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What does Holographic QCD predict for anomalous $(g-2)_\mu$?

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Abstract. In this talk we discuss how holographic models of QCD have been applied to the study of the light-by-light contribution to the muon anomalous magnetic moment. After a review of the holographic procedure, we discuss the approach we followed in a previous work, in which, using a certain set of holographic model as a "Theory Space", we gave our estimate for the π^0 exchange diagram. Our result depended also on the value of the quark magnetic susceptibility. Thus, in the last part of the talk, we concentrate on the attempts which have been done in the literature to apply holographic methods to the evaluation of this important order parameter of QCD.

1 Introduction

Holographic QCD (HQCD) is a set of *simple* models, aimed at reproducing relevant aspect of Large-N_c QCD in non-perturbative regime. By simple, we mean that they provide explicit analytic expressions for correlation functions of QCD operators, covering the full energy range, with surprisingly good results at lowest energy, where their predictions can be compared with those of Chiral Lagrangians, Vector Meson Dominance, etc.

HQCD models are based on classical Lagrangians, although defined in a (curved) five-dimensional space, contain a small number of parameters, and approximate well the almost conformal behavior of QCD at large momenta. They naturally encode some non-perturbative features of Large-N_c QCD, such as, for instance, the appearance of an infinite number of narrow resonance, saturating amplitude channels.

Many features are, however, model-dependent. Chiral symmetry breaking (ChSB) is differently realized in different HQCD models, each realization having its problematic issues. The mass spectrum of vector mesons do not follow a Regge behavior in some models. Subleading terms in the large momentum expansion of two-point correlation functions, related to QCD condensates, are badly reproduced or completely absent in some models. Moreover, the simplicity of HQCD models can be easily be spoiled, when further aspect of real QCD are tried to be enforced into them.

The simplicity of HQCD makes them useful for the discussion of the Hadronic Ligh-by-Light scattering contribution to the muon anomalous magnetic moment, as was also emphasized by M.Knecht in this workshop [1].

Indeed, it is possible to extract predictions for HLbL scattering using HQCD models, which turn out to be compatible with those obtained using different models and techniques. This strongly suggests that those (partial) aspects of QCD which have been encoded in the holographic models are really (among) the ones relevant for HLbL scattering contributions to $(g-2)_\mu$. Although similar conclusions had been obtained using other models and methods, HQCD offers an additional independent confirmation.

In the following, we briefly review the basics of holographic models in Sect.2, introduce the basic tools for an holographic calculation of the pion-pole contribution in Sect. 3 and illustrate the strategy we used in [2] *(i)* to work with several holographic models as a "Theory Space" and *(ii)* to take into account a new high energy constraint involving the quark magnetic susceptibility χ_0. In Sect.4, we review the various attempts to use or to extend holographic models in order to obtain predictions on χ_0. Conclusions are drawn at the end.

2 Holographic models of QCD

HQCD models have been constructed along the lines of the original conjectured AdS/CFT duality (equivalence) between a four dimensional (conformal) gauge theory at strong coupling and a (classical) five-dimensional field theory in a curved gravitational background with Anti-de-Sitter metric [3–5]. The conjecture can be boldly summarized as follows: for any 4D operator O_Δ of QCD, there exists a corresponding field $\Phi(x,z)$, living in a curved 5D space with (asymptotically) AdS metric, whose value on the 4D boundary of the 5D space, $\Phi(x,0)$, coincide with the classical source $s(x)$ coupled to O_Δ. The generating functional of correlation functions of the 4D theory is equal to the 5D action of $\Phi(x,z)$, evaluated *on-shell*

$$\begin{aligned}\exp(iW[s(x)]) &\equiv \langle \exp\left(i\int d^4x\, s(x)O_\Delta(x)\right)\rangle_{QCD}\\ &= \exp\left(i\,S_5(\Phi_0(z,x))\right),\end{aligned} \tag{1}$$

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where $\Phi_0(z,x)$ is the solution of the 5D equation of motion satisfying the boundary condition $\Phi(x,0) = s(x)$.

The operator-field correspondence can be further specified leading to an holographic dictionary, relating 4D and 5D quantities as exemplified in Tab. 1

Table 1. Example of holographic dictionary. The last three lines contain the operators relevant for HQCD (a is a flavour index).

4D operator	5D dual field
source $s(x)$ coupled to $O_\Delta(x)$	on-shell $\Phi_0(x,z)$
conformal dimension Δ	mass m_Φ
global symmetry	gauge symmetry
conserved current $\bar{q}\gamma_\mu t^a\, q$	gauge field V^a_M
axial current $\bar{q}\gamma_\mu\gamma_5 t^a\, q$	gauge field A^a_M
quark bilinear $\bar{q}t^a\, q$	scalar field X^a

with the following relation between the 5D mass m_Φ of a 5D p-form field and the conformal dimension Δ of the corresponding 4D operator:

$$m^2_\Phi = (\Delta - p)(\Delta + p - 4). \tag{2}$$

One can easily check Eq.(2) for the case of 5D gauge fields, while the tachyonic mass obtained for the 5D scalar field is above the unitarity lower bound valid in an AdS space.

In the case of holographic models of QCD, additional ingredients have to be added in order to encode the effects of confinement and chiral symmetry breaking (ChSB). Different proposals have led to different models. Confinement is realized by effectively compactifying the 5D extradimension. As a consequence of the compactification, vector and axial vector meson resonances appear, corresponding to an infinite set of normalizable 5D states. One HQCD model was proposed by Sakai and Sugimoto [6, 7], who actually derived it from a more sophisticated string theory setting. The SS model is characterized by a curved but not AdS$_5$ metric background, contains 5D gauge fields invariant under a local chiral symmetry, which reduces to global chiral flavour transformation on the 4D boundary. In the Hard Wall (HW) models of [8, 9] and [10] the extradimension of the AdS$_5$ space is cut-off at a finite size, while in the Soft-Wall model of [11] the extradimension is infinite, but one still recovers infinite numbers of 4D vector and axial vector states corresponding to 5D normalizable modes in an effective potential generated by an additional 5D dilaton background. Figs. 1 and 2 show cartoons of HW and SW models.

The relevant 5D Lagrangian of an HQCD model contains vector and axial vector gauge fields, V^a_M and A^a_M dual to vector and axial quark currents $\bar{q}\gamma_\mu t^a q$and t $\bar{q}\gamma_\mu\gamma_5 t^a q$ respectively. In some models, in particular in the HW model considered in [8] and [9], one has also the scalar field X^a corresponding to the quark bilinear $\bar{q}t^a q$.

The relevant 5D Lagrangian is the following

$$S_5 = S_{\rm YM} + S_{\rm X} + S_{\rm CS}\,, \tag{3}$$

where

$$S_{\rm YM} = -{\rm tr}\int d^4x \int_0^{z_0} dz\, e^{-\Phi(z)}\frac{1}{8g_5^2}w(z)\left[\mathcal{F}_L^2 + \mathcal{F}_R^2\right], \tag{4}$$

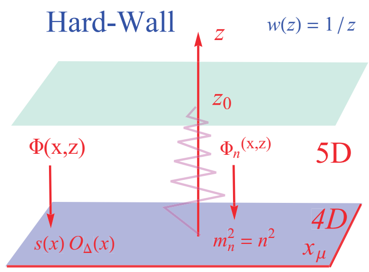

Figure 1. A pictorial representation of a Hard-Wall model. The 5D metric is AdS$_5$ with warp factor $w(z) = 1/z$. The extradimension z is sharply cut-off at a value z_0. This compactification generates an infinite series of 5D normalizable modes $\Phi_n(x,z)$ corresponding to an infinite tower of 4D resonances with masses $m_n^2 \propto n^2$.The mass scale is fixed by the value of the dimensional parameter z_0.

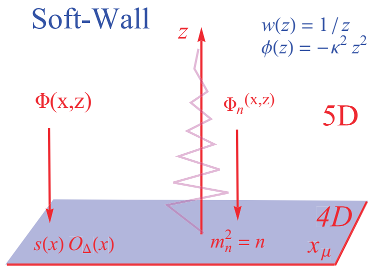

Figure 2. A pictorial representation of the Soft-Wall model.The 5D metric is AdS$_5$ with warp factor $w(z) = 1/z$ and with unbounded extradimension z. In presence of the dilaton background $\phi(z) = -\kappa^2 z^2$, an infinite series of 5D normalizable modes $\Phi_n(x,z)$ is produced, and hence 4D resonances with masses, $m_n^2 \propto n$, following a Regge behavior. The mass scale is fixed by the value of the dimensional parameter κ.

$$S_X = {\rm tr}\int d^4x \int_0^{z_0} dz\, e^{-\Phi(z)} w(z)^3\left[D^M X D_M X^\dagger + V(X^\dagger X)\right], \tag{5}$$

$$S_{\rm CS} = \frac{N_c}{24\pi^2}\int {\rm tr}\left(\mathcal{A}_L\mathcal{F}_L^2 - \frac{i}{2}\mathcal{A}_L^3\mathcal{F}_L - \frac{1}{10}\mathcal{A}_L^5\right) - (L\to R)\,. \tag{6}$$

In these equations:

- $w(z)$ is the warp factor of the 5D metric $ds^2_5 = w(z)^2\left(dx_\mu^2 - dz^2\right)$. For HW and SW models the metric is AdS with $w(z) = 1/z$.
- $\mathcal{F}_{MN} = \partial_M\mathcal{A}_N - \partial_N\mathcal{A}_M - i[\mathcal{A}_M, \mathcal{A}_N]$ and $\mathcal{A}_{L,R} = V \mp A$,
- In HW models the extradimension is IR cut-off at a finite value of z_0, whilst or $z_0 = \infty$ in SW models.
- In SW models, there is also a background dilaton field $\Phi(z) = -\kappa^2 z^2$, absent in the HW models.
- In the HW model of [8] and [9], (from here on dubbed HW1), the 5D scalar field $X(x,z)$, dual to $\bar{q}q$, induces ChSB, by acquiring a non trivial 5D profile $X = v(z)$. Such scalar field, and its action S_X, do not appear in the HW of [10] (which we shall call HW2), where chiral symmetry between vector and axial vector fields is bro-

ken by assigning different boundary conditions for V_μ and A_μ on the IR wall z_0.

- One may notice that the SS model [6] can be reformulated as an HW2 model, with a 5D metric which is not even asymptotically AdS for $z \to 0$.
- Both in the SS and in the HW2 model, the chiral field $U(x)$ realization of Goldstone bosons of the spontaneous ChSB is obtained. Indeed, in these models $U(x)$ is the remnant of non trivial 5D Wilson lines, induced by the ChSB boundary conditions. This makes it possible to derive predictions for the coefficients of the low-energy chiral Lagrangian in a straightforward way.

3 HLbL pion exchange diagram in HQCD

The main motivation of this conference was the forthcoming improvement in the precision of the experimental measure of the muon anomalous magnetic moment, scheduled at Fermilab, as illustrated by D. Herzog's talk [15]. At present, there is a discrepancy between the experimental value [16] and the prediction of the Standard Model. To asses whether this discrepancy is real or not is of paramount importance to understand if new physics is required to fill the gap.

Hadronic light-by-light scattering (HLbL) is the hadronic contribution to the muon anomaly which is less under control from the theoretical side, as illustrated in many reviews on the subject and in the talks by D. Melnikov [17] and M. Knecht [1] at this conference. To-date, the best estimates of the HLbL contributions are given in Ref.[18, 19] as being $(11\pm4)\times10^{-10}$ or in Ref.[20] as being $(10.5\pm)\times10^{-10}$. The two different values and errors show the difficulty to produce a world average from the plethora of different theoretical methods that have been employed in the literature, some of them illustrated in talks at this conference (this one included, of course !).

HQCD models can be used to extract a prediction for the neutral pion exchange contribution to the HLbL diagram of Fig.3. In this diagram the key role is played by the pion form factor $F_{\pi^0\gamma^*\gamma^*}(Q_1^2, Q_2^2)$, with Q_1 and Q_2 the two photon momenta. Using the HQCD action (3) explicit

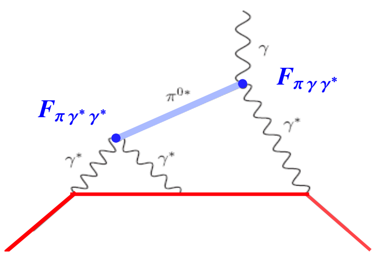

Figure 3. The neutral pion exchange diagram in the hadronic light-by-light scattering contribution to $(g-2)_\mu$

(analytic) expressions for $K(Q_1^2, Q_2^2$ defined by

$$F_{\pi^0\gamma^*\gamma^*}\left(Q_1^2, Q_2^2\right) = -\frac{N_C}{12\pi^2 f_\pi} K(Q_1^2, Q_2^2) \,,$$

can be obtained. For different HQCD models, they were worked out in a series of papers [12–14].

The relevant anomalous 3-point function $\langle AVV \rangle$ is obtained from the 5D Chern-Simons action (6), which contains the following trilinear term

$$S^{(\pi VV)}_{\rm CS} = \frac{N_c}{4\pi^2}\epsilon^{\mu\nu\rho\sigma}\int_0^{z_0} dz\,(\partial_z\alpha(z))\int d^4x\,\pi^a\left(\partial_\rho V^a_\mu\right)\left(\partial_\sigma \hat{V}_\nu\right) \,.$$

Then, using the holographic recipe (1)

$$K(Q_1^2, Q_2^2) = -\int_0^{z_0} \partial_z\alpha(z)\mathcal{J}(Q_1, z)\mathcal{J}(Q_2, z)\,dz, \qquad (7)$$

which contains *(i)* the vector bulk-to-boundary propagator $\mathcal{J}(Q, z)$, which allows to express the 5D action *on-shell*, and *(ii)* the "pion wave function" $\alpha(z)$ coming from the pion pole in the axial bulk-to-boundary propagator.

It should be stressed that any HQCD model provides explicit expressions for $\mathcal{J}(Q, z)$ and $\alpha(z)$, obtained from solutions of second order ordinary differential equations (the 5D equation of motion) in the extra dimension z, and subject to suitable boundary conditions. This leads to explicit expressions for $K(Q_1^2, Q_2^2)$ for any value of the two virtualities. In particular, one can extract the limiting behavior at low momenta and in the deep Euclidean region.

3.1 An Hybrid Strategy: HQCD models as a "Theory Space"

In Ref. [2], the questions we addressed, using HQCD, were the following:

- Which parameters of $F_{\pi^0\gamma^*\gamma^*}$ mostly affect the predicted value of $(g-2)_\mu$?
- What does HQCD predicts for linear and quadratic slopes (low-Q^2 expansion) of $F_{\pi^0\gamma^*\gamma^*}$
- Can we discriminate some HQCD models with respect to others?
- Is Vector Meson Dominance a valid approximation or more resonances give sizable contributions?
- Can we implement QCD low- and high-energy constraints in HQCD models ?

This led us to consider a certain set of HQCD models as a sort of "Theory Space" in which to move trying to answer the questions listed above. Thus, our approach was different from the one followed in [21], were the pion form factor was evaluated in a specific holographic model HW1, *i.e.* HW1, the Hard-Wall model with chiral symmetry breaking induced by a 5D scalar field.

Eventually, our strategy was as follows: from the expressions for $F_{\pi^0\gamma^*\gamma^*}$ obtained from different HQCD models we extracted the corresponding predictions for linear slope $\hat{\alpha}$ at low values of the two photon momenta:

$$K(Q_1^2, Q_2^2) \simeq 1 + \hat{\alpha}\,(Q_1^2 + Q_2^2) + \hat{\beta}\,Q_1^2 Q_2^2 + \hat{\gamma}\,(Q_1^4 + Q_2^4,) \quad (8)$$

Different HQCD models gave different results for the linear slope $\hat{\alpha}$, as shown in Table 2.

Table 2. HQCD predictions for the linear slope. The first three holographic models, HW1, HW2 and SW, give results is in good agreement with the experimental value $\hat{\alpha}_{exp} = -1.76 \pm 0.22$ GeV^{-2}, while this does not hold for the SS model and a simpler version of HW2 with 5D flat space used for further comparison.

Model	$\hat{\alpha}$ (GeV^{-2})
HW1	-1.60
HW2	-1.81
SW	-1.66
SS	-2.04
HW2 (Flat)	-1.37

Somewhat surprisingly, a good agreement with the experimental value of $\hat{\alpha}$ is obtained only for models - HW1, HW2 and SW - which reproduce the partonic log at large-Q^2, thus matching the UV behavior of QCD. Therefore, we restricted our further analysis only to those three models. From those model we also extracted a prediction for the curvatures $\hat{\beta}$ and $\hat{\gamma}$

$$\hat{\beta} = 3.33(32)\,\mathrm{GeV}^{-4}, \quad \hat{\gamma} = 2.84(21)\,\mathrm{GeV}^{-4}. \tag{9}$$

The investigation of (Lowest) Vector Meson Dominance, using the resonance expansion of the bulk-to-boundary propagator

$$\mathcal{J}(Q,z) = \sum_{n=1}^{\infty} \frac{f_n}{Q^2 + m_n^2} \psi_n(z) \tag{10}$$

and consequently of the form factor

$$K(Q_1^2, Q_2^2) = \sum_{k,l=1}^{\infty} \frac{B_{kl}}{(Q_1^2 + m_k^2)(Q_2^2 + m_l^2)}, \tag{11}$$

led to the conclusion that dominant contributions to $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$ were obtained from first terms in (10), suggesting that truncating the sums in $K(Q_1^2, Q_2^2)$ to double poles was already a quite good approximation.

The full analytic expressions for $F_{\pi^0\gamma^*\gamma^*}$, which encode also the different spectrum of resonance of each model, would generically lead to complicated integrals in the evaluation of a_μ. More importantly, there is a further QCD high-energy constraint, which we want to impose, but which none of these HQCD models can take into account.

A list of high energy constraints on $F_{\pi^0\gamma^*\gamma^*}$ contains some constraints which are satisfied in HQCD models with asymptotic AdS metric

$$\lim_{Q^2\to\infty} K(Q^2, Q^2) = \frac{8\pi^2 f_\pi^2}{N_c} \frac{1}{Q^2} \tag{12}$$

$$\lim_{Q^2\to\infty} K(0, Q^2) \sim \frac{1}{Q^2}, \tag{13}$$

where the second one is the Brodsky-Lepage scaling [22].

In Ref.[23], the importance a new constraint was stressed

$$\lim_{Q^2\to\infty} F_{\pi_0^*\gamma^*\gamma^*}(Q^2, Q^2, 0) = \frac{f_\pi}{3} \chi_0 + \cdots$$

where f_π is the pion decay constant and χ_0 is a new order parameter of QCD: the quark magnetic susceptibility, which we shall deal with in greater detail in the next Section. In literature

$$0 \le \chi_0 \le 9\ \mathrm{GeV}^{-2}.$$

All these reasons led us to propose an interpolator for the pion form factor (hereafter dubbed DIP and introduced in kaon physics in [24]) to encode both HQCD low-energy predictions and QCD short-distance constraints, and whose expression was also simple enough to allow us to use the formulas of [25] for the evaluation of the one pion exchange to HLbL contribution to a_μ:

$$\begin{aligned} K^{(DIP)}(q_1^2, q_2^2) &= 1 + \lambda \left(\frac{q_1^2}{q_1^2 - m_V^2} + \frac{q_2^2}{q_2^2 - m_V^2} \right) \\ &+ \eta \frac{q_1^2 q_2^2}{(q_1^2 - m_V^2)(q_2^2 - m_V^2)}. \end{aligned} \tag{14}$$

Low-energy parameters $\hat{\alpha}, \hat{\beta}$ and $\hat{\gamma}$ are easily related to λ, η and m_V and, moreover, it is easy to implement some high- and low-energy constraints and yield predictions for the remaining parameters. For instance, imposing

$$1 + 2\lambda + \eta = 0, \quad \lambda + \eta = -\frac{4\pi^2 f_\pi^2}{3m_V^2} \qquad (OPE), \tag{15}$$

$$\frac{\lambda}{m_V^2} = \hat{\alpha} = -1.76 \pm 0.22 \tag{16}$$

one gets

$$m_V = (0.64^{+0.10}_{-0.13})\mathrm{GeV}, \tag{17}$$

$$\chi_0 = \frac{N_c}{4\pi^2 f_\pi^2}(1 + \lambda) = (2.42 \pm 0.17)\,\mathrm{GeV}^{-2}, \tag{18}$$

with the value of m_V in the DIP close to the value of the mass of the first vector resonance, ρ.

3.2 Numerical results

We defer the reader to Ref. [2] for the details of the numerical analysis, which we did in order to extract our prediction for a_μ and estimate the error. Here, we only mention that in the numerical analysis a (mild) generalization of the interpolator was also considered, showing, however, that numerical values were stable and very close to those obtained with the original DIP. Our final number for the pion-exchange HLBL contribution was

$$a_\mu^{\pi^0} = 65.4(2.5) \cdot 10^{-11} \tag{19}$$

with the error mainly driven by the linear slope of $F_{\pi^0\gamma^*\gamma^*}$. We stress that the reported error in (19) has to be interpreted as indicating the allowed range of values of a_μ in the "Theory Space" of HQCD models that we had considered. Another caveat is that, although large values of the magnetic susceptibility χ_0 are disfavored, in the absence of stronger bounds on χ_0, an additional $(10-15)\%$ systematic uncertainty on the previous value for $a_\mu^{\pi^0}$ cannot be excluded.

Table 3. A partial list of estimates of $a_\mu^{\pi_0}$ obtained using different models for $F_{\pi^*\gamma^*\gamma^*}$

Models for $F_{\pi^*\gamma^*\gamma^*}$	$a_\mu^{\pi_0} \times 10^{-11}$
Modified ENJL [26–28]	59(9)
VMD/HLS [29–32]	57(4)
VMD+V ($h_2 = 0$) [33]	58(10)
VMD+V ($h_2 = -10\,\mathrm{GeV}^2$) [33]	63(10)
VMD+V (const.$F_{\pi_0\gamma^*\gamma}$) [37]	77(7)
DSE [34, 36]	58(7)
Non local χQM [38]	65(2)
AdS/QCD[21, 39]	69
AdS/QCD/DIP [2]	65.4(2.5)
RχT[40]	65.8(1.2)
CχQM [41]	68(3)

Table 3 shows a partial list of values of $a_\mu^{\pi^0}$ obtained using different models, some of them discussed in other talks at this conference. Our numerical analysis showed also that the values of χ_0 hovered around $2\,\mathrm{GeV}^{-2}$, as in the case shown in (18), with higher values disfavored although not definitely excluded. Higher values, e.g. $\chi_0 \sim 9\,\mathrm{GeV}^{-2}$, would shift the result (19) of at most 15%. Given the precision needed for $(g-2)_\mu$, a better control on χ_0 is compelling. In the next section, we shall discuss the present knowledge about χ_0 and some attempt to pinpoint its value using (extended) HQCD models.

4 Beyond the DIP ?

Can we go beyond the approach we followed in [2] and make a more thorough application of HQCD to the evaluation of $a_\mu^{\pi^0}$?

Some possible options could be the following:

- Given one of the HQCD models, we could use the full analytic expressions for $K(Q_1^2, Q_2^2)$ for the evaluation of $a_\mu^{\pi^0}$ as it was done for HW1 in [21].
- We could try to understand if a given model HQCD effectively comply with all the QCD constraints on the pseudoscalar exchange, included those advocated in [37], which led to an enhancement of the value of $a_\mu^{\pi^0}$, with respect to the other competing models.
- We could try to use the extensions of HQCD models proposed in [42, 43, 63], in which additional 5D tensor fields were added, corresponding in the AdS/CFT dictionary to quark tensor operators, in order to produce an independent prediction on the magnetic susceptibility χ_0.

On the other hand, one would readily led to some problematic issues.

- As we stressed in the introduction, none of the HQCD models that have been proposed is free from problems. That is why we tried the "Theory Space" approach, illustrated in the previous Section. For instance, it would be nice to have a model, with resonances following Regge trajectories, as happens in the SW model, and at the same time recovering the full chiral field for the pion at low momenta, which is the appealing feature of HW2. An attempt in this direction, following some ideas in [45], is currently under study.
- A technical problem, directly related to the evaluation of the HLbL contribution, is that the analytical form of $K(Q_1^2, Q_2^2)$ could not allow the use of the simpler 2-loops integrations as it was the case with DIP. (However, more general expression exists, see for instance the talk of M. Procura in this workshop [46].)
- The extended models of HQCD with 5D Lagrangians containing also to tensor fields, that we mentioned above and that one could try to use in order to get independent determinations of χ_0, seem to have serious problems in reproducing a satisfying QCD phenomenology.

In the following, we shall briefly consider this last issue.

4.1 Some facts about χ_0

In presence of a magnetic background, the tensor current

$$O^{T,a}_{\mu\nu}(x) = \bar{q}(x) t^a \sigma_{\mu\nu} q(x) \tag{20}$$

can acquire a non vanishing VEV

$$\langle \bar{q} t^a \sigma_{\mu\nu} q \rangle = e_q \chi_q \langle \bar{q} q \rangle F^{cl}_{\mu\nu} \tag{21}$$

where $F^{cl}_{\mu\nu} = \partial_\mu A^{cl}_\nu - \partial_\nu A^{cl}_\mu$.

The operator O^T is odd under C but contains both parity even and odd parts and thus can create create both $J^{PC} = 1^{--}$ and 1^{+-} states $\rho^{(n)}, b_1^{(n)}$:

$$\begin{aligned}
\langle 0|O^{T,a}_{\mu\nu}(x)|b_1^{(n),c}(k)\rangle &= \frac{i}{\sqrt{2}} f_b^{(n)} \epsilon_{\mu\nu\alpha\beta} \varepsilon^{\perp(n)\alpha}_{(b)} k^\beta \delta^{ac} e^{-ik\cdot x} \\
\langle 0|O^{T,a}_{\mu\nu}(x)|\rho^{(n),c}(k)\rangle &= \frac{i}{\sqrt{2}} f_\rho^{T,(n)} (\varepsilon^{\perp(n)}_{(\rho)\mu} k_\nu - \varepsilon^{\perp(n)}_{(\rho)\nu} k_\mu) \delta^{ac} e^{-ik\cdot x}.
\end{aligned}$$

χ_0 can be estimated using QCD sum rules, because it enters in the mixed VT 2-point function

$$\lim_{q^2\to 0} \Pi_{VT}(q^2) = -\chi_0 \langle \bar{q} q \rangle \tag{22}$$

Since it first evaluation, done in [47] using QCD sum rules, many different approaches have been followed to calculate χ_0. The comparison of the various results, however, is not immediate, partly because χ_0 depends on the renormalization scale. In comparing the results, it is common to write the magnetic susceptibility in terms of a dimensionless parameter c_χ as

$$\chi_0 = -c_\chi \frac{N_c}{16\pi^2 f_\pi^2} \tag{23}$$

where $f_\pi = 92.4$ is the pion decay constant.

Table 4 shows an incomplete list of values c_χ, obtained using different theoretical approaches.

The first four raws in Table 4 contain the results obtained applying sum rules in different contexts: analysing the of nucleon magnetic moments and $\Delta \to N\gamma$ radiative transitions [47], studying photon distribution amplitudes in QCD [50] or radiative heavy meson decays [51].

Table 4. Values of c_χ obtained with different theoretical methods.

Method (renormalization point)	c_χ
Sum rules (1 GeV) [47]	3.86 ± 0.11
Sum rules (0.5 GeV) [48]	2.56
Sum rules (1 GeV) [49]	1.98 ± 0.18
Sum rules (1 GeV) [50]	1.41 ± 0.13
Sum rules (1 GeV) [51]	1.28 ± 0.22
OPE + Pion Dominance (0.5 GeV) [52]	2
Instanton vacuum (1 GeV) [53]	1.12 ± 0.07
Zero mode of Dirac Op. (1 GeV) [54]	1.58
Lattice (1 GeV) [55]	0.695(1)
Lattice (1 GeV) [56]	1.90 ± 0.08
NJL model (0.63/ GeV) [57]	-1.93
QM model (0.56 GeV) [57]	-2.36
$\langle AVV\rangle$ in HQCD ($\ll 1$ GeV) [58]	2.15
$\langle AVV\rangle$ in HQCD [59]	2
HQCD + Tensors [60]	0.06
HQCD +Tensors [61]	1–1.5

The result $c_\chi = 2$ was obtained by Vaishtein in [52], studying the QCD $\langle AVV\rangle$ 3-point function, when two vitualities of the external legs are large and one vector current represents the constant external electromagnetic field strength, and then using OPE and pion pole dominance. This result is extremely important, in particular for the comparison with the predictions obtained from HQCD models.

Further results displayed in Table 4 were obtained by using instantons [53], zero mode solutions of quark Dirac equations in QCD [54], lattice [55, 56] and other models in [57].

4.2 HQCD predictions for χ_0

The last four raws in Table 4 show the results of applying (extended) HQCD models to evaluation of the quark magnetic susceptibility.

The first two results were obtained from the study of the anomalous three-point function $\langle AVV\rangle$ in HW1 model [58], and in both HW1 and HW2 (SS) model in [59]. In particular the result of [59] exactly coincides with the one obtained in [52] using QCD OPE, non renormalization theorems and pion pole dominance. The authors of [59] based their results on an anomaly matching for resonances valid in HQCD models that, if true also in real QCD, reproduced Vainshtein result. However, their assumptions were criticised as being actually not true in QCD [62].

The last two raws in Tab 4 show the values of for c_χ obtained in HW1 model extended to include an antisymmetric tensor dual to the QCD operator $O^{T,a}_{\mu\nu}(x)$ in (20).

There is a great discrepancy between the result found in [60] and those found in [58, 59], which were instead very close or coinciding with the one of Vainshtein [52]. The authors of [61], who used the same 5D model HW1 with tensor fields as in [60], in order to make a global fit on low energy phenomenological parameters, raised the question about the real necessity to match the UV behavior of holographic models with the QCD one. They advocated that by relaxing this constraint, a better fit to low energy properties could be obtained. In particular, regarding the quark magnetic susceptibility, they were led to a range of values much closer to the other ones reported in Table 4.

Actually, that was not the first time that a more phenomenological use of the HQCD models was proposed in the literature. In [61], it was argued that the UV matching should be done only in the case of conserved currents, whose scaling dimensions are not modified by the RG flow; this latter property does not hold for the tensor currents $O^T_{\mu\nu}(x)$. Indeed, in Ref. [63], using the same model, with holographic correlation functions of the vector and tensor operators reproducing the large-Q^2 behavior expected from QCD, a poor phenomenological agreement of the resulting spectrum of resonances was obtained. Notice that already in [42], we were led to modify the UV boundary conditions, altering the rule in Eq.(2) of the holographic dictionary in the case of the antisymmetric tensor field, in order to get a realistic resonance spectrum.

5 Conclusions

HQCD models allow many analytic calculations of physical quantities which are relevant for non perturbative regime of QCD. In particular, they capture important aspect of QCD at low and intermediate energies, where QCD is at strong coupling, reproducing features like chiral symmetry breaking, vector meson dominance, including whole families of vector and axial vector resonances of increasing masses.

However, there is not yet a model prevailing on the others: any of them has some problematic issue, eventually not shared by the others. This state of affairs led us to consider, in the evaluation of HLBL contribution to $(g-2)_\mu$, the idea of playing in the "Theory Space" of a set of HQCD models, in order to extract generic predictions. The result for the pion exchange contribution to the hadronic ligh-by-light that we obtained is compatible with those obtained in other models.

Our result depends on an additional UV constraint containing the quark magnetic susceptibility χ_0. We have reviewed the attempts that have been made in the literature to get predictions for this quantity, by extending some Hard-Wall models through the introduction of tensor fields dual to QCD tensor operators. These models, however, have not led to an improvement of predictive power. The results for χ_0 can be considered at most preliminary.

We still have to understand how far we can go trying to strike the right balance between the simplicity of the HQCD models and their predictivity.

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