DOI: 10.1051/epjconf/20148105010

Extracting excited mesons from the finite volume

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Abstract. As quark masses come closer to their physical values in lattice simulations, finite volume effects dominate the level spectrum. Methods to extract excited mesons from the finite volume are discussed, like moving frames in the presence of coupled channels. Effective field theory can be used to stabilize the determination of the resonance spectrum.

1 Introduction

One of the present issues in QCD lattice simulations is to determine the spectrum of excited hadrons. Increased computational power allows to extract not only the ground state levels but also a tower of excited states, in combination with techniques like multi-hadron operators in generalized eigenvalue problems. As quark masses drop, excited levels are often in kinematic regions where two or more particles can be simultaneously on-shell, i.e. where resonances acquire a width and can decay to one or more channels.

For resonances with a single two-body decay channel, one often uses Lüscher's approach to extract phase shifts from the discrete energy levels in the box [1, 2]. This method allows to extract the phase at the energy of the eigenvalue, but phases at different energies are needed to determine the precise resonance properties, in particular the width as demonstrated for the $\rho(770)$ [3–6].

If more than one two-body decay is possible, the reconstruction of the resonance, or in general the multi-channel scattering amplitude, becomes more complicated. For example, in isoscalar S-wave meson-meson-scattering, there are at least two relevant channels, $\pi\pi$ and $K\bar{K}$. Obviously, close to and above the $K\bar{K}$ threshold, one has three independent transitions $\pi\pi \to \pi\pi$, $K\bar{K} \to K\bar{K}$ and $\pi\pi \to K\bar{K}$, $K\bar{K} \to \pi\pi$ with the off-diagonal transitions being equal due to time-reversal invariance. To determine the amplitude, one needs, thus, three measurements at the same energy, instead of one as in the one-channel case. It is little practical or even impossible to tune different lattices such that three independent measurements at a given energy can be taken. Interpolations in energy can provide the needed information, with minimal assumptions on the smoothness of the interaction kernel.

In data analysis it is advantageous to separate the known non-analytic threshold openings from the full amplitude T. This can be achieved by schematically writing [7]

$$T = V + VGT \tag{1}$$

where the kernel V and the two-body propagator G are matrices in channel space. The kernel V is a smooth function at the (in)elastic thresholds and G contains non-analyticities. If G is only given

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Article available at http://www.epj-conferences.org or http://dx.doi.org/10.1051/epjconf/20148105010

by the phase space factor (purely imaginary above threshold and real below), one has a K-matrix approach. Dispersive real contributions to G can be included by writing a dispersion relation, but also then the amplitude can be recast in the K-matrix formulation [8]. The imaginary part of G ensures also coupled-channel two-body unitarity. In any case, the fact that V is a smooth function permits to develop it locally in a power series of the energy. In the Chiral Unitary Approach the series is matched to CHPT order-by-order. When it comes to the analysis of data, this helps to stabilize the extraction of the amplitude.

In the finite volume, momenta are restricted to discrete values due to the periodic boundary conditions of the box. This modifies both V and G, but some changes are suppressed exponentially with the box size L, $e^{-M_\pi L}$. The connection between the notation implied in Eq. (1) and the Lüscher formalism can be found, e.g., in Ref. [7]. The integration over momenta in the two-body propagator G is reduced to a sum due to the periodic boundary conditions in the finite volume, $G \to \tilde{G}$. The kernel $\tilde{V}$ contains only exponentially suppressed contributions to finite-volume corrections that can be neglected for sufficiently large box sizes, $\tilde{V} = V$. Then, the eigenvalues in the finite volume are given by the poles of $\tilde{T}$ defined by $\tilde{T} = V + V\tilde{G}\tilde{T}$, from which V can be deduced and inserted in Eq. (1) to provide the T-matrix in the infinite volume. Phase shifts and inelasticities follow. This procedure is independent of the regularization scheme for G.

2 Excited mesons and multi-channel scattering

A practical example of how the expansion of the kernel V can be used to determine the amplitude is given in Ref. [7] for the $f_0(980)$. This case is particularly interesting because the resonance is situated closely to the S-wave $K\bar{K}$ threshold. The latter manifests itself in the same avoided level crossing as a narrow resonance. Indeed, in Ref. [7] it has been shown that by removing the $f_0(980)$ the amplitude changes drastically but the level spectrum in the finite volume shows only rather small changes and the qualitative L-dependence of the levels stays the same. In Refs. [7, 8] it has been proposed to modify the boundary conditions for the strange quark to remove the influence of the threshold. In the notation used here, this results in a modified $\tilde{G}$. In particular, instead of allowed momenta $\vec{q} = \frac{2\pi}{L}\vec{n}, \quad \vec{n} \in \mathbb{Z}^3$ for periodic boundary conditions, one has the momenta $\vec{q} \to \vec{q} + \vec{\theta}/L$ where $\vec{\theta}$ is the twisting angle that can be chosen to vary the $\bar{K}K$ threshold. While for periodic boundary conditions, $\tilde{G}$ has a pole at threshold ($\vec{q} = (0,0,0)$), this is obviously not any more the case for $\vec{\theta} \neq (0,0,0)$.

Using the Chiral Unitary approach of Ref. [10] one can generate realistic lattice eigenvalues for different box sizes L as shown in Fig. 1 to the left. The data $E_n(L)$ are fitted by using linear functions in the potential V_{ij},

$$V_{ij}(s) = a_{ij} + b_{ij}(s - 4M_K^2) \tag{2}$$

where i, j are indices for the $\pi\pi$ and the $K\bar{K}$ channels with $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$ for $i \neq j$. The V are indeed smooth functions of s by construction. With a, b determined, the original scattering equation (1) can be solved. The extracted infinite-volume phase shift is shown in the middle panel of Fig. 1, as a function of $E = \sqrt{s}$. To the right in Fig. 1 the complex plane of scattering energy is shown with the reconstructed pole position of the $f_0(980)$ (shaded areas). As the data had been originally generated from the approach of Ref. [10], the exact pole position is known (triangle in the figure). The figure shows thus, that it is indeed possible to reliably reconstruct the resonance.

While the a, b in Eq. (2) are free real fit parameters, one can use Chiral Perturbation Theory to stabilize the resonance extraction, as demonstrated in Ref. [9], by using the fit potential

$$V^{\rm fit} = \left(\frac{V_2 - V_4^{\rm fit}}{V_2^2}\right)^{-1}, \quad V_4^{\rm fit} = a + b(s - s_0) + c(s - s_0)^2 + d(s - s_0)^3 + \cdots \tag{3}$$

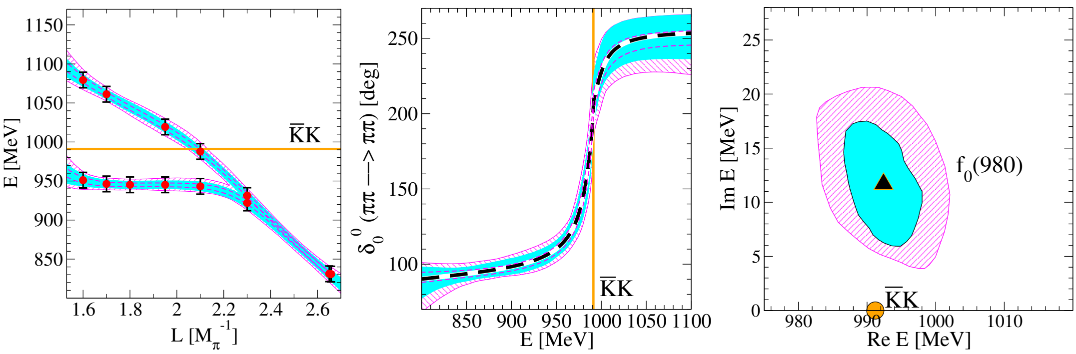

Figure 1. Left: generated synthetic lattice data for the energy region of the $f_0(980)$. Shown are two sets. The one with the visible plateau is evaluated with periodic boundary condition, the other one with twisted boundary conditions of $\vec{\theta} = (\pi, \pi, \pi)$. Center: Extracted phase shift close to the $K\bar{K}$ threshold. Right: complex energy plane. The reconstructed $f_0(980)$ pole position (colored areas) matches with the exact value (triangle). See text for further explanations.

that has been written for the one-channel case for simplicity. Here, $V_2 \equiv V_{\rm LO}$ is the fixed lowest order (LO) contribution of the chiral expansion; one may also fit low-energy constants directly to lattice eigenvalues.

The Lüscher formalism [2] was derived for a pair of particles with total zero-momentum. The generalization to moving frames has been done in [11]. In Ref. [13], the concepts of moving frames and coupled-channel scattering were brought together. The disentanglement of partial waves and channels was demonstrated for $I = 1/2$ πK, ηK scattering in S- and P-wave, and for $I = 0$ $\pi\pi$, πK scattering in S- and D−wave. See Ref. [12] for the first coupled-channel calculation on the lattice.

3 Excited baryons

Recently, first results on the spectrum of excited baryons including the decay dynamics have appeared [14]. A few phase shift points could be extracted using Lüscher's method, and first evidence for the $N(1535)1/2^-$ and the $N(1650)1/2^-$ resonances could be found. Results from Ref. [15] show that when quark masses drop further, an unexpected behavior for the chiral extrapolation could occur. For this, the next-to-leading order chiral unitary calculation of Ref. [16] was extrapolated to different lattice setups. As quark masses rise from the physical point, thresholds start moving as well. In particular, thresholds move faster than the resonance poles, in this case. The prominent resonance structure visible in the amplitude [15] is not induced by any of the original $N(1535)1/2^-$ or $N(1650)1/2^-$ resonance poles. Rather, it results from a shadow pole that is invisible at physical quark masses and becomes uncovered at unphysical quark masses. In conclusion, coupled channel dynamics induces here a complication of a different type than those discussed before. In Ref. [15], the finite-volume spectrum is predicted as well, and different strategies to extract the resonances are discussed, in particular using twisted boundary conditions.

Pioneering work for the coupled-channel $\bar{K}N$, $\pi\Sigma$ system and the $\Lambda(1405)$ in the finite volume has been carried out in Ref. [17]; the lattice levels for the $\Lambda(1405)$ quantum numbers were predicted, for the first time, in Ref. [18] using a dynamical coupled channel model in the finite volume; strategies to determine the two $\Lambda(1405)$ states from lattice results were discussed in Ref. [19], using moving frames. First results of actual lattice calculation have appeared recently [20].

4 Summary and outlook

As quark masses drop, resonance can decay in the finite volume. Lüscher's method provides the model-independent framework to extrapolate amplitudes to the infinite volume. As scattering problems become more complex and new decay channels open, the multi-channel extension of the method will become relevant. The formalism can be combined with moving frames and modified boundary conditions, to obtain more lattice eigenvalues, to disentangle partial waves, or to disentangle resonances from threshold effects. To improve the accuracy of the result, interpolations in energy, with minimal assumptions on the smoothness of the amplitude, allow to relate eigenvalues at different energies to each other. Such fits can be stabilized by Chiral Parturbation Theory.

Three-body scattering problems in the finite volume start to become of high interest, for example triggered by recent JLab calculations of excited mesons, partly with exotic quantum numbers [21], that decay to multi-meson final states. See Ref. [22] for a first calculation of the a_1 meson decaying into $\pi\rho$. Also, the consequences for the finite-volume spectrum of the Roper resonance, coming from its large $\pi\pi N$ inelasticties deserve attention, possibly explaining the unusual quark mass dependence of the eigenvalues. A practical scheme, yet with minimal assumptions, for three-body systems is called for.

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