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Mathematical model of non-stationary temperature distribution in the metal body produced by induction heating process

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Abstract. An induction heating problem can be described by a parabolic differential equation. For this equation, specific Joule looses must be computed. It can be done by solving the Fredholm Integral Equation of the second kind for the eddy current of density. When we use the Nyström method with the singularity subtraction, the computation time is rapidly reduced. This paper shows the method for finding non-stationary temperature distribution in the metal body with illustrative examples.

Key words. induction heating, integral equation of the second kind, non-stationary temperature distribution, Nyström method

1 Introduction

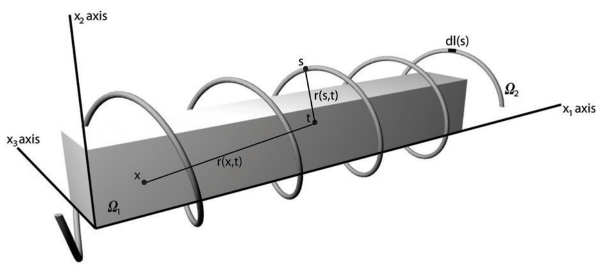

Figure 1. Heated body and coil.

A bounded metal cuboid body Ω_1 of sizes $l_1 \times l_2 \times l_3$ is heated by an external electromagnetic field produced by inductor Ω_2 (see figure 1). The inductor is formed by a conductor of general shape and position that carries the harmonic current I_{ext}.

2 Eddy current of density

Computation of the temperature distribution depends on the eddy current of density. It is a phasor

$$J_{eddy} = (J_{eddy,x_1}, J_{eddy,x_2}, J_{eddy,x_3}).$$

The x_1 component of $J_{eddy,x_1}(\boldsymbol{x})$ can be computed (see [4] and [5]) by Fredholm integral equation of the second kind

$$\iota J_{eddy,x_1}(\boldsymbol{x}) - \kappa(\boldsymbol{x}) \int_{\Omega_1} \frac{J_{eddy,x_1}(\boldsymbol{t})}{r(\boldsymbol{x},\boldsymbol{t})} dt_1 dt_2 dt_3 = I_{ext} F(\boldsymbol{x}), \quad (1)$$

where

$$F(\boldsymbol{x}) = \kappa(\boldsymbol{x}) \int_{\Omega_2} \frac{dl(\boldsymbol{s}).e_{x_1}}{r(\boldsymbol{x},\boldsymbol{s})}, \quad (2)$$

$$\kappa(\boldsymbol{x}) = \frac{\omega\gamma(T(\boldsymbol{x}))\mu_0}{4\pi}, \quad (3)$$

$r(\boldsymbol{x},\boldsymbol{t})$ is the Euclidean distance, $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{t} = (t_1, t_2, t_3)$ are points in the metal body, $\boldsymbol{s} = (s_1, s_2, s_3)$ is a point at the inductor, I_{ext} is harmonic current carried by the inductor, ω is angular frequency, $\gamma(T(\boldsymbol{x}))$ is electrical conductivity, μ_0 is permeability of vacuum, e_{x_1} is unite vector $e_{x_1} = (1, 0, 0)$ and ι is the complex unit.

For each bounded and continuous temperature distribution $T(\boldsymbol{x})$, $\kappa(\boldsymbol{x})$ is a real, positive, bounded and continuous function.

Formulas for the remaining components J_{eddy,x_2} and J_{eddy,x_3} can be obtained by mere interchanging of indices.

With the notation

$$J_R(\boldsymbol{x}) = Re J_{eddy,x_1}(\boldsymbol{x}) \quad (4)$$

$$J_I(\boldsymbol{x}) = Im J_{eddy,x_1}(\boldsymbol{x}) \quad (5)$$

$$I_R = Re(I_{ext}) \quad (6)$$

$$I_I = Im(I_{ext}) \quad (7)$$

we have system of integral equations

$$J_R(\boldsymbol{x}) - \kappa(\boldsymbol{x}) \int_{\Omega_1} \frac{J_I(\boldsymbol{t})}{r(\boldsymbol{x},\boldsymbol{t})} dt = I_I F(\boldsymbol{x}),$$

$$-J_I(\boldsymbol{x}) - \kappa(\boldsymbol{x}) \int_{\Omega_1} \frac{J_R(\boldsymbol{t})}{r(\boldsymbol{x},\boldsymbol{t})} dt = I_R F(\boldsymbol{x}). \quad (8)$$

The specific Joule losses at the point $\boldsymbol{x}$ in the body, which are needed to compute the temperature distribution are given by

$$\omega(\boldsymbol{x}) = \frac{1}{\gamma} J_e(\boldsymbol{x}) \overline{J_e(\boldsymbol{x})}, \quad (9)$$

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where

$$J_e(\boldsymbol{x}) =$$

$$\sqrt{[ReJ_{eddy,x_1}(\boldsymbol{x})]^2 + [ReJ_{eddy,x_2}(\boldsymbol{x})]^2 + [ReJ_{eddy,x_3}(\boldsymbol{x})]^2} +$$

$$+\iota\sqrt{[ImJ_{eddy,x_1}(\boldsymbol{x})]^2 + [ImJ_{eddy,x_2}(\boldsymbol{x})]^2 + [ImJ_{eddy,x_3}(\boldsymbol{x})]^2}$$

and $\overline{J_e(\boldsymbol{x})}$ is complex conjugate to $J_e(\boldsymbol{x})$.

3 The non-stationary temperature distrubution

The non-stationary distribution of the temperature $T(\boldsymbol{x}, t)$ at point $\boldsymbol{x}$ and time t in the metal body is described by a partial differential equation of the parabolic type

$$\mathrm{div}\,[\lambda(T(\boldsymbol{x}, t))\ \mathrm{grad}\ T(\boldsymbol{x}, t)] =$$

$$= \rho(T(\boldsymbol{x}, t))c(T(\boldsymbol{x}, t))\frac{\partial T(\boldsymbol{x}, t)}{\partial t} - \omega(\boldsymbol{x}, t), \qquad (10)$$

where $\lambda = \lambda(T(\boldsymbol{x}, t))$ denotes the thermal conductivity, $\rho = \rho(T(\boldsymbol{x}, t))$ the specific mass of the material, $c = c(T(\boldsymbol{x}, t))$ specific heat and $\omega(\boldsymbol{x}, t)$ the specific Joule losses given by (9) at time t.

The boundary condition along the whole surface of the body reads

$$-\lambda(T(\boldsymbol{x}, t))\frac{\partial T(\boldsymbol{x}, t)}{\partial n} = \alpha\left(T(\boldsymbol{x}, t) - T_{ext}\right), \qquad (11)$$

where α denotes the coefficient of the convective heat transfer, T_{ext} the temperature of the surrounding medium and n the direction of the outward normal.

4 Numerical solution

The metal body Ω_1 was assumed to be cuboid. Let us cover the body by n_1 subsuboides in the x_1 direction, n_2 subsuboides in the x_2 direction and n_3 subsuboides in the x_3 direction. The size of each subcuboid is $h_1 \times h_2 \times h_3$, where

$$h_1 = \frac{l_1}{n_1}, h_1 = \frac{l_2}{n_2}, h_3 = \frac{l_3}{n_3}.$$

We need to solve integral equation (1) to get Joule losses. Then we can put computed Joule losses to partial differential equation (10) to get the temperature distribution. The Joule losses must be recalculated at the time when changed.

For computation of Joule losses we will apply the Nyström method with the compound mid-cuboid rule. For time integration we will use the classical finite difference method for partial differential equations of the parabolic type.

4.1 Application of Nyström method

The Nyström method is based on approximation of the integral by the numerial integration rule. We will use the compound min-point rule. Let the node points be $\boldsymbol{x}_i$ defined as center of sub-cuboides. For the weight let

$$w_j = w = \frac{|\Omega_1|}{N}, j = 1, ..., N,$$

where $N = n_1 n_2 n_3$.

Here the numerical integration rule cannot be applied directly. The reason is that the function $r(\boldsymbol{x}, \boldsymbol{t})^{-1}$ is singular at $\boldsymbol{x} = \boldsymbol{t}$. One way to deal with such a problem is singularity subtraction. It was described in [3]. Let $r_N(\boldsymbol{x}, \boldsymbol{t})$ be an approximation of $r(\boldsymbol{x}, \boldsymbol{t})$ which coincide outside a certain neighborhood of $\boldsymbol{x} = \boldsymbol{t}$. The integrand in equation (1) is approximated by

$$\frac{J_{eddy,x_1}(\boldsymbol{t})}{r(\boldsymbol{x}, \boldsymbol{t})} \approx \frac{J_{eddy,x_1}(\boldsymbol{t}) - J_{eddy,x_1}(\boldsymbol{x})}{r_N(\boldsymbol{x}, \boldsymbol{t})} + \frac{J_{eddy,x_1}(\boldsymbol{x})}{r(\boldsymbol{x}, \boldsymbol{t})}. \qquad (12)$$

Since the first element of the approximation is zero when $\boldsymbol{x} = \boldsymbol{t}$, the exact constructioin of $r_N(\boldsymbol{x}, \boldsymbol{t})$ is immaterial. By using the numerical integration rule, approximation (12) and running $\boldsymbol{x}$ throung the node points we get a system of linear equations for numerical approximation of the x_1 component of the eddy current of density $\widetilde{J}_N$

$$\left[1 - \iota\kappa(\boldsymbol{x}_i)\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{w}{r_N(\boldsymbol{x}_i, \boldsymbol{x}_j)} = \iota\kappa(\boldsymbol{x}_i)\int_{\Omega_1}\frac{dt_1dt_2dt_3}{r(\boldsymbol{x}_i, \boldsymbol{t})}\right]\widetilde{J}_N(\boldsymbol{x}_i) +$$

$$+\iota\kappa(\boldsymbol{x}_i)\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{w\widetilde{J}_N(\boldsymbol{x}_j)}{r_N(\boldsymbol{x}_i, \boldsymbol{x}_j)} = -\iota I_{ext}F(\boldsymbol{x}_i), i = 1, ..., N. \qquad (13)$$

$\widetilde{J}_N(\boldsymbol{x}_i)$ is complex number. Let us define for each $i = 1, ..., N$

$$\widetilde{J}_i^{(R)} = \mathrm{Re}\widetilde{J}_N(\boldsymbol{x}_i) \text{ and } \widetilde{J}_i^{(I)} = \mathrm{Im}\widetilde{J}_N(\boldsymbol{x}_i). \qquad (14)$$

Then (13) is equivalent the real system of linear equations

$$\widetilde{J}_i^{(R)} + \kappa(\boldsymbol{x}_i)\left[\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{w}{r_N(\boldsymbol{x}_i, \boldsymbol{x}_j)} - \int_{\Omega_1}\frac{dt_1dt_2dt_3}{r(\boldsymbol{x}_i, \boldsymbol{t})}\right]\widetilde{J}_i^{(I)} -$$

$$-\kappa(\boldsymbol{x}_i)\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{w\widetilde{J}_j^{(I)}}{r_N(\boldsymbol{x}_i, \boldsymbol{x}_j)} = \mathrm{Im}I_{ext}F(\boldsymbol{x}_i), i = 1, ..., N$$

$$-\widetilde{J}_i^{(I)} + \kappa(\boldsymbol{x}_i)\left[\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{w}{r_N(\boldsymbol{x}_i, \boldsymbol{x}_j)} - \int_{\Omega_1}\frac{dt_1dt_2dt_3}{r(\boldsymbol{x}_i, \boldsymbol{t})}\right]\widetilde{J}_i^{(R)} -$$

$$-\kappa(\boldsymbol{x}_i)\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{w\widetilde{J}_j^{(R)}}{r_N(\boldsymbol{x}_i, \boldsymbol{x}_j)} = \mathrm{Re}I_{ext}F(\boldsymbol{x}_i), i = 1, ..., N. \qquad (15)$$

Following Nyström interpolation formula (for details see for example [1]) we get the numerical solution for eddy currents $\widetilde{J}_N$ outside the node points

$$\widetilde{J}_N(\boldsymbol{x}) = \frac{-\iota I_{ext}F(\boldsymbol{x}) - \iota\kappa(\boldsymbol{x})\sum_{j=1}^{N}\frac{w}{r_N(\boldsymbol{x}, \boldsymbol{x}_j)}\widetilde{J}_N(\boldsymbol{x}_j)}{1 - \iota\kappa(\boldsymbol{x})\sum_{j=1}^{N}\frac{w}{r_N(\boldsymbol{x}, \boldsymbol{x}_j)} + \iota\kappa(\boldsymbol{x})\int_{\Omega_1}\frac{1}{r(\boldsymbol{x}, \boldsymbol{t})}dt}, \qquad (16)$$

where

$$\widetilde{J}_N(\boldsymbol{x}_i) = \widetilde{J}_i^{(R)} + \iota \widetilde{J}_i^{(I)}.$$

Since Ω_1 is a cuboid, the integral in (15) and (16) can be computed analytically. The x_2 and x_3 components of the eddy current of density can be computed in an analogous way.

5 Application of the finite difference method

Let us use a finite difference method for the numerical solution of equation (10). Since thermal conductivity is a constant that depends on the temperature we can rewrite equation (10) into the form

$$\frac{\partial T(\boldsymbol{x},t)}{\partial t} = \frac{1}{\rho(T(\boldsymbol{x},t))c(T(\boldsymbol{x},t))}\left[\lambda(T(\boldsymbol{x},t)) \cdot \right.$$

$$\left. \cdot \left(\frac{\partial^2 T(\boldsymbol{x},t)}{\partial x_1^2} + \frac{\partial^2 T(\boldsymbol{x},t)}{\partial x_2^2} + \frac{\partial^2 T(\boldsymbol{x},t)}{\partial x_3^2}\right) + \omega(\boldsymbol{x},t)\right]. \quad (17)$$

Let the coordinates of the node point be

$$(x_{1_i}, x_{2_j}, x_{3_k}), i = 1, ..., n_1, j = 1, ...n_2, k = 1, ..., n_3.$$

With the notation

$$\begin{aligned} x_{i,j,k} &= x_{1_i}, x_{2_j}, x_{3_k} \\ T_{i,j,k}(t) &= T(x_{i,j,k}, t) \\ \omega_{i,j,k}(t) &= \omega(x_{i,j,k}, t) \end{aligned}$$

the x_1, x_2, x_3 derivatives are approximated by

$$\begin{aligned} \frac{\partial^2 T(\boldsymbol{x},t)}{\partial x_1^2} &\approx \frac{T_{i+1,j,k}(t) - 2T_{ijk}(t) + T_{i-1,j,k}(t)}{h_1^2} \\ \frac{\partial^2 T(\boldsymbol{x},t)}{\partial x_2^2} &\approx \frac{T_{i,j+1,k}(t) - 2T_{ijk}(t) + T_{i,j-1,k}(t)}{h_2^2} \\ \frac{\partial^2 T(\boldsymbol{x},t)}{\partial x_3^2} &\approx \frac{T_{i,j,k+1}(t) - 2T_{ijk}(t) + T_{i,j,k-1}(t)}{h_3^2}. \quad (18) \end{aligned}$$

Now let us approximate of the derivation by difference in the boudary condition (11) to define T_{ijk} for $i = 0$, $i = n_1 + 1$, $j = 0$, $j = n_2 + 1$, $k = 0$ and $k = n_3 + 1$ as

$$\begin{aligned} T_{0,j,k}(t) &= T_{1,j,k} - \frac{h_1\alpha}{\lambda(T_{1,j,k}(t))}(T_{1,j,k} - T_{ext}) \\ T_{n_1+1,j,k}(t) &= T_{n_1,j,k} - \frac{h_1\alpha}{\lambda(T_{n_1,j,k}(t))}(T_{n_1,j,k} - T_{ext}) \\ T_{i,0,k}(t) &= T_{i,1,k} - \frac{h_2\alpha}{\lambda(T_{i,1,k}(t))}(T_{i,1,k} - T_{ext}) \\ T_{i,n_2+1,k}(t) &= T_{i,n_2,k} - \frac{h_2\alpha}{\lambda(T_{i,n_2,k}(t))}(T_{i,n_2,k} - T_{ext}) \\ T_{i,j,0}(t) &= T_{i,j,1} - \frac{h_3\alpha}{\lambda(T_{i,j,1}(t))}(T_{i,j,1} - T_{ext}) \\ T_{i,j,n_3+1}(t) &= T_{i,j,n_3} - \frac{h_3\alpha}{\lambda(T_{i,j,n_3}(t))}(T_{i,j,1n_3} - T_{ext}). \quad (19) \end{aligned}$$

Let us define time step Δt and approximate time derivation by

$$\frac{\partial T(\boldsymbol{x},t)}{\partial t} \approx \frac{T(\boldsymbol{x},t+\Delta t) - T(\boldsymbol{x},t)}{\Delta t}. \quad (20)$$

From (17), (18), (19) and (20) we have numerical formula for temperature evolution

$$T_{i,j,k}(t+\Delta t) = T_{i,j,k}(t) + \frac{\Delta t}{\rho(T_{i,j,k}(t))c(T_{i,j,k}(t))} \cdot$$

$$\cdot \left[\lambda(T_{ijk}(t))\left(\frac{T_{i+1,j,k}(t) - 2T_{i,j,k}(t) + T_{i-1,j,k}(t)}{h_1^2} + \right.\right.$$

$$+ \frac{T_{i,j+1,k}(t) - 2T_{i,j,k}(t) + T_{i,j-1,k}(t)}{h_2^2} +$$

$$\left.\left. + \frac{T_{i,j,k+1}(t) - 2T_{i,j,k}(t) + T_{i,j,k-1}(t)}{h_3^2}\right) + \omega_{i,j,k}(t)\right].$$

Joule losses need to be updated when time changes. The whole calculation is started with an initial condition. It is the starting temperature of the body and is equal to temperature of the air of the surroundig medium.

6 Example 1

A brass cuboid body with the size $0.15 \times 0.01 \times 0.01$ m is heated with a stationary inductor starting at room temperature 20 °C. The inductor has the form of a coil which turns around the heated body in the x_1-direction in 6 loops. The radius of the coil is 0.015 m, exciting current 500 A and frequency 150 kHz. The length of the coil is 0.15 m. The cuboid is partitioned by 75 elements in x_1 direction, 10 elements in x_2 and x_3 directions. Figures 2 - 4 show temperature distribution during time evolution (1, 10 and 60 s) at the surface of the heating body. Figures 5 - 7 show temperature distribution during time evolution (1, 10 and 60 s) at cuts of the body when $x_2 = 0$ and $x_3 = 0$.

Computation was made by Matlab. The integration for computing $F(x_i)$ is done by the Matlab's method *quad*. It uses an aplication of the adaptive Simpson quadrature. Improper integral is computed analytically.

7 Example 2

To see the dependence of the parameters let us show examples with different parameters. Let the parameters be the same as in example 1. Figure 8 shows temperature distribution at 30 s. Other figures have change in one parameter.

- at figure 9 the coil rotates over the heating body at 3 loops.
- at figure 10 the size of the body is $0.15 \times 0.005 \times 0.005$ m.
- at figure 11 the coil is exciting current 1000 A.
- at figure 12 the angular frequency is 300 kHz.

8 Conclusion

In example 2, the temperature changed as expected. It was shown in [5], that the time to compute the Joule losses is reduced to approximately 10 percent compared to the collocation method. Since the Joule losses need to be recalculated 3 – 10 times during time evolation to 60 s is this method a big improovement.

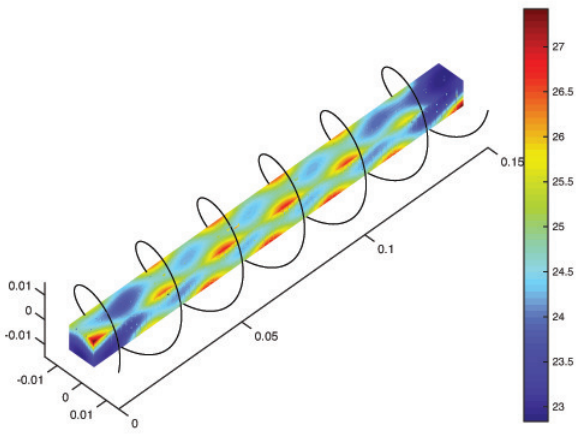

Figure 2. Example 1, t = 1 s, body

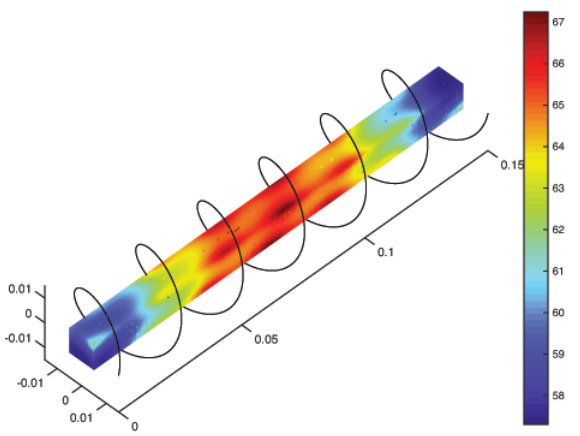

Figure 3. Example 1, t = 10 s, body

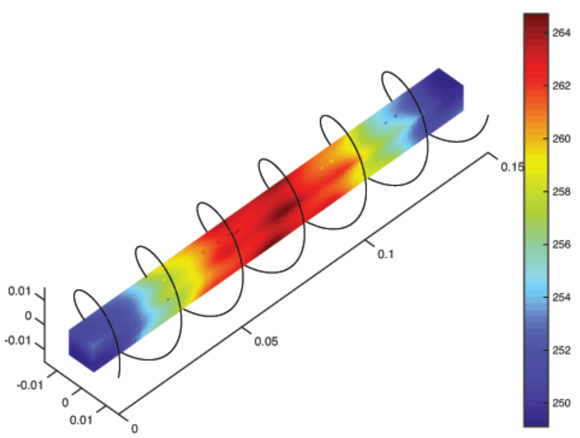

Figure 4. Example 1, t = 60 s,body

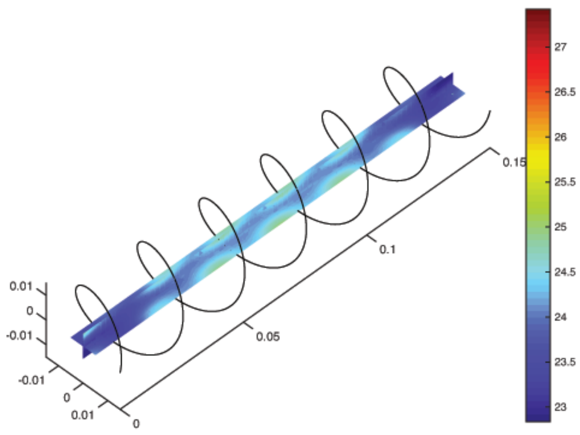

Figure 5. Example 1, t = 1 s, body cut

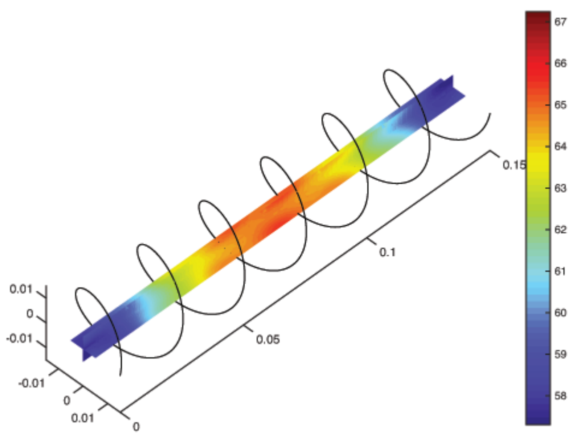

Figure 6. Example 1, t = 10 s, body cut

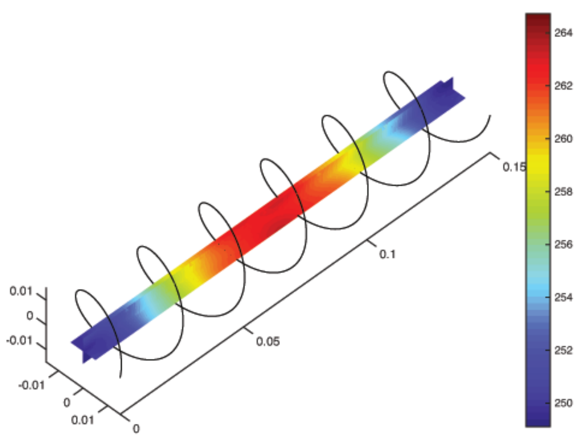

Figure 7. Example 1, t = 60 s, body cut

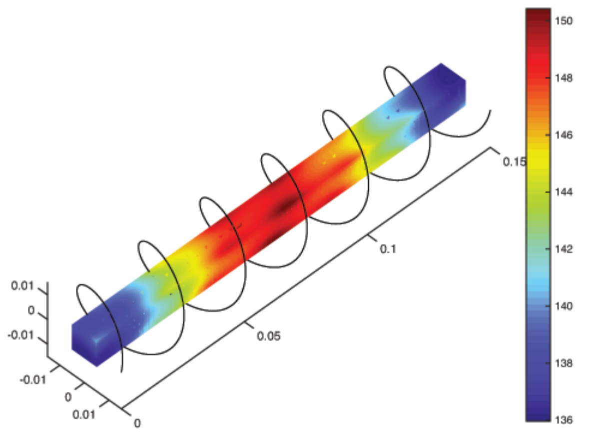

Figure 8. Example 2, t = 30 s, original parameters

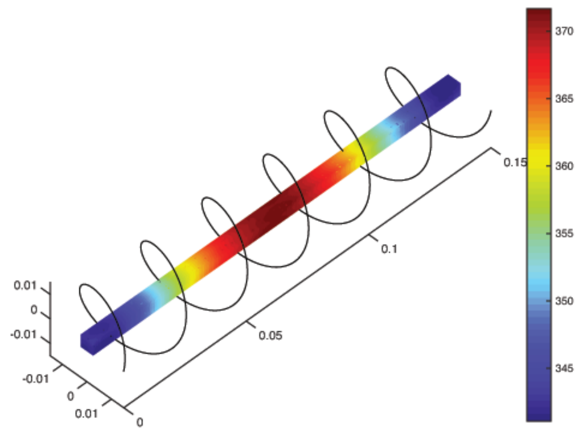

Figure 10. Example 2, t = 30 s, size 0.15 × 0.005 × 0.005 m

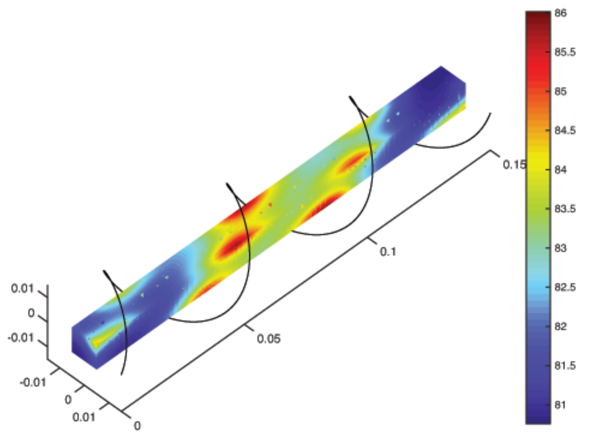

Figure 9. Example 2, t = 30 s, 3 loops of the coil

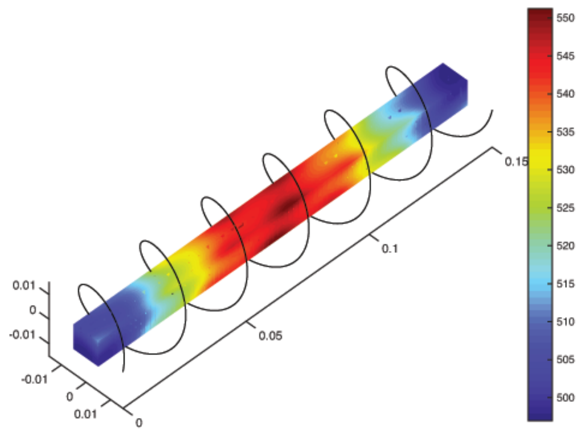

Figure 11. Example 2, t = 30 s, harmonic current 1000 A

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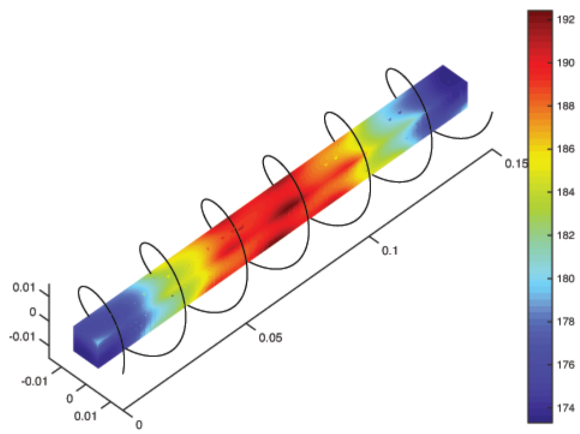

Figure 12. Example 2, t = 30 s, angular frequency 300 kHz