

Chiral liquids

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Abstract. We review briefly properties of chiral liquids, or liquids with massless fermionic constituents. We concentrate on three effects, namely, the low ratio of viscosity η to entropy density s , chiral magnetic and vortical effects. We sketch standard derivations of these effects in the hydrodynamic approximation and then concentrate on possible unifying approach which is based on consideration of the (anomalously) conserved axial current. The point is that the conservation of chirality is specific for the microscopic, field-theoretic description of massless fermions and their interactions. On the macroscopic side, the standard hydrodynamic equations are not consistent, generally speaking, with conservation of a helical macroscopic motion. Imposing extra constraints on the hydrodynamics might resolve this “clash-of-symmetries” paradox.

1 Introduction

By chiral liquids one understands media with massless fermionic constituents interacting in a chiral invariant way. Coupling of the fermions to gauge fields results in a chiral anomaly. Mostly, one considers $U(1)$ anomaly, and we will concentrate on this case as well. Theory of the chiral liquids attracted a lot of attention recently, for reviews see, e.g., the corresponding volume of Springer lectures in physics [1]. One of the main reasons for such an interest is that one expects that there are macroscopic manifestations of the chiral anomaly, which by itself is a loop, or quantum phenomenon. What is most intriguing, the chiral effects are expected to persist, say, at high temperature, or, more generally, be described by standard hydrodynamics, see [2–4] and further references therein. In other words, dynamics of chiral liquids on microscopical level is not similar to that of superfluids. Nevertheless, in both cases one expects dissipation-free flow of currents in equilibrium.

There are a few specific effects revealing the chiral nature of the underlying field theory. The chiral magnetic effect seems to be best known, for a recent review and further references see [5]. Namely, there is a flow of electric current along external magnetic field B_μ :

$$\mathbf{j}_\mu^{el} = \mu_5 \frac{e^2}{2\pi^2} \mathbf{B}_\mu, \quad (1)$$

where μ_5 is the chiral chemical potential, $\mu_5 = \mu_R - \mu_L$, e is the electric charge of a massless Dirac fermion of the underlying field theory, $B_\mu = 1/2\epsilon_{\mu\nu\rho\sigma}u^\nu F^{\rho\sigma}$, u^ν is the 4-velocity of an element of the liquid and $F^{\rho\sigma}$ is the external electromagnetic field strength tensor.

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Another important effect is the chiral vortical effect, for a concise review and references see, e.g., [5]. In this case, the theoretical prediction is that axial current j_μ^5 is contributed by macroscopic helical motion of the liquid ¹:

$$j_\mu^5 = \frac{\mu^2}{2\pi^2} \omega_\mu, \quad (2)$$

where μ is the chemical potential $\mu = \mu_R + \mu_L$ (assuming now $\mu_5 = 0$) and $\omega_\mu = 1/2\epsilon_{\mu\nu\rho\sigma} u^\nu \partial^\rho u^\sigma$. There are other effects of similar nature but consideration of the predictions (1) and (2) suffices for our purposes.

Eqs (1), (2) can be derived in various ways. In particular, Eq (2) emerged first within a holographic approach [3]. Later, however, it was found that the effect is not specific for the holographic description and can be derived within the standard hydrodynamic approach [2]. However, there is another prediction which remains specific for holography and has good chances to be true phenomenologically. We mean the ratio η/s :

$$\frac{\eta}{s} = \frac{1}{4\pi}. \quad (3)$$

Holography predicts that this ratio is the same as for the fictitious liquid [6] living on the stretched horizon of black holes. Field-theoretic approach to derivation of (3) in case of the black-hole physics was elaborated first in [7] while in Ref. [8] it was argued that in fact the ratio (3) represents a universal lower bound consistent with the uncertainty principle.

2 Derivation of the effects

As has already been mentioned there exist various techniques to derive the chiral magnetic and vortical effects. The most economical approach seems to be to use exclusively hydrodynamic approximation, – as an expansion in derivatives,– and fundamental conservation laws of the underlying field theory [2]. As is well known the hydrodynamic equations of motion are nothing else but conservations of the energy-momentum tensor and of the Noether currents relevant to the problem considered. In our case of presence of external electromagnetic fields:

$$\begin{aligned} \partial^\mu T_{\mu\nu} &= e F_{\mu\nu} j_{el}^\mu, \\ \partial^\mu j_\mu^a &= \frac{e^2}{4\pi^2} E^\nu B_\nu, \\ \partial^\mu j_\mu^{el} &= 0, \end{aligned} \quad (4)$$

where E^ν is the electric field in the medium, $E^\nu = F^{\nu\mu} u_\mu$. One more condition is to be added [2], that is, growth of entropy, or $\partial^\mu s_\mu \geq 0$, where s_μ is the entropy current.

Substituting the standard derivative expansion for the energy-momentum tensor and currents, the authors of Ref. [2] (see also [9]) were able to fix the currents in the approximation of ideal liquid and reconstruct Eqs (1) and (2). Keeping further terms in the derivative expansions one apparently could evaluate cotrections the results obtained. The derivation [2] is far from being trivial and, in fact, somewhat puzzling as well. The point is that the numerical coefficients in the right-hand sides of (1) and (2) are fixed, within the approach considered, in terms of the coefficient determining the anomalous piece in the equation for divergence of the axial current in (4). For the anomaly to be effective one has to include second order effects in the electromagnetic coupling $\sqrt{\alpha_{el}}$ and introduce a

¹For simplicity we ignore temperature dependences, which could be present, see original papers. This simplification does not affect our conclusions

non-vanishing electric field. On the other hand, the chiral vortical effect (2) survives in the zero-order approximation in $\sqrt{\alpha_{el}}$ and exists independent of introduction of the electric field.

A possible explanation is that hydrodynamic expansion is not just an effective field theory, as is commonly recognized. Indeed, introducing an effective field theory implies integrating out high-frequency degrees of freedom. By applying the long-wave approximation, or expansion in derivatives one does follow the standard effective-field-theory prescription. However, thermodynamics assumes also a change in the original Hamiltonian H_0 :

$$H_0 \rightarrow H_0 - \mu Q, \quad (5)$$

where Q is a conserved charge and μ is the chemical potential associated with this charge. The substitution (5) does not correspond to any integration over hard degrees of freedom. Moreover, the very notion of the chemical potential can be introduced only on average, in the context of thermodynamics.

Rewriting (5) in a formally Lorentz invariant way one comes to the conclusion [10], that the chiral anomaly in the hydrodynamic approximation contains more terms than its standard field theoretic expression. Indeed, we notice that for small μ there is a controllable change in the Lagrangian and its density:

$$\begin{aligned} \delta L &= -\delta H = \mu \cdot Q = \mu \cdot \int d^3x j_0 \\ \delta S &= \int dt \delta L \rightarrow \int d^4x \mu \cdot u^\mu j_\mu. \end{aligned} \quad (6)$$

Treating δS as a new perturbative term, along with the standard electromagnetic interaction we conclude that the new perturbative series, relevant to hydrodynamics can be obtained from the standard field theory results by the substitution:

$$eA_\mu \rightarrow eA_\mu + \mu u_\mu, \quad (7)$$

where both the electromagnetic potential A_μ and the 4-velocity u_μ are treated as external fields.

As a result of the extension (7) the chiral vortical effect (2) arises automatically. Moreover, in terms of the fundamental current the vorticity term (2) is to be considered as a matrix element of a conserved axial current, since the anomaly is of higher order in the electromagnetic coupling. The fact that there exists a field-theoretic route to derive (2) directly through substitution (5) looks encouraging. However, there arises a new question, what is the physical meaning of conservation of vorticity seemingly implied by the conservation of the fundamental axial current. We will address this issue in the next section. Now, we notice that the chiral anomaly does not have corrections, by virtue of the Adler-Bardeen theorem. Seemingly the same is to be true for the chiral vortical effect (2) since it is related to the standard chiral anomaly through the substitution (7). It is not obvious, however, whether this constraint is consistent with the general hydrodynamic expansion in derivatives.

3 Conservation of helicity

Let us first recall that the standard chiral anomaly in an abelian case can be rewritten as conservation of a new axial charge Q^a which is ascribed also to external electromagnetic field:

$$Q^a = Q_{naive}^a + \frac{e^2}{2\pi^2} \mathcal{H}_{magn}, \quad \frac{Q^a}{dt} = 0, \quad (8)$$

where Q_{naive}^a is the original axial charge of the constituents, or matrix element of $\int d^3x \bar{\Psi} \gamma_0 \gamma_5 \Psi$ and \mathcal{H}_{magn} is the so called magnetic helicity,

$$\mathcal{H}_{magn} = \int d^3x \vec{A} \cdot \vec{B} . \quad (9)$$

Note that the magnetic helicity has been discussed in the context of ordinary magneto-hydrodynamics, with dynamical electromagnetic field in many papers, see, e.g. [11] and references therein.

The argumentation presented in the preceding section implies that in the hydrodynamic approximation the conserved axial charge includes further terms [12]:

$$Q_{hydro}^a = Q_{naive}^a + \frac{1}{2\pi^2} \mathcal{H}_{fluid} + \frac{e}{2\pi^2} \mathcal{H}_{mixed} + \frac{e^2}{2\pi^2} \mathcal{H}_{magn}, \quad \frac{dQ_{hydro}^a}{dt} = 0 , \quad (10)$$

where

$$\mathcal{H}_{fluid} = \int d^3x \epsilon_{ijk} (\mu u^i) \partial^j (\mu u^k), \quad \mathcal{H}_{mixed} = \int d^3x \epsilon_{ijk} A^i \partial^j (\mu u^k) = \int d^3x \epsilon_{ijk} (\mu u^i) \partial^j A^k , \quad (11)$$

and the magnetic helicity is defined in Eq (9). Again, both the fluid and mixed helicities have been studied since long, see [11] and references therein. It is worth noting that the helicities introduced above can be viewed as linking numbers of various defects [11]. In particular, the fluid helicity is related to the linking number of vortices, while the mixed helicity can be expressed in terms of the linking number of vortices and magnetic tubes.

The simplest way to ensure the required conservation of the hydrodynamic axial charge is to impose condition of conservation of each term in the sum in the r.h.s. of Eq (10). It has been shown [11] that conservation of the magnetic helicity requires infinite conductivity,

$$\sigma_E \rightarrow \infty , \quad (12)$$

or complete screening of the electric field in plasma:

$$E^\mu B_\mu \rightarrow 0 . \quad (13)$$

Another implication of (12) is that the plasma is *dissipation free*.

The fluid helicity (11) is also conserved under certain conditions [11]. In particular, a necessary condition seems to be vanishing of the viscosity, or dissipation:

$$\eta \rightarrow 0 . \quad (14)$$

Further constraints might also be necessary according to [11]. Of course, in all these cases we discuss classical physics, while the lower bound (3) is a consequence of the uncertainty principle. The right-hand side of (3) is in fact proportional to the Planck constant which is not manifested because of the choice of units.

4 Conservation of axial charge

Let us pause here and discuss, why we have started to impose constraints on hydrodynamics to ensure (anomalous) conservation of the axial charge. Indeed, this is quite unusual and nobody would dream to derive constraints like (12) or (14) from conservation of, say, electric charge.

The point is that the conservation of electric charge is common both for the microscopic underlying theory and effective theory of hydrodynamics. This is not so in case of chiral symmetry. We do assume the chiral invariance to be a property of the underlying fundamental theory since it is built on interactions of massless fermions. On the other hand, the general hydrodynamic approximation is not incorporating any chiral symmetry since it is not a universal symmetry, like rotational or translational invariance. One can talk about possible “clash of symmetries” of the fundamental theory and its macroscopic, or effective counterpart. The logic promoted so far is that conservation of chirality on the fundamental level implies conservation of macroscopic helical motions, or helicities. In turn, conservation of helicities requires dissipation-free limit, or ideal liquid.

The problem can be made more precise if one uses explicit field theoretic approach to hydrodynamics of ideal liquids, see, in particular, Ref. [14] and references therein. Within this approach, the effective infrared degrees of freedom are represented by massless scalars φ^I , $I = 1, 2, \dots, d - 1$ where d is the dimension of space-time. The conserved entropy current s_α can be immediately constructed in terms of the scalars φ^I . For example, in case of $d = 4$:

$$s_\alpha = (const)\epsilon_{\alpha\beta\gamma\delta}\epsilon_{IJK}\partial^\beta\varphi^I\partial^\gamma\varphi^J\partial^\delta\varphi^K . \tag{15}$$

This current is automatically conserved and does not receive any corrections of higher order in derivatives.

Moreover, further scalars are introduced if there are conserved charges. In particular, to realize the chemical-shift symmetry behind (5) one introduces an extra massless scalar ψ , see [14] and references therein. One can develop intuition on the transformation properties and interactions of ψ if ψ is treated as a relativistic generalization of the phase of the wave function of the superfluid component in case of superfluidity, $\Psi \sim \sqrt{\rho_s} \exp(i\psi)$, where ρ_s is the density of the superfluid component. No superfluidity is assumed, however, in the general case. In terms of the scalar ψ the corresponding conserved current appears to be a Noether current conserved only with account of equations of motion of the ideal liquid, or on mass shell. In Ref [13] this programm was realized also in case of anomalous axial current which we considered in the preceding section.

For this purpose, it is crucial to use the so called entropy frame [13, 14] where the entropy current (15) takes the form:

$$s_\alpha \equiv s u_\alpha , \tag{16}$$

where s is the entropy density. This relation serves as definition of the 4-velocity u_α . It is in this frame that the conserved axial charge (10) receives no further contributions from the hydrodynamic expansion in derivatives [13]. As is mentioned above the possibility to introduce such a charge is implied [10] by the hydrodynamic generalization of the Adler-Bardeen theorem.

5 Conclusions

Theory of chiral liquids is an exciting subject since chiral liquids possess remarkable properties. We mentioned, in particular, the chiral magnetic effect (1), chiral vortical effect (2) and the low value (3) of the ratio η/s . So far, the low value of viscosity has been predicted only within holographic approach in those cases when theory in Minkowski space does have a holographic counterpart in extra dimensions.

Following Ref. [12] we argued that the viscosity η vanishing in the classical limit, see (14) might be required by the (anomalous) conservation of the axial charge (10). In other words, there is a possibility that the conservation of the axial charge might be realized only at price of constraining hydrodynamics of the chiral liquids. Conservation of the axial current is so specific because the

hydrodynamics in its generality does not incorporate any chiral symmetry which can be a property only of interactions of massless fermionic constituents in the underlying fundamental theory. In this sense conservation of helicities is foreign to the general formulation of hydrodynamics.

At the moment, however, one cannot rule out that conservation of the axial current is kinematic in nature and is not associated with any Noether current within field theoretic approach to describe liquids. An example of such a construction is provided in Ref [13]. The physics behind could be that the non-trivial hydrodynamic charges associated with the fluid and mixed helicities, see (10) imply use of non-inertial, rotating frames. An analogy might be provided by, say, the Unruh effect. An accelerated observer sees thermal excitations but the energy-momentum tensor associated with the thermal radiation is not conserved separately. Instead, it is conserved only in conjunction with the energy-momentum tensor of the vacuum fluctuations:

$$\langle T_{\mu\nu} \rangle_{Minkowski} = \langle T_{\mu\nu} \rangle_{thermal} + \langle T_{\mu\nu} \rangle_{vacuum} , \quad (17)$$

where $\langle T_{\mu\nu} \rangle_{Minkowski} = 0$ is the Minkowskian expectation value of the energy-momentum tensor normalized, by definition, to zero, $\langle T_{\mu\nu} \rangle_{thermal}$ is the energy-momentum tensor associated with the thermal radiation seen by an accelerated observer, $\langle T_{\mu\nu} \rangle_{vacuum}$ is the energy-momentum tensor associated with the vacuum quantum fluctuations in the Rindler frame. Note that the sum in the r.h.s. of (17) is vanishing by virtue of the general covariance.

The mechanism of conservation of the axial charge $Q^a = Q_{naive}^a + (1/2\pi^2)\mathcal{H}_{fluid}$ could be similar. Within our analogy, the term Q_{naive}^a would be then similar to the energy $\langle T_{\mu\nu} \rangle_{thermal}$ seen by the observer while the \mathcal{H}_{fluid} would be similar to $\langle T_{\mu\nu} \rangle_{vacuum}$. We will dwell on this possible analogy elsewhere.

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