EPJ Web of Conferences **95**, 03034 (2015)
DOI: 10.1051/epjconf/20159503034

Quantum measurements and Landauer's principle

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Abstract. Information processing systems must obey laws of physics. One of particular examples of this general statement is known as Landauer's principle - irreversible operations (such as erasure) performed by any computing device at finite temperature have to dissipate some amount of heat bound from below. Together with other results of this kind, Landauer's principle represents a fundamental limit any modern or future computer must obey.
We discuss interpretation of the physics behind the Landauer's principle using a model of Unruh-DeWitt detector. Of particular interest is the validity of this limit in quantum domain. We systematically study finite time effects. It is shown, in particular, that in high temperature limit finiteness of measurement time leads to renormalization of the detector's temperature.

1 Introduction

The 3D topology of modern physics frontiers resembles that of Swiss cheese. The cheese itself represents the land of "known" - with the Standard Model being its perhaps most impressive part. Looking from inside the piece of cheese, there is an obvious "external border" with exciting new phenomena and corresponding yet undiscovered theoretical structures lying behind, in still unexplored space : new hypothetical particles and forces, unity of gravity and quanta etc. On the other hand, the Swiss cheese is not a simply connected object and there are also "internal borders" of the holes inside the piece, representing our lack of understanding of physical phenomena deeply inside the domain of validity of established theories. One well known example of this kind is confinement problem in quantum chromodynamics. There are no doubts how the fundamental microscopic theory looks like and using ordinary personal computer one can prove confinement numerically in a few minutes. On the other hand, from theoretical point of view, the problem has been staying unsolved for many decades and still it is, despite tremendous invested efforts - both in "fundamental" (lack of rigorous proof of the mass gap in Yang-Mills theory) and in "applied" sense (lack of theoretical method for analytic computation of dimensionless ratios of hadron masses with controlled accuracy). Our understanding of quantum chromodynamics and developed techniques how to work with it are still inadequate to the confinement problem.

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Sonoluminescence and ball lightning are another examples of this kind. Despite no new discoveries of Nobel scale are expected there, the topics are certainly worth exploring from basic as well as from applied science points of view.

The present text is devoted to discussion of another hole-in-the-cheese like phenomena, centered around the so called Landauer's principle. The subject lies at the heart of intimate relation between physics and information and its roots go back to the XIX-th century. Seminal insights of J.C.Maxwell, L.Boltzmann, J.von Neumann, L.Szilard, L.N.Brillouin and many others shaped this area of research. In modern science, the work of C.Shannon [1] put the intuitive idea of information on firm mathematical grounds. The crucial (but still looking trivial for many) point is that information is physical and all information processing systems ("hardware") of today or tomorrow are to obey the laws of physics. Much less trivial is a question about intrinsic energy or entropy cost of "software" - an algorithm of computation. Are there any fundamental limits from this point of view? How typical question of this kind looks like: does the amount of energy one has to spend to copy some information depend on the amount of this information? Is there any minimal energy needed to copy one bit? Minimal time to copy? To erase? Et cetera. Such problems were under discussion for decades (see [2, 3]) and are of prime importance for information theory understood as the one among other natural sciences. Of course, answers to these questions for real computers could be very different quantitatively from the ones for idealized devices.

It is easy to understand that in general one can indeed expect various limits of this kind. For example, limiting character of the velocity of light puts relativistic bounds on the computing speed of any device having finite size (as should be the case for any realistic one). Speed of quantum evolution is limited by the so called Margolis-Levitin bound [4], inclusion of gravity puts its own limits [5] etc.

In his seminal paper [6], Rolf Landauer made an important contribution to this discussion. He formulated the statement known as the Landauer's principle: erasure of one bit of information leads to dissipation of at least $k_B T \log 2$ of energy. Roughly speaking, forgetting is costly.[1] By "erasure" one understands (almost) any operation that does not have single-valued inverse. One the other hand, logical operations such as copying can be performed, at least in principle, with zero energy consumption. Speaking more rigorously, it means that for any chosen $\epsilon > 0$ one can suggest copying protocol which requires an amount of energy less that ϵ.

Landauer's principle is considered by many as a key to solution of the famous Maxwell demon paradox (see [2] and nice introductory review to the subject [8]). The crucial point is the necessity to restore initial state of the demon's mind ("erase its memory"), and this operation dissipates just the amount of heat one gained during the previous steps of the demon's activity. The discussions still go on.

It is rather challenging to explore this kind of physics experimentally and direct evidence of the Landauer's principle came quite recently [9, 10]. The validity of the principle was demonstrated, but two important comments are worth making. First, the bound is valid only statistically, while in a single event fluctuations can drive the system well below it. Second, the bound is valid only in stationary case, dissipated heat exceeds the bound and increases with inverse erasure time.

More refined formulation of the principle was given recently in [11]. The authors considered a system immersed into the thermal bath. Initially, the density matrices of the system

[1] "Even if you're not burning books, destroying information generates heat" [7].

Table 1. The Landauer's principle

	$\Delta S > 0$	$\Delta S < 0$
$\Delta Q > 0$	**Cristallization.** *Erasure dissipates energy*	**Explosion** *Internal energy converts to heat.*
$\Delta Q < 0$	**Perpetuum mobile of the second kind.** *Cannont order system using thermal energy*	**Melting.** *Use of thermal energy increase entropy*

("detector") and the bath are uncorrelated:

$$\rho = \rho_s \otimes \rho_b \tag{1}$$

where $\rho_b \sim \exp(-\beta H_b)$ and β is initial inverse temperature. After unitary evolution the density matrix ρ' has no longer the form of direct product $\rho'_s \otimes \rho'_b$ and one defines $\rho'_s = \mathrm{Tr}_b \rho'$, $\rho'_b = \mathrm{Tr}_{s'} \rho'$. Then one can prove that

$$\beta \Delta Q \geq \Delta S \tag{2}$$

where $\Delta S = S(\rho_s) - S(\rho'_s)$, $\Delta Q = \mathrm{Tr}[H\rho'_b] - \mathrm{Tr}[H\rho_b]$ with the standard definition of entropy $S[\rho] = -\mathrm{Tr}[\rho \log \rho]$. The equation (2) is a precise formulation of the stationary Landauer's principle. It is worth noting that factorization of the initial state density matrix and thermal character of the bath are important, without these assumptions one could design gedanken experiments violating (2).

The physical meaning of the Landauer's principle is represented in the Table 1. In its original form Landauer's principle corresponds to $\Delta Q > 0$, $\Delta S > 0$ case. But it can be applied to other case as well being trivial in $\Delta Q > 0$, $\Delta S < 0$ case and equivalent to the second law of thermodynamics (no perpetuum mobile of the second kind is possible) for $\Delta Q < 0$, $\Delta S > 0$ case.

2 Finite time measurements

Before we come to our main topic, let us make a few general comments concerning the measurement problem in quantum field theory (QFT). Typical QFT is defined by the partition function

$$Z = \int \mathcal{D}\phi \, e^{iS} \tag{3}$$

The "dynamics" is usually assumed to be encoded in the action S. But this is a conditional statement and in fact the dynamics can be shifted from measure to action and back. Moreover, the measure can encode some *a priopri* known (or assumed) results of measurements. Nice example is provided by Casimir boundary conditions for ideal conductor:

$$\int \mathcal{D}A_\mu \to \int \mathcal{D}A_\mu \, \delta \left[n^\alpha(z(x,y)) \tilde{F}_{\alpha\beta}(z(x,y)) \right] \tag{4}$$

where the function $z = z(x,y)$ encodes the boundary profile, $n^\alpha(z)$ is the normal vector at the point z and $\tilde{F}_{\alpha\beta}$ is dual field strength tensor. Another example is UV-regularization

$$\int \mathcal{D}\phi \to \int\limits_{|k|<\Lambda} \mathcal{D}\phi \tag{5}$$

In renormalizable theories all physics above Λ can be encoded just in a couple of numbers (coefficients in front of marginal operators) like $1/137$.

But in most cases in particle physics we assume that dynamics, described by the "action" is uncorrelated with dynamics of "measure" and for good reasons: typical scale of the former is given by strong interaction distance of $\sim 10^{-15}$ meters and even smaller for weak interactions, while detectors are macroscopic objects having sizes of the order of dozens of microns and larger. All the standard perturbative quantum field theory machinery (asymptotic states; propagators computed in plane waves basis etc.) is based on this assumption. Even if the detector dynamics is relevant, like in case of Unruh effect and similar phenomena - one usually tries to disentangle "beautiful" field theoretic part (universal response functions etc) from "ugly" detector part (concrete models of the detector). On the other hand, to what extent it is possible is undoubtedly quantitative question which should be analyzed in each particular problem.

We have all reasons to believe that our understanding of measurement procedure in QFT is not deep enough. Indeed, suppose one use lattice with the link a to measure some local quantities like $\langle T_{00}\rangle$ or $\langle G_{\mu\nu}G^{\mu\nu}\rangle$. The dominant term in the answer is given by proper power of a, and it characterizes the measuring device (i.e. lattice in the chosen example), but not the physics described by these operators. However we know from experiment that *there is some physics* behind these vacuum averages:

$$\begin{array}{cccl} \langle T_{00}\rangle & \sim & a^{-4}+..+10^{-29}\,\mathrm{g/cm^3} & \text{dark energy} \\[2ex] \langle G_{\mu\nu}G^{\mu\nu}\rangle & \sim & a^{-4}+..+0.012\,\mathrm{GeV}^4 & \text{gluon condensate} \\[2ex] M_H^2 & \sim & a^{-2}+..+(125)\,\mathrm{GeV}^2 & \text{Higgs mass} \end{array}$$

where the story with the Higgs boson mass is generally the same. No systematic way of disentangling "the physics of the detector" from "the physics of the physics" is known in the above (and many other) cases.[2]

In the present talk we analyse simple model of time-dependent measurement by point-like Unruh-DeWitt detector. Namely, we re-examine the Landauer's principle for finite time measurement protocols. The detector-field system state is described as a vector from the space $|n\rangle\otimes|\Phi\rangle$, where the index n encodes discrete state of the detector, while Φ stays for state of the field subsystem. For simplicity of notation we will use the symbol $|n,\Phi\rangle$ for the state of detector-field system, having in mind that there is no one-to-one correspondence between states of these two subsystems. For two-level Unruh-DeWitt detector immersed in the heat bath the index n takes values 0 or 1 and Φ_β stays for thermal state of free massless scalar field with inverse temperature $\beta=(k_BT)^{-1}$.

The evolution operator reads:

$$U_\chi = \mathrm{T}\exp\left[ig\int d\tau\;\chi(\tau)\mu(\tau)\phi(x(\tau))\right] \tag{6}$$

Here $x(\tau)$ parameterizes the detector's world-line, τ is a proper time along it, $\mu(\tau)$ - monopole transition operator for the detector and the latter is assumed to evolve with Hamiltonian H_d

[2]Needless to say that these are just famous "cosmological constant problem" and "hierarchy problem" viewed from different angle.

having discrete spectrum $\{E_n\}$. The field-detector coupling constant is g and the field $\phi(x(\tau))$ is assumed to be elementary or composite scalar field. Operators are ordered along the world-line by the standard T-ordering recipe. In what follows we do not consider moving detectors, so the proper time can be chosen to coincide with the usual time, $x(\tau) = (\tau, 0, 0, 0)$.

The important ingredient is the window function $\chi(\tau)$. It parameterizes the measurement procedure. In equivalent way one can speak about $g\chi(\tau)$ as about time-dependent coupling, which is different from zero in a given proper time intervals. We assume the function $\chi(\tau)$ to be real non-negative for any τ with the property:

$$\chi(\tau) \sim 1 \text{ for } |\tau - \bar{\tau}|/\tau_m \lesssim 1 \;\; ; \;\; \chi(\tau) \to 0 \text{ for } |\tau - \bar{\tau}|/\tau_m \to \infty \tag{7}$$

Here $\bar{\tau}$ is natural to identify with *the moment* of the measurement, while τ_m is the measurement's *duration*. The simplest case is given by $\chi_{12}(\tau) = \theta(\tau - \tau_1) - \theta(\tau - \tau_2)$ where $\tau_2 > \tau_1$ and the standard θ-function is 1 for $x > 1$ and 0 for $x < 0$.

The amplitude for the detector's transition to the state $|n\rangle$ from another state $|m\rangle$ is given by the following expression:

$$\mathcal{A}_{m\to n} = \langle \Phi', n|U_\chi|m, \Phi\rangle \tag{8}$$

The state $|\Phi'\rangle$ represents final (after the detector's transition) state of the field subsystem, while initially it is supposed to be in the initial state $|\Phi\rangle$. Transition probability summed over final states of the field subsystem reads:

$$\mathcal{P}_{m\to n} = \sum_{\Phi'} |\mathcal{A}_{m\to n}|^2 \tag{9}$$

and unitarity dictates $\sum_n \mathcal{P}_{m\to n} = 1$. For two-level detector it reads

$$\mathcal{P}_{0\to 0} = \langle 0, \Phi|U_\chi^\dagger|0\rangle\langle 0|U_\chi|0, \Phi\rangle \tag{10}$$

where the sum over intermediate states of the field subsystem has been already taken and due to completeness relation $|0\rangle\langle 0| + |1\rangle\langle 1| = 1$ the probability is normalized to unity as it should be.

One can show that the analyzed system thermalizes in Gaussian approximation [12]. This allows to make use of formal expansion over g giving at the leading order:

$$A_{m\to n} = ig\int d\tau\, \chi(\tau)\langle n|\mu(\tau)|m\rangle \cdot \langle \Phi'|\phi(x(\tau))|\Phi\rangle \tag{11}$$

and

$$P_{m\to n} = \lambda_{mn} \int d\tau \int d\tau'\, \chi(\tau)\chi(\tau')\, e^{-i\Omega_{mn}s}\, G(\tau, \tau') \tag{12}$$

where we denote $s = \tau - \tau'$, $\lambda_{mn} = g^2|\langle n|\mu(0)|m\rangle|^2$, $\Omega_{mn} = E_n - E_m$ and the Wightman function

$$G(\tau, \tau') = \langle \Phi|\phi(x(\tau))\phi(x(\tau'))|\Phi\rangle \tag{13}$$

In what follows we always assume $\lambda_{mn} = \lambda_{nm}$ and focus our attention on two-state detector, when the ground state's energy of the detector denoted as E_0 and energy of the excited state as E_1, with $\Omega = E_1 - E_0 > 0$ and $\lambda_{01} = \lambda_{10} = \lambda$.

Most papers dealing with expression (12) and alike, concentrate on effects resulting from the choice of the detector's trajectory $x(\tau)$. The best known result in this respect is celebrated

Unruh effect [13], which established the correspondence between two response functions computed from (12) in the leading order in g for infinite time measurement: one for $x(\tau)$ describing moving detector with constant acceleration a and interacting with free massless scalar field $\phi(x(\tau))$ at zero temperature; and another for $x(\tau)$ describing detector at rest in the thermal bath of free massless scalar field $\phi(x(\tau))$ at temperature $T = \frac{\hbar a}{2\pi k_B c}$. Much less attention has been paid to results, corresponding to nontrivial function $\chi(\tau)$ (see however [14, 15]). As is argued above, the (un)importance of finite time effects should be examined in every particular case from scratch. This is just the analysis we are going to present here.

All the above expressions can be greatly simplified in time-uniform case when the Wightman's function depends only on time difference, $G(\tau, \tau') = G(s)$, and does not depend on $\bar{\tau} = (\tau + \tau')/2$. Then one can define, developing the ideas of [14], the following operator:

$$D_\chi = D_\chi(\partial^2/\partial\Omega^2) = \frac{1}{\tau_m} \int d\bar{\tau}\, \chi\left(\bar{\tau} + (i/2)\partial/\partial\Omega\right) \chi\left(\bar{\tau} - (i/2)\partial/\partial\Omega\right) \tag{14}$$

where the measurement time τ_m is given by $\tau_m = \int d\bar{\tau}\, (\chi(\bar{\tau}))^2$. The transition probability (12) is rewritten as

$$P_{0\to1} = \lambda \cdot \tau_m \cdot D_\chi F(\Omega) \tag{15}$$

where

$$F(\Omega) = \int ds\, e^{-i\Omega s}\, G(s) \tag{16}$$

An advantage of the expression (15) is clear separation between the dynamics of the field subsystem, encoded in $F(\Omega)$, and the dynamics of measurement process, encoded in the operator (14). However a word of caution is to be said here. The operator (14) is, generally speaking, a differential operator of infinite order. For the interchange of integration over s and differentiation $\partial/\partial\Omega$ to be a valid operation, analytical properties of $F(\Omega)$ as the function of Ω are crucial. In this respect the presence of infrared regulator (e.g. nonzero temperature T) is important. Only at the final step one can remove it (e.g. to take zero temperature limit $\beta \to \infty$). Notice also that the definition (14) is independent on the normalization of $\chi(\tau)$ and without loss of generality one can take $\chi(0) = 1$.

The most interesting object is given by the ratio

$$\xi = \frac{P_{0\to1}}{P_{1\to0}} = \frac{D_\chi\, F(\Omega)}{D_\chi\, F(-\Omega)} \tag{17}$$

This probability ratio is finite, well defined and has direct physical meaning. In particular, for the two-state detector interacting with the heat bath with inverse temperature β (the corresponding Wightman function is denoted $G_\beta(s)$ in what follows) in thermal equilibrium for infinite time measurement one should have $\xi = \exp(-\beta\Omega)$.

Let us now study the general structure of the operator D_χ. The simplest example is given by infinite time measurement case $\chi(\tau) \equiv 1$. Then, obviously, $D_\chi = 1$. At large times one can expand D_χ as

$$D_\chi(\partial^2/\partial\Omega^2) = 1 + \frac{1}{2\tau_{eff}^2}\frac{\partial^2}{\partial\Omega^2} + O\left(\frac{\partial^4}{\partial\Omega^4}\right) \tag{18}$$

where we define positive quantity

$$\frac{1}{\tau_{eff}^2} = \frac{\int d\bar{\tau}\, (\chi'(\bar{\tau}))^2}{\int d\bar{\tau}\, (\chi(\bar{\tau}))^2} \tag{19}$$

having the meaning of effective interaction time. By no means should it coincide with the measurement time τ_m, even in parametric sense. The instructive examples are given by one-parametric Gaussian

$$\chi_G(\tau) = e^{-\frac{(\tau-\bar{\tau})^2}{\tau_0^2}}, \tag{20}$$

and Cauchy-Lorentz

$$\chi_L(\tau) = \frac{\tau_0^2}{(\tau-\bar{\tau})^2 + \tau_0^2}, \tag{21}$$

shapes, and, as more complex example, two-parametric window:

$$\chi_\theta(\tau) = \frac{\tanh(\lambda(\tau-\tau_1)) + \tanh(\lambda(\tau_2-\tau))}{2\tanh(\lambda\tau_0/2)} \tag{22}$$

In the latter case $\tau_{1,2}$ define the moments of proper time when the detector is switched on and off, respectively, and we assume $\tau_0 = \tau_2 - \tau_1 > 0$. The parameter $\lambda = 1/\delta\tau$ is the inverse switching time. It is usually reasonable to assume "large window" approximation $z = \lambda\tau_0 \gg 1$.

In all these cases the operator D_χ can be explicitly computed, the results read:

$$D_{\chi_G} = e^{(1/2\tau_0^2)\partial^2/\partial\Omega^2} \tag{23}$$

$$D_{\chi_L} = \frac{1}{1-(1/2\tau_0^2)\partial^2/\partial\Omega^2} \tag{24}$$

and

$$D_{\chi_\theta} = \frac{1}{2}\frac{\mathrm{sh}z}{\mathrm{sh}d}\frac{1}{z\,\mathrm{cth}z-1}\left[\frac{z-d}{\mathrm{sh}(z-d)} - \frac{z+d}{\mathrm{sh}(z+d)}\right] \tag{25}$$

where $z = \lambda\tau_0$ and $d = i\lambda\partial/\partial\Omega$. The latter formula simplifies significantly in large z limit, which we always assume, up to exponentially suppressed terms, it reads:

$$D_{\chi_\theta} = 1 + \frac{1}{z-1}\left[1 - \lambda\frac{\partial}{\partial\Omega}\,\mathrm{ctg}\,\lambda\frac{\partial}{\partial\Omega}\right] \tag{26}$$

All the expressions above should be understood as asymptotic expansions at large measurement time τ_0. For Gaussian and Cauchy-Lorentz shapes one has

$$\tau_m \sim \tau_{eff} \sim \tau_0 \tag{27}$$

On the other hand for two-parametric hyperbolic tangential shape (22) one gets (in soft switching limit):

$$\tau_{eff}^2 = \frac{3}{2}\,\tau_0 \cdot \delta\tau \tag{28}$$

where $\delta\tau$ is switching time. Notice that in abrupt switching limit (small $\delta\tau$) $\tau_m = \tau_0$ up to exponentially suppressed terms while τ_{eff} is not defined. The scaling $\tau_{eff} \sim \tau_0^{1/2}$ remarkably demonstrates that the system has a memory about its switching history even in the limit $\tau_0 \gg \delta\tau$. Important conclusion is that universal character of $1/\tau_{eff}^2$ asymptotic (18) could well mean actual $1/\tau_m$ (and not $1/\tau_m^2$) dependence for the leading finite time correction.

The opposite limit, $\tau_m \to 0$, is less trivial in a sense that in this limit the universality is lost and everything depends of the detailed microscopic model of the detector. Speaking formally, it is clear that this limit corresponds to small-s expansion of the Wightman function $G(s)$. One can expand the integrals of interest as a series in powers of τ_0:

$$D_\chi F(\Omega) = G(0) \cdot \int ds D_\chi(-s^2) + (G''(0) - 2i\Omega G'(0) - \Omega^2 G(0)) \int ds \frac{s^2}{2} D_\chi(-s^2) + .. \quad (29)$$

Some regularization is assumed in order to have $G(0)$ and its derivatives finite. Since $\int ds D_\chi(-s^2) \sim \tau_0$ at small τ_0, the leading term for the transition probability is

$$P \sim \tau_0^2 \cdot G(0) \quad (30)$$

to be compared with constant asymptotic for the ratio P/τ_m at large times (and hence, besides other things, exponential decays of excited states). This $\sim \tau^2$ behavior is crucial from quantum Zeno effect point of view, and the corresponding Zeno time is proportional to $1/\sqrt{G(0)}$.

As discussed in details in [14] the physical requirement that transition probability must vanish in $\tau_0 \to 0$ limit (i.e. when the detector is switched on for zero time, which should be equivalent to not switching it on at all) dictates the correct order of taking limits in (30): first to send τ_0 to zero and then to remove UV-regulator of $G(s)$.

It is instructive to look at the simplest case of free massless scalar field. The Wightman function reads:

$$G_\beta(s) = -\frac{1}{4\pi^2} \sum_{l=-\infty}^{\infty} \frac{1}{(s - i\zeta + i\beta l)^2} = -\frac{1}{4\beta^2} \frac{1}{\sinh^2(\pi(s - i\zeta)/\beta)} \quad (31)$$

where the infinitesimal parameter $\zeta > 0$ physically represents the finite size of any realistic detector. Then

$$F_\beta(\Omega, \zeta) = \int ds\, e^{-i\Omega s}\, G_\beta(s) = \frac{\Omega}{2\pi} \frac{e^{\zeta\Omega}}{e^{\beta\Omega} - 1} \quad (32)$$

In zero temperature limit

$$F_\beta(\Omega, 0) \to F_\infty(\Omega, 0) = -\Theta(-\Omega)\, \frac{\Omega}{2\pi} \quad (33)$$

There is a following important relation between the functions $F_\beta(\Omega, \zeta)$ and $F_\beta(-\Omega, \zeta)$

$$F_\beta(-\Omega, \zeta) - F_\beta(\Omega, \zeta) = \frac{\Omega}{2\pi} - 2\sinh\frac{\zeta\Omega}{2}\, [F_\beta(\Omega, \zeta/2) + F_\beta(-\Omega, \zeta/2)] \quad (34)$$

The second term in the right hand side of (34) vanishes in the limit $\zeta \to 0$. As is mentioned above to keep $\zeta \neq 0$ is important in order to have correct zero measurement time limit $\tau_0 \to 0$. However if one is not interested in $\tau_0 \to 0$ limit, one can put $\zeta = 0$ and to simplify notation we drop the second argument of $F_\beta(\Omega, \zeta)$ and denote $F_\beta(\Omega) = F_\beta(\Omega, 0)$ in what follows. Then expression (17) can be rewritten as

$$\xi = 1 - \frac{\Omega}{2\pi} \cdot \frac{1}{D_\chi\, F_\beta(-\Omega)} \quad (35)$$

where we also have taken into account that

$$D_\chi(\partial^2/\partial\Omega^2)\frac{\Omega}{2\pi} = \frac{\Omega}{2\pi} \quad (36)$$

It is instructive to consider concrete examples. The simplest one is given by infinite time measurement case $\chi(\tau) \equiv 1$. Then $D_\chi = 1$ and inserting (32) into (35) one gets expected result $\xi = \exp(-\beta\Omega)$ - the standard thermal distribution. The leading correction to this stationary case is suppressed at large τ_m. This correction is universal in the sense that it does not depend on the exact profile of $\chi(\tau)$ function but only on some integral moment of it. Indeed, plugging the expansion (18) into (35) one finds

$$\xi = e^{-\beta\Omega} \left(1 + \frac{\beta^2 g(\beta\Omega)}{2\tau_{eff}^2}\right) \tag{37}$$

where $g(x) = \mathrm{cth}\frac{x}{2} - \frac{2}{x}$. In high temperature limit $\beta\Omega \ll 1$ one has $g(x) \sim x/6$, which means that at this order the distribution remains to be thermal with effective renormalization of the temperature $\beta \to \beta^*$:

$$T^* = T\left(1 + \frac{\hbar^2}{12\tau_{eff}^2 (k_B T)^2}\right) \tag{38}$$

If we consider the detector as a thermometer, one can say that it returns higher values if the system's temperature measurement takes finite time. Most of thermometers people use in everyday life need quite some time (from seconds to minutes) to thermalize with a system being measured and indicate accurate result, the expression (38) defines a quantum limit for this kind of processes. It is remarkable that at the leading order it has no explicit Ω-dependence (coding the thermometer microstructure, in a sense).

Making use of thermodynamic identity

$$\frac{d}{d\beta} S(\beta) = -\beta\, \mathrm{var}_\beta\,(H) \tag{39}$$

one conclude that the detector's entropy change is given by

$$S^* - S = \frac{\hbar^2}{12\tau_{eff}^2 (k_B T)^4}\ \left(\langle E^2\rangle - \langle E\rangle^2\right) \tag{40}$$

This result reminds another well known phenomenon of time-dependent coarse graining - having a given amount of Shannon information I (for example, some image), one can typically restore only coarse grained version of it (i.e. an image containing less information $I^* < I$) for a finite time, if information transferring channel capacity is bounded. The difference $I - I^*$ approaches zero for infinite time.

Low temperature limit $\beta \to \infty$ is more tricky since the expansion (37) takes $\beta/\tau_{eff} \ll 1$ as small parameter. In any case due to finite energy gap $\Omega > 0$ the ratio ξ is exponentially suppressed in this limit.

One can also pose another question - what is the softest possible mode for switching, i.e. optimal time profile for $\chi(\tau)$ which minimizes the finite-time corrections given fixed measurement time. In general case this is a complicated variational problem in the spirit of Pontryagin's optimal control theory. However in large time limit it gets simple - according to (19) one should minimize $\int d\tau(\chi'(\tau))^2$ with fixed measurement time $\int d\tau(\chi(\tau))^2$. Since τ_{eff} is independent on normalization of $\chi(\tau)$, without loss of generality the answer reads:

$$\chi_{opt}(\tau) = e^{-|\tau|/\tau_0} \tag{41}$$

Another interesting question about possibility to reproduce thermal distribution by some choice of the switching function. In other words, we are looking for $\chi(\tau)$ such that the resulting detector levels distribution is quasi-thermal at inverse temperature $\bar{\beta}$, despite the original bath is at zero temperature. One can see from (15) that this corresponds to the following condition

$$D_\chi F_\infty(\Omega) = F_{\bar{\beta}}(\Omega) \tag{42}$$

In principle, nothing guarantees the existence of regular solutions to (42). This however happens to be the case. Expanding in Fourier components $\chi(\omega) = \int d\tau \chi(\tau) e^{i\omega\tau}$ with time-symmetric switching function, which corresponds to real $\chi(\omega)$, and taking into account that

$$\frac{\partial^2 F_\infty(\omega+\Omega)}{\partial\omega^2} = \frac{\delta(\omega+\Omega)}{2\pi} \tag{43}$$

one obtains the following result

$$\chi_{\bar{\beta}}(\tau) = \int d\omega \left(\tau_m \, \frac{\partial^2 F_{\bar{\beta}}(\omega)}{\partial\omega^2}\right)^{1/2} e^{-i\omega\tau} \tag{44}$$

where the factor ${\tau_m}^{1/2}$ provides correct normalization. This "heating by switching" has much in common with heating by acceleration which is at the heart of Unruh effect.

3 Finite time Landauer's principle

The aim of this section is to apply the formalism developed above to erasure process. In this section we consider the single detector as point-like two-level system with energy gap Ω encoding one classical bit of information. Alternatively, this detector can be seen as a pixel of the macro-detector. There are two different setups - in the first case detector is prepared in either excited or ground state and since it does not interact with the environment it remains in this eigenstate. At some moment detector gets in weak contact with the thermal bath for finite time. If this interaction time is large enough one expects that the detector will be finally thermalized, i.e. speaking in terms of macro-detector the number of excited micro-detectors with respect to non-excited ones will be $e^{-\beta\Omega}$. This obviously corresponds to erasure of initial information [16].

There is another case, with switching off fully thermalized detector. In principle, for finite switching time the final distribution of the detector's levels will not be thermal. In order to prevent perpetuum mobile of the second kind the process must cost some energy and Landauer's principle puts a lower bound on it.

Speaking more formally, let initial state of the detector is described by the following density matrix

$$\rho = p\,|1\rangle\langle 1| + (1-p)\,|0\rangle\langle 0| \tag{45}$$

and the corresponding entropy reads $S(p) = -p\log p - (1-p)\log(1-p)$. Energy balance has the following form:

$$E^*_\phi + E^*_d = E_\phi + E_d + \delta E \tag{46}$$

where E_d, E_ϕ and E^*_d, E^*_ϕ are the initial and the final energies of the detector and the bath, respectively, while δE is the work done by external force responsible for turning the detector on/off. The Landauer principle dictates

$$\beta\Delta Q - \Delta S \geq 0 \tag{47}$$

where

$$\Delta Q = E_\phi^* - E_\phi \ \ ; \ \ \Delta S = S_d - S_d^* \tag{48}$$

or, in other words

$$\kappa = \beta(E_d - E_d^* + \delta E) - (S_d - S_d^*) \geq 0 \tag{49}$$

where change in the detector's energy is given by

$$E_d^* - E_d = \Omega(p^* - p) \tag{50}$$

Let's first check the infinite time measurement case, when final probability corresponds to pure thermal distribution:

$$p_\infty^* = \frac{F_\beta(\Omega)}{\Sigma_\beta(\Omega)} = \frac{1}{1 + e^{\beta\Omega}} \tag{51}$$

and no external work is done, $\delta E = 0$. Straightforward computation shows that

$$\kappa = \log\left(1 + e^{-\beta\Omega}\right) + \beta\Omega p - S(p) \tag{52}$$

where $S(p)$ is defined above. This quantity is indeed nonnegative for any choice of initial state p as it should be. The lower bound $\kappa = 0$ is trivially reached when the initial state is the same thermal state as the final one, i.e. when $p = p_\infty^*$.

For finite time measurement the situation becomes more complicated because of the term δE, accounting for the work done by external force. The Landauer principle can be reformulated in this case as a lower bound on this work δE. Consider again large time limit. Then one has

$$p^* = \frac{\xi}{1 + \xi} \tag{53}$$

The bound of interest take the form:

$$\beta\delta E \geq S(p) - \beta\Omega p + (\beta\Omega + \log\xi)\,\frac{\xi}{1 + \xi} - \log\left(1 + \xi\right) \tag{54}$$

and of special interest is the case of thermal initial state, then

$$\beta\delta E \geq (\beta\Omega + \log\xi)\,\frac{\xi}{1 + \xi} + \log\left(\frac{1 + e^{-\beta\Omega}}{1 + \xi}\right) \tag{55}$$

The physical meaning of the above results is transparent - if in the course of interaction between the detector and the thermal bath the detector state becomes (due to finite time effects) non-thermal (and more ordered in this sense) than it used to be initially, this must be compensated by external work, exceeding the limit (55). Taking into account (37) the leading term reads:

$$\delta E \geq \frac{k_B T}{8}\left(\frac{\hbar}{k_B T \cdot \tau_{eff}}\right)^4 \cdot \frac{e^x g^2(x)}{(1 + e^x)^2} \tag{56}$$

with $g(x)$ defined above and $x = \beta\Omega$. Last factor in (56) reaches its maximum ≈ 0.01 at $x \approx 2.17$. The existence of such bound is important as a matter of principle, however quantitatively it is very tiny due to strong suppression - both parametrical (τ_{eff}^{-4} instead of τ_{eff}^{-2}) and numerical.

4 Conclusion

Any realistic measurement (or erasure) needs finite time. Landauer's bound implicitly assumes quasi-stationarity and gets corrected at finite time. The simple model we used to study finite time effects in measurement procedure demonstrates the pattern which may be of much more general character. One can transmit and process only finite amount of information in finite time. Moreover, pattern recognition by human brain works even in more intricate way – when human sees something looking similar to a snake, two processes starts in parallel. The first one is crude but fast analysis of visual perception by brain's limbic system – is it dangerous or not, should I run away or no – this is life preserving mechanism, common for all mammals. At the same time, more precise analysis by neocortex is going and it needs more time – is it a snake at all, if yes, what kind of snake it is etc. Expressions like (38),(40) put fundamental quantum mechanical limits to this kind of processes. Non-stationary Landauer's principle from this point of view becomes a kind of entropy-time uncertainty relation, to some extent resembling energy-time uncertainty relation. It is presumably not universal but still can be interesting and useful.

Acknowledgements

The author acknowledges partial support from Russian Foundation for Basic Research, grant RFBR-ofi m 14-22-03030.

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