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# Fermi energy of electrons in neutron stars with strong magnetic field and magnetars

Qiu-He Peng[1,a], Jie Zhang[2], and Chih-Kang Chou[3]

[1] *Department of Astronomy, Nanjing University, Nanjing, 210093, China*
[2] *Normal University in the west China, Nanchong, China*
[3] *National Astronomical Observatory, Chinese Academy of Sciences, Beijing, China*

## 1 Origing of strong magnetic fields for magnetars

Since there is $^3P_2$ neutron superfluid in a neutron star interior, it can be treated as a system of magnetic dipoles. When there is a background magnetic field, the magnetic dipoles tend to align in the same direction.

The total difference of the $^3P_2$ neutron Cooper pair number with paramagnetic and diamagnetic moment is

$$\Delta N = N_n(^3P_2\_pair) f(\frac{\mu_n B}{kT}) = \frac{1}{2} N_A m(^3P_2) q f(\frac{\mu B}{kT}). \tag{1}$$

The total induced magnetic moment of the anisotropic neutron superfluid is

$$\mu_{pair}^{(tot)}(^3P_2) = 2\mu_n \times \Delta N = \mu_n N_A m(^3P_2) q f(\frac{\mu_n B}{kT}). \tag{2}$$

Where $m(^3P_2)$ is the mass of the anisotropic neutron superfluid in the neutron star, $N_A$ is the Avogadro constant. Where the Brillouin function, $f(\mu_n B/kT)$, is introduced to take into account the effect of thermal motion.

$$f(x) = \frac{2\sin h(2x)}{1 + 2\cos h(2x)} \tag{3}$$

We note that $f(x)(< 1)$ is an increasing function, in particular, $f(x) \approx 4x/3$, for $x \ll 1$ and $f(x) \to 1$, when $x \gg 1$. $f(\mu_n B/kT)$ increases with decreasing temperature. And eq.(3) is the mathematical formula for the B-phase of the $^3P_2$ superfluid.

Due to the energy gap the combination of neutrons into the $^3P_2$ Cooper pairs occurs only in a thin layer at the Fermi surface with thickness of $k_\Delta$, and $k_\Delta = \sqrt{2m_n\Delta_n(^3P_2)}$. $\Delta$ is the binding energy of the Cooper pair. The fraction of the neutrons that combined into the $^3P_2$ Cooper pairs, $q$, is

[a] e-mail: qhpeng@nju.edu.cn

$$q = \frac{4\pi k_F^2 k_\Delta}{4\pi k_F^3/3} = 3\frac{k_\Delta}{k_F} = 3[\frac{\Delta_n(^3P_2)}{E_F(n)}]^{1/2} \tag{4}$$

The magnetic moment with the dipolar magnetic field is $|\mu_{NS}| = B_p R_{NS}^3/2$(Shapiro and Teukolski, 1984). Here $B_p$ is the polar magnetic field strength while $R_{NS}$ is the radius of the neutron star. The induced magnetic field is then

$$B^{(in)} = \frac{2\mu^(tot)_{pair}(^3P_2)}{R_{NS}^3} = \frac{2\mu_n N_A m(^3P_2)}{R_{NS}^3} q f(\frac{\mu_n B}{kT}). \tag{5}$$

When the temperature is lower than $10^7$ K, the strong magnetic fields of the magnetars may originate from the induced magnetic moment of the $^3P_2$ neutron Cooper pairs in the anisotropic neutron superfluid. And this gives an intuitive interpretation of the existence of strong magnetic field for magnetars.

## 2 The Fermi energy of electrons in neutron stars with a strong magnetic field

The energy of free electrons moving in the direction perpendicular to a strong magnetic field (MF) is quantized or discrete according to the idea of Landau quantization. The energy of an electron in intense MF can be expressed as follows:

$$E^2 = m_e^2c^4 + p_z^2c^2 + p_\perp^2c^2, \tag{6}$$

where $p_z$ and $p_\perp$ denote the momentum of electron along and perpendicular to the magnetic field respectively. For a given $p_z$ (it may take a continuous variation), $p_\perp$, takes the quantized values:

$$(\frac{p_\perp}{m_e c})^2 = (2n + 1 + \sigma)b,\ b = \frac{B}{B_{cr}}, \tag{7}$$

$B_{cr} = m_e^2c^3/e = 4.414 \times 10^13$ is the critical magnetic field [1], n and $\sigma$ are the quantum numbers for Landau level and electron spin respectively. $n = 0, 1, 2, 3...$, $\sigma = -\frac{1}{2}$ for $n = 0$, and $\sigma = \pm\frac{1}{2}$ for $n \geq 1$. For a Boltzmann gas, we have : a) the quantum numbers of Landau level, n, tend to infinity; b) the probability for an electron, at a high energy level, in a strong magnetic field, will be very small because it is proportional to the Boltzmann factor $\exp\{-(\frac{E}{kT})\}$ ($\left(\frac{E}{kT}\right) \gg 1$), and it can be negligible . For simplification, we can assume that the free electrons whose motion is perpendicular to the magnetic field are almost completely concentrated to a few lowest energy levels ($n = 0, 1, 2, 3$).

However, for a complete degenerate electron gas in neutron stars (and white dwarfs) with a strong magnetic field, the proprieties are different. The Fermi sphere in a weak magnetic field, however, will be shaped into a Landau column in a strong magnetic field. In the x-y plane perpendicular to the magnetic field, electrons are populated in the discrete Landau levels, although $p_z$ changes continuously. For a given $p_z$there is a maximum orbital quantum number ( or Landau level number ) $n_{max}(p_z, b, \sigma)$(Fig.1).

$$n_{max}(p_z, b, \sigma = +1) \approx n_{max}(p_z, b, \sigma = -1) = n_{max}(p_z, b) = Int\{\frac{1}{2b}\left[\left(\frac{E_F}{m_ec^2}\right)^2 - 1 - \left(\frac{p_z}{m_ec}\right)\right]\}.$$

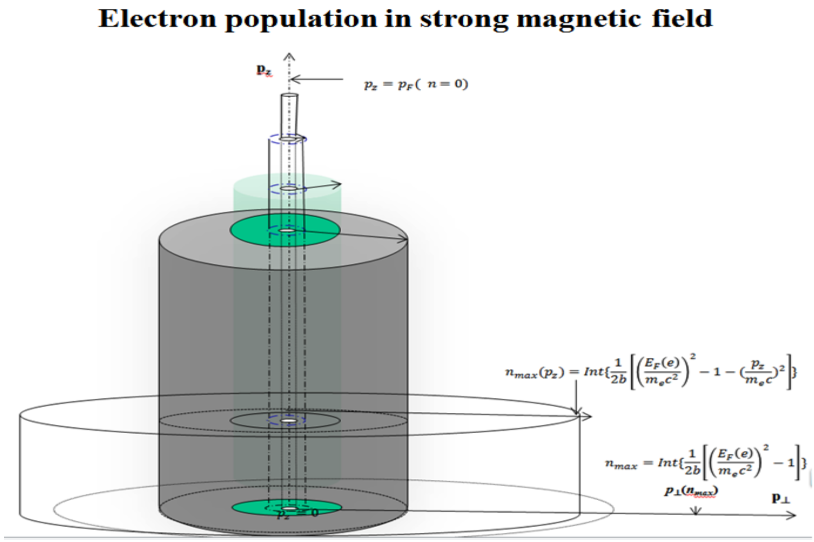


**Figure 1.** The Landau column in a strong magnetic field for a complete degenerate electron gas.

The maximum number, $n_{max}(p_z, b)$, is smaller for a strong magnetic field. When $B \gg B_{cr}$ or $b \gg 1$, the Landau column becomes a very long , narrow cylinder along the magnetic field, and the overwhelming majority of electrons congregate in the lowest levels, $n_L = 0, 1$, etc. According to the Pauli exclusion principle, for a given number density of electrons in the interior of neutron stars, the stronger the magnetic field, the fewer the number of states occupied by electrons across the magnetic field,. Therefore, the maximum momentum along the magnetic field $p_z(max)$ would increase with increasing magnetic field, and Fermi energy of electrons would also increase, where the relation $p_z(max) \approx p_F \approx E_F(e)/c$ for ultra-relativistic electrons is used[1,2]. Consequently, the stronger the magnetic field, the higher the Fermi energy of electrons.

Unfortunately, we are surprised to find that the generally accepted idea is just on the contrary, namely, "the stronger the magnetic field, the lower Fermi energy of electrons". Most seriously, this popular idea has been propagated in many papers for more than four decades, e.g. refs.[2-5]. In view of the importance of this relevant issue, we endeavor to solve this dilemma.

## 3 A serious error in some statistical physics textbooks

We find that in almost all the literatures directly or indirectly cite the internationally popular classic statistical physics textbooks (see e. g., Refs [6] and [7]), however, in which the calculation of electronic micro state number density for a strong magnetic field is not correct. In those statistical physics textbooks, the authors used the following method to calculate the statistical weight (degeneracy) of the Landau level n: In the interval of momentum $p_z \to p_z + \mathrm{d}p_z$ along the magnetic field direction, the possible microscopic state number density for the electron gas is $N_{phase}(p_z) = h^{-2}\int \mathrm{d}p_x \mathrm{d}p_y = h^{-2}\pi(p_\perp^2)_n^{n+1} = 4\pi m\mu_B B/h^2$. This method of calculating the microscopic state number for the level (n+1) in the plane perpendicular to magnetic field is essentially equal to the area of the torus from n to (n+1). However, this is obviously inconsistent with the important concept of Landau orbital quantization in a strong magnetic field. Really no quantum states between $p_\perp(n) \to p_\perp(n+1)$ in the momentum space are allowed. Therefore the method used in these statistical physics textbooks as elaborated above is inappropriate and this is just the basic reason for the dilemma.

In order to see the inconsistency of the treatment for the usual approach, we briefly mention the relevant steps. The incorrect result from the textbooks is the same as Landau's conclusion for the non-relativistic case [1]. When this result is extended to calculate the number of microscopic states for almost completely degenerate electron gas in the neutron stars, it leads to the popular result: the electron Fermi energy decreases with increasing magnetic field. The reason is as follows: the number density of microscopic states is $N_{phase} = \int_0^{p_F} N_{phase}(p_z)\mathrm{d}p_z = \frac{eB}{4\pi\hbar^2}\frac{E_F}{c^2}$, by using the conventional method of statistical physics. According to the Pauli Exclusion Principle, the number density of microscopic states is equal to electrons number density in a complete degenerate electron gas, $N_{phase} = n_e = N_A \rho Y_e$, where $Y_e$ and $\rho$ are electron fraction (5-8%) and mass density. This derivation implies the inevitable conclusion: $E_F(e) \propto B^{-1}$.

Unfortunately, this incorrect result has been used by many authors in the last four decades. For example, In the Page 12 of Ref. [2], authors clearly cited the result of the statistical physics textbooks edited in 1965. Many relevant papers directly or indirectly cited reference [3] as a starting point since 1991. In fact, all the theories so far have not been tested by astronomical observations yet, and it is especially difficult to explain the observed phenomena of magnetars or neutron stars with strong magnetic field by their theories.

## 4 Magnetic field dependence of the Fermi energy for relativistic electrons: Our method and results

To re-examine the method of calculating the microscopic state number density for the electrons in strong magnetic fields, we introduce the Dirac -$\delta$ function to rigorously describe the Landau orbital quantization in the direction perpendicular to the magnetic field.

The following formulae (4) is valid when the magnetic field stronger than the Landau critical value, $B \geq B_{cr}$. The total number density for the electrons in a strong magnetic field is given by

$$N_{phase} = 2\pi(\frac{m_e c}{h})^3 \times \int_0^{\frac{p_F}{m_e c}} \mathrm{d}\left(\frac{p_z}{m_e c}\right) \sum_{n=0}^{n_{max}(p_z,b,\sigma)} g_n \int_0^{\frac{p_F}{m_e c}} \delta[(\frac{p_\perp}{m_e c} - \sqrt{2(n+\frac{1}{2}+\sigma)b})](\frac{p_\perp}{m_e c})\mathrm{d}(\frac{p_\perp}{m_e c}). \quad (8)$$

Where $g_n$ is the degeneracy of electron spin. $g_0$ for n=0, and $g_n = 2$ for $n \geq 1$.

After a simple calculation, $N_{phase} = \frac{\pi^2}{4b}(\frac{m_e c}{h})^3(\frac{E_F}{m_e c})^4$, where $(\frac{h}{m_e c}) = \lambda_e$ is the Compton wavelength of electrons. According to the Pauli Exclusion Principle, the number of microscopic states in the unit volume is equal to the number density of completely degenerate electrons, i.e., $N_{phase} = n_e = N_A \rho Y_e$, consequently we obtain

$$E_F(e) \approx 42.9(\frac{Y_e}{0.05})^{1/4}(\frac{\rho}{\rho_n uc})^{1/4}(\frac{B}{B_{cr}})^{1/4}\ \mathrm{MeV} \quad (B \geq B_{cr}). \quad (9)$$

This means that the Fermi energy of the relativistic electrons in strong magnetic fields is proportional to the one forth power of the field strength.

## 5 A physical mechanism of high x-ray luminosity for magnetars

In terms of our new result of the Fermi energy with a strong magnetic field, eq.(9), we can easily explain the physical origin for high x-ray luminosity for magnetars. The main idea is as follows:

since the magnetic fields in magnetars are much stronger than that of the critical magnetic field, , the Fermi energy of the electrons in magnetars is very high. According to Eq.(9), if the Fermi energy of the electrons significantly exceeds that of the neutrons ($E_F(n) > 60$ MeV), electrons can be captured by the protons near the proton Fermi surface leading to the production of neutrons(i.e., $e^- + p \rightarrow n + \nu_e$). The resulting energy of the outgoing neutrons ( $> 60$ MeV) is much higher than the binding energy of the $^3P_2$ neutron Cooper pair (0.05MeV) for the anisotropic superfluid region in a neutron star interior . The outgoing neutrons may destroy $^3P_2$ neutron Cooper pair by strong nuclear interactions, $(n + (n \uparrow, n \uparrow) \rightarrow n + n + n)$. We note that the binding energy of $^3P_2$ neutron Cooper pair is compensated by the energy of the outgoing neutron.

When the Cooper pairs are destroyed, the spin of the two neutrons that form the $^3P_2$ neutron Cooper pairs is no longer parallel (they are in the state of random thermal motion and chaotic). The magnetic moments of the $^3P_2$ neutron Cooper pairs, which is twice of the neutron anomalous magnetic moment, $2\mu_n$, would disappear, and then the induced magnetic moment generated by the $^3P_2$ Cooper pair also disappear. The magnetic moments of the $^3P_2$ neutron Cooper pairs have the tendency to the reverse direction of the magnetic field. The energy of the magnetic moments, $2\mu_n B$, could be transformed into the chaotic thermal energy when the destruction of the Copper pairs takes place. The heat released by each $^3P_2$ neutron Cooper pair is $kT = 2\mu_n B \approx 10B_{15}$ keV, where $B_{15} = B/10^{15}$ gauss. The thermal energies then become the origin of the x-ray radiation for the magnetars. If the entire $^3P_2$ neutron Cooper pairs are destroyed, the total energy released is $E = qN_A m(^3P_2) \times 2\mu_n B \approx 1 \times 10^{47} \frac{m(^3P_2)}{0.1m_\odot}$ ergs, where $m(^3P_2)$ is the total mass of the superfluid region with anisotropic $^3P_2$ neutrons, and q is the ratio of the number of neutron in $^3P_2$ Cooper neutron pairs to the total number of neutron in the superfluid region (about 8.7%)[8]. According to the current x-ray observations ($L_X \sim 10^{34} - 10^{36}$ erg s$^{-}$1), the duration of the magnetic star activity can be maintained for $\sim 10^4 - 10^6$ yrs.

We calculate in detail the x-ray luminosity of magnetars in terms of the electron capture process, and make comparison with the observations. We note that up to now, there is no other known theoretical models that can fully delineate the observed high x-ray luminosities of the magnetars. However, using the physical mechanism we proposed, the theoretical prediction is basically consistent with the observed data of x-ray luminosities for magnetars [9] (Fig.2).

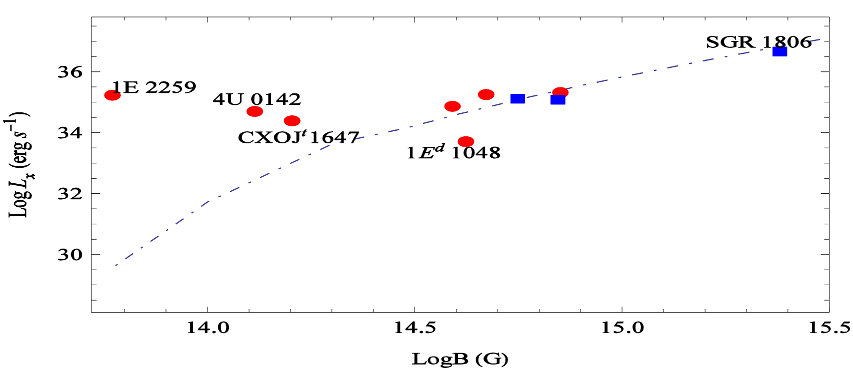


**Figure 2.** Comparing the observed magnetar $L_X$ with the calculations: Red cycle-SGR; Blue cycle-AXP. Some accretion have detected for the left three AXP.

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