Nonextensive critical effects in the NJL model

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Abstract. It is shown how nonextensive Nambu-Jona-Lasinio (NJL) calculations in the critical region in the vicinity of phase transition to quark matter allow to determine a nonextensive parameter q from a specific heat capacity C_q.

1 Introduction

We present a possible extension of the usual relativistic nuclear mean field models widely used to describe nuclear matter towards accounting for the influence of the possible intrinsic fluctuations caused by the environment. Rather than individually identifying their particular causes we concentrate on the fact that such effects can be summarily incorporated in the changing of the statistical background used, from the usual (extensive) Boltzman-Gibbs (BG) one to the nonextensive taken in the form proposed by Tsallis [1] with a dimensionless nonextensivity parameter q responsible for the above mentioned effects (for $q \rightarrow 1$ one recovers the usual BG case). We illustrate this proposition by the example of the QCD-based Nambu - Jona-Lasinio (NJL) model ([2]) of a many-body field theory describing the behavior of strongly interacting matter using its nonextensive version. We check the sensitivity of the usual NJL model to the departure from BG scenario expressed by the value of $|q - 1|$, in particular in the vicinity of critical points.

2 Formalism

2.1 The q extension of the NJL model - the q-NJL

The q-statistics is introduced by using the q-form of quantum distributions for fermions - quarks. There are also similar modifications of nuclear mean field ([3]) for nucleons (with higher baryon resonances with strangeness) ([4, 5]). This non-extensive statistics is introduced following a prescription provided in [6], namely by replacing n and $\bar{n}$ by

$$n_{qi} \quad = \quad \frac{1}{\tilde{e}_q(\beta(E_{qi} - \mu_i)) \pm 1}, \tag{1}$$

(notice that $E_{qi} = \sqrt{M_{qi}^2 + p^2}$ and M_{qi} are both q-dependent quantities). Denoting $x = \beta(E - \mu)$ one has that for $q > 1$

$$\tilde{e}_{q>1}(x) \quad = \quad [1 + (q - 1)x]^{\frac{1}{q-1}} \quad \text{if} \quad x > 0 \quad \text{and} \quad [1 + (1 - q)x]^{\frac{1}{1-q}} \quad \text{if} \quad x \leq 0, \tag{2}$$

$$\tilde{e}_{q<1}(x) \quad = \quad [1 + (q - 1)x]^{\frac{1}{q-1}} \quad \text{if} \quad x \leq 0 \quad \text{and} \quad [1 + (1 - q)x]^{\frac{1}{1-q}} \quad \text{if} \quad x > 0. \tag{3}$$
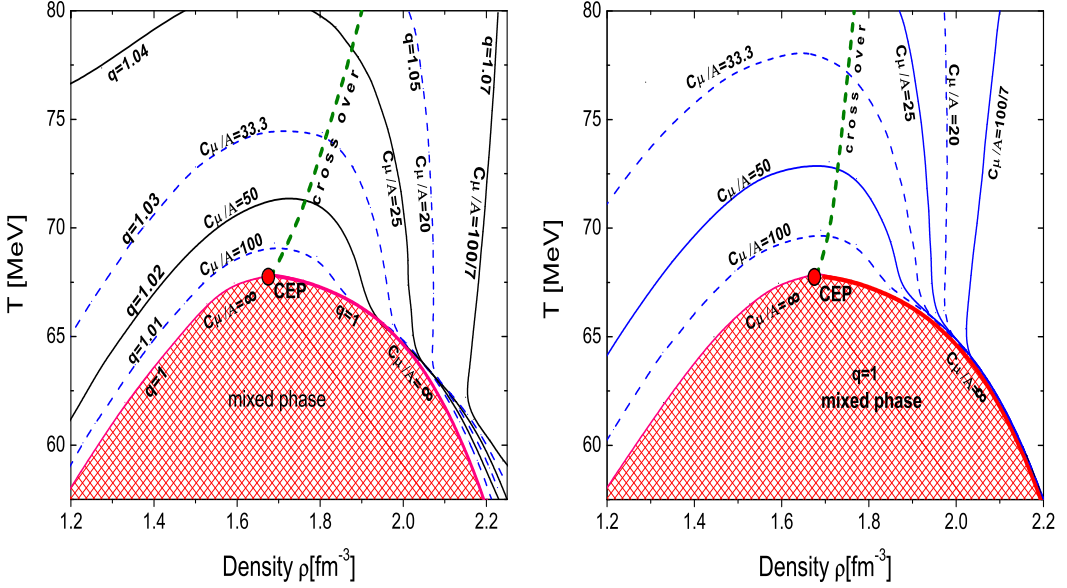
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Figure 1. The specific heat C_μ/A calculated with extensive in n_q density, $\varrho = \Sigma n_q/V$, in the vicinity of the critical values of temperature T and density as function of density ρ and temperature. The shadowed area is the mixed phase with infinite boundaries of C_μ/A. In the left panel, the non-extensive parameter q was determined from the self-consistent equation (10). The right panel presents, for comparison, results for $q = 1$ (BG).

In this way one can consistently treat on the same footing quarks and antiquarks for all values of x. Notice the particle-hole symmetry observed in the q-Fermi distribution in plasma containing both particles and antiparticles: $n_q(E, \beta, \mu, q) = 1 - n_{2-q}(-E, \beta, -\mu)$. It means, that in our system, which contains both particles and antiparticles, both q and $2 - q$ occur (or, that one can encounter both $q > 1$ and $q < 1$ at the same time, and they both have physical meaning). This differentiates our q-NJL model from the model presented in [7]. Notice that for $q \to 1$ one recovers the standard FD distribution, $n(\mu, T)$. It is important to realize that for $T \to 0$ one always gets $n_q(\mu, T) \to n(\mu, T)$, irrespectively of the value of q [7]. This means that we can expect any nonextensive signature only for high enough temperatures [8].

Our q-NJL model is obtained by replacing the usual NJL formulas using the FD distributions with their q-counterparts [10]. Additionally, when calculating energies and condensates we follow now [11] and use the following q-versions of quark condensates:

$$\langle \bar{q}_i q_i \rangle_q = -\frac{N_c}{\pi^2} \sum_{i=u,d,s} \left[\int \frac{p^2 M_{qi}}{E_{qi}} (1 - n_{qi}^q - \bar{n}_{qi}^q) \right] dp \tag{4}$$

$$M_{qi} = m_i - 2g_s \langle \bar{q}_i q_i \rangle_q - 2g_D \langle \bar{q}_j q_j \rangle_q \langle \bar{q}_k q_k \rangle_q, \tag{5}$$
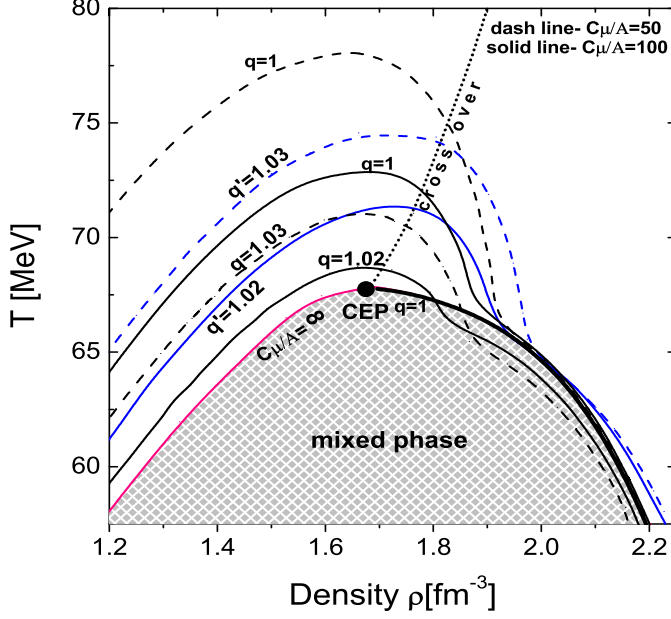
Figure 2. The specific heat C_μ/A calculated in the vicinity of the critical values of temperature and density as function of density ρ and temperature. The shadowed area is a mixed phase with infinite boundaries of C_μ/A. Here we compare the lines for the constant heat capacity, $C_\mu/A = \infty, 100, 50$, for three different cases: the extensive BG case, marked by the value $q = 1$, case of extensive in n_q density, $\varrho = \Sigma n_q/V$, marked by $q' = 1.02, 1.03$ and case of nonextensive density $\varrho = \Sigma n_q^q/V$ marked by the respective q values.

$$E_q = -\frac{N_c}{\pi^2}V\sum_{i=u,d,s}\left[\int p^2 dp \frac{p^2 + m_i M_{qi}}{E_{qi}}(1 - n_{qi}^q - \bar{n}_{qi}^q)\right] - g_S V\sum_{i=u,d,s}\left(\langle \bar{q}_i q_i\rangle_q\right)^2 - 2g_D V\langle \bar{u}u\rangle_q\langle \bar{d}d\rangle_q\langle \bar{s}s\rangle_q.$$

Baryon densities ϱ are introduced in two different ways: (i) using simply (extensive in n_q) density, $\varrho = \Sigma n_q/V$ (as in |citeLPQ), and (ii) by using nonextensive density defined in [15] as $\varrho_q = \Sigma n_q^q/V$ (and satisfying thermodynamical consistency condition). The pressure for a given q is calculated using the energy E_q and the q-entropy (cf. [6]):

$$
\begin{aligned}
S^q &= -\frac{N_c}{\pi^2}V\sum_{i=u,d,s}\int p^2 dp \cdot \tilde{S}_q, \\
\tilde{S}_q &= \left[n_{qi}^q \ln_q n_{qi} + (1 - n_{qi})^q \ln_q(1 - n_{qi})\right] + \left\{n_{qi} \rightarrow 1 - \bar{n}_{qi}\right\}.
\end{aligned}
\tag{6}
$$

Eqs. (5,6) together with the q-version of the gap equation, Eq.(4), are the basic equations from which we deduces results presented here.

The single particle specific heat for the constant chemical potential μ defined as the entropy derivative, $C_\mu/A = T\partial S/A\partial T|_\mu$, reads:

$$\frac{C_\mu}{A} = \frac{-N_c T}{\pi^2 \rho} \sum_{i=u,d,s} \int p^2 dp \left.\frac{d\tilde{S}_q}{dT}\right|_\mu = \frac{q\beta N_c}{\pi^2 \rho} \sum_{i=u,d,s} \int p^2 dp * [(E_{qi} - \mu_i) n_{qi}^{q-3} \frac{\tilde{e}_q(\beta(E_{qi} - \mu_i))}{1 + (q-1)\beta(E_{qi} - \mu_i)} +$$

$$+ (E_{qi} + \mu_i) \bar{n}_{qi}^{q-3} \frac{\tilde{e}_q(\beta(E_{qi} + \mu_i))}{1 + (q-1)\beta(E_{qi} + \mu_i)}] (\frac{\partial E_{qi}}{\partial T} - \frac{E_{qi}}{T}). \tag{7}$$

The largest contribution in the critical region comes from the quark mass derivative of the temperature:

$$\frac{\partial E_{qi}}{\partial T} = \frac{2M_{qi}}{E_{qi}} \frac{\partial M_{qi}}{\partial T}. \tag{8}$$

We would like to find a connection between the nonextensivity parameter q and the specific heat C_μ. To this end we consider small energy, $\epsilon \ll E$, fluctuations in a reservoir. Following [13] we use the q version of the Einstein's formula [13] for probability of states, $W \sim \exp_q\left(S_q\right)$ [1], and expand it to the second order in $\epsilon \ll E$ [14]:

$$\tilde{e}_q[S_q(E_q - \epsilon) - S(E_q)] \simeq 1 - \epsilon S_q{}'(E_q) + \frac{\epsilon^2}{2}(qS_q{}'(E_q)^2 + S_q{}''(E_q)) - ... \tag{9}$$

where $S_q{}' = 1/T$ and $S_q{}'' = -C_\mu/T^2$, (valid also in nonextensive thermodynamics [15]). The partial derivative $\partial M_{qi}/\partial T$ is singular at the critical point and therefore the specific heat C_μ/A is also singular there. Physically it means that the temperature cannot increase unless the effective quark will stabilize its mass with the quark condensates $< \bar{q}_i q_i >_q$ created out of the vacuum. This resembles situation in the ideal gas with the ideal heat bath, which can lose or gain any amount of energy without changing the temperature. In such a case the specific heat for the extensive (q=1) distribution is infinite. The self-consistent condition which connects finite C_μ/A with the non-extensive parameter q is therefore obtained from the requirement that, for the ideal reservoir, the expression (9) is independent of the parameter q,

$$q \rightarrow 1 + 1/C_\mu. \tag{10}$$

After that substitution the energy fluctuations in (9) became independent of q and of C_μ. Our condition (10) is then an alternative to the usual $q \rightarrow 1$ limit of BG statistics. This relation was first discussed in the high energy hadron spectra [12].

3 Results

In Fig.1 we present self-consistent calculations of the heat capacity for the extensive in n_q density $\varrho = \Sigma n_q/V$. In the mixed (shadowed) region the heat capacity is infinite due to the coexistence of the hadronic and quark phases. On the border of that region quark mass changes rapidly and C_μ develops a singularity. Two cases, $q > 1$ (Tsallis) and $q = 1$ (BG), are shown for comparison. The resulting C_μ/A are visible concentrated near the border of the mixed phase. In Fig.(2) we compare results from Fig.1 with the distribution obtained for thermodynamically consistent non-extensive density, $\varrho = \Sigma n_q^q/V$. Concentrations of C_μ/A near the border of the mixed phase is much stronger in this case. Consequently, the area with "ideal" reservoir, i.e., with large values of the heat capacity, is much

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[1]However, for simplicity, we shall neglect higher order terms in $| q - 1 |$ by using usual algebra, not its q-version used in [13].

smaller with respect to the temperature and density. This concentration will be even stronger for the nuclear matter with fixed density ϱ because the heat capacity $C_\varrho < C_\mu$ and will enforce a stronger nonextensivity of the compressed quark system.

We have found that model with thermodynamically consistent [15] nonextensive statistics changes significantly the distribution of the heat capacity in the critical region of phase transition to compressed quark matter.

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