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Modeling Open Transitions of the "Horn" Type between Open Planar Waveguides

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Abstract. The generalization of the incomplete Galerkin method to the description of the transitions of the "horn" type between open planar waveguides is discussed. The obtained result is characterized by a high degree of analyticity of the derived equations and this is expected to enhance the efficiency of the eigenwave modeling in open irregular waveguides.

1 Introduction

This study deals with the waveguide propagation of a polarized monochromatic electromagnetic radiation of light wavelength range in an irregular dielectric transition along the axis Oz between two regular planar dielectric waveguides. In such transitions the TE- and TM- modes are deformed, but not depolarized, which allows considering only the Helmholtz equation for $E_y(x, z)$ in the case of the TE-mode, and for $H_y(x, z)$ in the case of the TM-mode, avoiding thus the use of the complete Maxwell equations. The description of the electromagnetic field of a guided TE-mode in the waveguide transition is done by partial separation of variables similar to the description of the regular waveguide allowing separation of variables.

The electromagnetic field of the guided waveguide TE-mode is described by the Maxwell equations, with suitable continuity conditions at the interface and boundary conditions at infinity. Through the transformations of the Maxwell equations similar to [1], a waveguide problem is obtained for the Helmholtz equation with respect to the tangential component $E_y(x, z)$.

2 Mathematical model of guided waveguide modes of regular planar waveguide

The propagation of TE-modes of the form

$$\begin{pmatrix} E_y \\ H_x \\ H_z \end{pmatrix}(x\,,z\,,t) = \begin{pmatrix} E_y \\ H_x \\ H_z \end{pmatrix}\exp\left(i\omega t - ik_0\beta z\right) \tag{1}$$

in a planar regular waveguide is considered for the geometry shown in figure (1).

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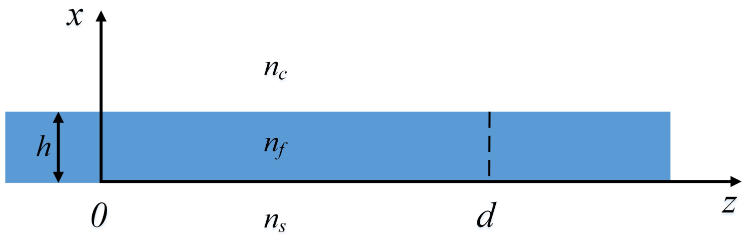

Figure 1. Planar regular waveguide

The electromagnetic field of the guided modes is described by the Helmholtz equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2\right) u = 0, \tag{2}$$

where $u(x, z) = E_y(x, z)$, k is the wave number, $k = k_s$, at $x < 0$, $k = k_f$ at $0 \le x \le h$, and $k = k_c$ at $x > h$. At the interfaces, u and $\left(\vec{\nabla} u\ ,\ \vec{n}\right)$ are continuous, and at infinity the following condition is fulfilled

$$u \underset{|x| \to \infty}{\to} 0. \tag{3}$$

The solution of (2) over $0 \le x \le h$ is sought in the form:

$$u(x, z) = V(z)\, \varphi(\xi), \tag{4}$$

where $\xi = \xi(x) \equiv \frac{x}{h}$.

In the areas of the substrate and the coating layer, the solution is sought in a form similar to (4), with indexes corresponding to the area of interest: $u_s(x, z) = V_s(z)\, \varphi_s(\xi)$ and $u_c(x, z) = V_c(z)\, \varphi_c(\xi)$.

Substituting the ansatz (4) into (2), after some algebraic transformation and the separation of the variables, we obtain the following relation for the waveguide layer:

$$\frac{V''}{V}(z) = -\left(\frac{\varphi''}{\varphi}\xi'^2 + k_f^2\right)(\xi) = -k_0^2\beta^2. \tag{5}$$

From (5) we obtain the equations fulfilled by the unknown functions V and φ:

$$V'' + k_0^2\beta^2 V = 0, \tag{6}$$

$$\varphi'' + k_0^2 h^2 \left(n_f^2 - \beta^2\right)\varphi = 0. \tag{7}$$

For V_c, φ_c and V_s, φ_s, corresponding to $x > h$ and $x < 0$, the equations are similar to (6) and (7) except for the constant: $n^2 = n_c^2$ at $x > h$ and $n^2 = n_s^2$ at $x < 0$.

The solutions of the equations of type (7) at $x < 0$ and $x > h$, satisfying (3), are

$$\varphi_s(\xi) = A_s e^{k_0 h p_s \xi}, \tag{8}$$

$$\varphi_c(\xi) = A_c e^{-k_0 h p_c \xi}, \tag{9}$$

where $p_s^2 = \beta^2 - n_s^2 > 0$, $p_c^2 = \beta^2 - n_c^2 > 0$ and therefore $\beta > \max\{n_s\,,\, n_c\}$.

Using (8) and (9), the problem of determining the form of the field inside the planar three-layer waveguide can be reduced to the solution of a Sturm-Liouville problem with the conditions of the third kind, for a nonlinear spectral parameter β, and limited to the area of the waveguide layer:

$$\varphi'' + k_0^2 h^2 p_f^2 \varphi = 0, \tag{10}$$

$$\begin{cases} \left(\varphi' - k_0 h p_s \varphi\right)\big|_{\xi=0} = 0, \\ \left(\varphi' + k_0 h p_c \varphi\right)\big|_{\xi=1} = 0, \end{cases} \tag{11}$$

where $p_f^2 = n_f^2 - \beta^2 > 0$, that is $\beta < n_f$.

For all allowable β values satisfying the problem (10), (11) we solve an equation of the form $V'' + k_0^2 \beta^2 V = 0$, i.e., a Helmholtz equation. The solution of (10) is reached in the form:

$$\varphi(\xi) = A \sin\left(k_0 h p_f \xi\right) + B \cos\left(k_0 h p_f \xi\right). \tag{12}$$

Substituting it into the boundary conditions (11), we get a system of equations, the condition of solvability of which i the nonlinear transcendental equation

$$\operatorname{tg}\left(k_0 h p_f\right) = \frac{p_f\left(p_s + p_c\right)}{p_f^2 - p_s p_c}, \tag{13}$$

the roots of which define the admissible values β. There is a finite number of β_j values, which are the roots of (13) in the interval $\left(\max\{n_s, n_c\}\ ;\ n_f\right)$. The eigenfunctions φ_j of the form (12) take the following form:

$$\varphi_j(\xi) = p_s\left(\beta_j\right) \sin\left(k_0 h p_f\left(\beta_j\right)\xi\right) + p_f\left(\beta_j\right) \cos\left(k_0 h p_f\left(\beta_j\right)\xi\right). \tag{14}$$

3 Mathematical model of guided waveguide modes in an irregular waveguide transition of the “horn” type

In this section we consider the waveguide propagation of TE-mode in a homogeneous open dielectric waveguide transition of the “horn” type shown in the figure (2), where $h(z)$ is a linear function: $h'(z) = \text{Const} > 0$, $h''(z) = 0$.

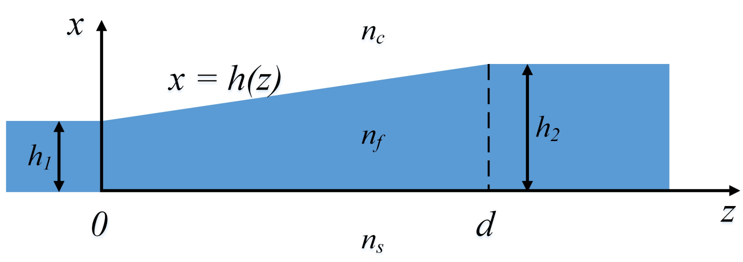

Figure 2. Waveguide transition of the “horn” type

The Helmholtz equation at $0 \le z \le d$, writes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2\right) u = 0, \tag{15}$$

where $u(x, z) = E_y(x, z)$, k is the wave number, $k = k_s$ at $x < 0$, $k = k_f$ at $0 \le x \le h(z)$, and $k = k_c$ at $x > h(z)$. This structure associates two semi-infinite waveguides of constant cross-section corresponding to the intervals $z < 0$ and $z > d$, and the waveguide transition at $0 \le z \le d$. At the interfaces, u and $\left(\vec{\nabla} u\,,\ \vec{n}\right)$ are continuous, and at infinity the condition (3) is fulfilled.

The solution of (15) at $0 \le x \le h(z)$ and $0 \le z \le d$ is sought in the form:

$$u(x, z) = V(z)\,\varphi(\xi)\,, \tag{16}$$

where $\xi = \xi(x\,, z) \equiv \frac{x}{h(z)}$.

In the areas of the substrate and the coating layer at the interval $0 \le z \le d$ the solutions is sought in a form similar to (16), with indexes which are characteristic for the corresponding area: $u_s(x, z) = V_s(z)\,\varphi_s(\xi)$ and $u_c(x, z) = V_c(z)\,\varphi_c(\xi)$.

Substituting the form of the solution (16) into (15) and carrying out the transformation separating the variables, we obtain the following relation:

$$\frac{V''}{V}(z) = -\left(\frac{\varphi''}{\varphi}\left[\left(\xi'_x\right)^2 + \left(\xi'_z\right)^2\right] + \frac{\varphi'}{\varphi}\left[\xi''_{xx} + 2\xi'_z\frac{V'}{V} + \xi''_{zz}\right] + k_f^2\right)(z, \xi) = -k_0^2\beta^2(z)\,. \tag{17}$$

The separation of variables obtained in (17) is partial, since the left side is a function of z, and the right side, of z and ξ. From (17) we get the following equations for the unknown functions V and φ:

$$V'' + k_0^2\beta^2(z)\,V = 0 \tag{18}$$

and

$$f(\xi, z)\,\varphi''_{\xi\xi} + g(\xi, z)\,\varphi'_{\xi} + k_0^2 h^2(z)\left(n_f^2 - \beta^2(z)\right)\varphi = 0, \tag{19}$$

where

$$f(\xi, z) = h'^2\xi^2 + 1, \tag{20}$$

$$g(\xi, z) = \left(2h'^2 - 2\alpha h' h - h'' h\right)\xi, \tag{21}$$

and besides, $\alpha(z) = V'(z)/V(z)$.

For V_c, φ_c and V_s, φ_s corresponding to $x > h(z)$ and $x < 0$, equations similar to (18) and (19) are obtained except for the constants: $n^2 = n_c^2$ at $x > h(z)$ and $n^2 = n_s^2$ at $x < 0$.

The discussion which follow holds in the case when $h'(z)$ is a small quantity

$$h'(z) = \delta \ll 1. \tag{22}$$

We consider the equation (19) for the areas of the substrate ($n = n_s$) and the coating layer ($n = n_c$) in the zero-order approximation in the small parameter δ:

$$\varphi''_0 + k_0^2\,h^2(z)\left(n^2 - \beta_0^2(z)\right)\varphi_0 = 0, \tag{23}$$

where $\varphi_0 = \varphi_0(\xi\,;\,z)$ is the zeroth order approximation of the eigenfunction, $\beta_0^2(z)$ is the zeroth order approximation of the eigenvalue.

The equation (23) is similar to the equation (7) for a regular waveguide except for the parametric dependence of the eigenfunctions and of the eigenvalues on the component z. Differentiation in (23) is over ξ, the values which depend on z are independent of ξ, and the reasoning is similar to that of (7)–(11).

The approximation of the zeroth order results in the following Sturm-Liouville problem with boundary conditions of the third kind and z-parametric dependences of the eigenvalues and the eigenfunctions:

$$\varphi''_0 + k_0^2 h^2(z)\, p_f^2(z)\, \varphi_0 = 0, \tag{24}$$

$$\begin{cases} \left(\varphi'_0 - k_0 h(z)\, p_s(z)\, \varphi_0\right)\big|_{\xi=0} = 0, \\ \left(\varphi'_0 + k_0 h(z)\, p_c(z)\, \varphi_0\right)\big|_{\xi=1} = 0, \end{cases} \tag{25}$$

where $p_f^2(z) = n_f^2 - \beta_0^2(z) > 0$, that is $\beta_0(z) < n_f$. The eigenvalues $\beta_{0j}^2(z)$ are the roots of the equation:

$$\operatorname{tg}\left(k_0 h(z)\, p_f(z)\right) = \frac{p_f(z)\left(p_s(z) + p_c(z)\right)}{p_f^2(z) - p_s(z)\, p_c(z)} \tag{26}$$

and the eigenfunctions $\varphi_{0j}(\xi; z)$ take the following form:

$$\varphi_{0j}(\xi; z) = p_{sj}(z) \sin\left(k_0 h(z)\, p_{fj}(z)\, \xi\right) + p_{fj}(z) \cos\left(k_0 h(z)\, p_{fj}(z)\, \xi\right), \tag{27}$$

where $p_{sj}^2(z) = \beta_{0j}^2(z) - n_s^2$, $p_{fj}^2(z) = n_f^2 - \beta_{0j}^2(z)$.

Having found the allowed values of $\beta_{0j}^2(z)$, the functions $V_{0j}(z)$ are got as solutions of the following problem:

$$V''_{0j} + k_0^2 \beta_{0j}^2(z)\, V_{0j} = 0, \tag{28}$$

$$\begin{aligned} \left(V'_{0j} + i k_0 \gamma_j V_{0j}\right)_{z=0} &= 2 i k_0 \gamma_m \delta_{jm}, \\ \left(V'_{0j} - i k_0 \Gamma_j V_{0j}\right)_{z=d} &= 0, \end{aligned} \tag{29}$$

where $\gamma_j = \beta_{0j}(0)$, $\Gamma_j = \beta_{0j}(d)$.

Remark. The zeroth order approximation in the small parameter makes the equations of the amplitude functions and of the eigenfunctions independent.

4 Numerical results and Conclusion

The proposed approach to the solution of the waveguide problem in the zeroth order approximation allows its reduction to a waveguide transition of the "horn" type in each cross-section with known eigenvalues and eigenfunctions. The admissible modulated amplitudes of the electric field are obtained as solutions of a boundary value problem for ordinary second order differential equations with variable coefficients and the boundary conditions of the third kind.

The zeroth order approximation of the generalization of the incomplete Galerkin method leads to a Sturm-Liouville problem, which is similar to that defining modes of the regular planar waveguide. This fact allows us to conclude, that the given zeroth order approximation of the method can describe the waveguide propagation in an irregular dielectric waveguide with comparable accuracy to that in the regular planar waveguides.

Goos–Hanchen shifts, shown in figure (3), describe the area which contains majority of electromagnetic field energy in the open waveguide. The waveguide upper and lower boundaries, enlarged by these shifts, create new boundaries, which represent a virtual structure – the closed waveguide with new enlarged boundaries. This waveguide has the same field representation as the given open waveguide in the area of the waveguiding layer. This fact allows the comparison of the reduced Galerkin's method of [1] with the current approach using Goos-Hanchen shifts.

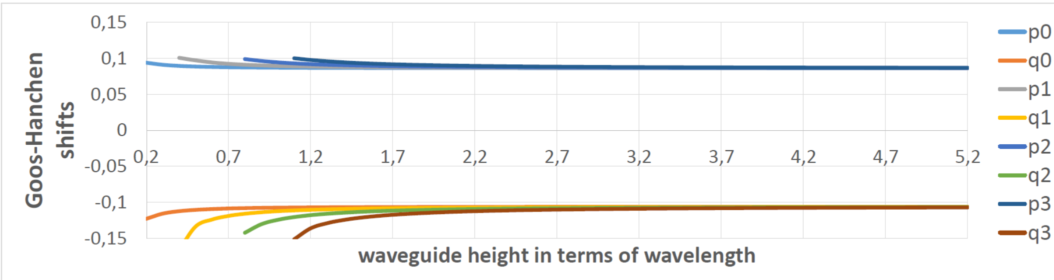

Figure 3. Goos-Hanchen shifts for the waveguide transition of the "horn" type

Acknowledgements

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