EPJ Web of Conferences **118**, 01020 (2016) DOI: 10.1051/epjconf/201611801020
FCCP2015

Theory review of the muon $g-2$

Kirill Melnikov[1,a]

[1] *Institute for Theoretical Particle Physics, Karlsruhe Institute of Technology, Karlsruhe, Germany*

Abstract. I discuss the current situation with the muon anomalous magnetic moment. I argue that a mistake in the theoretical predictions is a very unlikely explanation of the current discrepancy between the Standard Model value of the muon magnetic anomaly and its measured value.

1 Introduction

I was charged by the organizers of the conference with reviewing the theory of the muon anomalous magnetic moment. I will begin by summarizing the current situation with the muon $g-2$ and then discuss the various contributions to the muon magnetic anomaly starting with the QED one and then moving on to the hadronic vacuum polarization and the hadronic light-by-light scattering contributions. I will not talk in any detail about the electroweak corrections to the muon magnetic anomaly since they are well understood by now.

The enormous interest in the muon magnetic anomaly in the particle physics community originates from the fact that the latest and most precise measurement of this quantity by an experiment E821 performed in Brookhaven National Laboratory left us with a persistent difference between the result of the measurement [1] and the theoretical prediction in the Standard Model [2]. The difference

$$\Delta a_\mu = a_\mu^{\rm exp} - a_\mu^{\rm th} = (261 \pm 80) \times 10^{-11}, \qquad (1)$$

amounts to 3.3 standard deviations. The uncertainty of the measured value of $g-2$ is 63×10^{-11}; it is comparable to the uncertainty of the theoretical result, 50×10^{-11}.

The new experiment at the Fermi National Laboratory that will start operating in 2017 aims at reducing the experimental uncertainty by a factor four; if these plans succeed, the experimental error will be brought down to 16×10^{-11} [3]. If we assume no changes in central values of all the contributions and no changes in the theoretical uncertainty, the final discrepancy will be more than five standard deviations. By the standards of high-energy physics, this will qualify as a discovery of physics beyond the Standard Model. Hence, we see that the stakes are quite high. We have, on one hand, a clean electroweak observable that shows a surprising discrepancy between expected and measured values and, on the other hand, a forthcoming experiment that has a potential to increase the significance of the discrepancy further, to the point of a discovery. These prospects make the physics of muon $g-2$ quite an exciting and intriguing field for the next few years. At this point in time, the best thing that the $g-2$ theorists can do is to keep scrutinizing the theory predictions to ensure that theoretical results are sound and robust, at least at the level of the accepted uncertainties. On the other hand, it is important to realize that the current discrepancy shown in Eq.(1) is simply too large to be explained by a blunder on the theory side. This still leaves us with quite a few options to explain Δa_μ in Eq.(1); we will describe them below.

We begin with the discussion of the magnitude of the different contributions to $a_\mu^{\rm th}$ and their uncertainties. For definiteness, we will use results as compiled in Ref.[2]; they are shown in Table 2. The QED contribution is by far the largest one; it is known extremely precisely. For a proper perspective, it is useful to note that the uncertainty of the QED contribution is just $10^{-4} \times \Delta a_\mu$, i.e. completely irrelevant at the current level of discrepancy. The electroweak contribution is small [4], see Table 2. It is comparable to the current discrepancy between theory and experiment and its uncertainty is a factor of ten smaller than the expected precision of the FNAL experiment. The situation with hadronic contributions to the muon magnetic anomaly is very different. The so-called leading order hadronic vacuum polarization contribution is large and its uncertainty is sizable. The NLO hadronic vacuum polarization is rather small. Since its calculation uses the same non-perturbative input as the leading order hadronic vacuum polarization, we believe that this error estimate is justified and, for this reason, it plays no role in the current $g-2$ discussion. The last contribution is the so-called hadronic light-by-light scattering. It is relatively small but for a number of reasons that we discuss below it is considered to be quite uncertain.

If we put everything together, we arrive at the theoretical prediction for the muon $g-2$ in the Standard Model

$$a_\mu^{\rm th} = 116591830(50) \times 10^{-11}. \qquad (2)$$

The largest contributors to the uncertainty are the leading order hadronic vacuum polarization contribution and

[a] e-mail: kirill.melnikov@kit.edu

the hadronic light-by-light. The uncertainties of QED and electroweak corrections and the hadronic vacuum polarization at next-to-leading order play a minor role.

Before discussing physics of the muon magnetic anomaly in detail, I emphasize that if we choose the current discrepancy between theory and experiment as a measure of how well the various contributions to the muon $g-2$ need to be known, we have to conclude that there is basically not a single one that requires refinements. Moreover, we can estimate the magnitude of all the different contributions to the muon magnetic anomaly using relatively simple physical considerations and in this way assure ourselves that we do not do anything particularly wrong. Of course, independent checks of the hadronic contributions are very welcome but, in my opinion, it is highly improbable that they will give us a different version of the $g-2$ story.

Therefore, given all the stress-tests that the theory of the muon $g-2$ has been subject to and the magnitude of the discrepancy Δa_μ in Eq.(1), I believe that there are three possible explanations of what is going on. They are as follows:

- the experimental result is wrong;
- several ingredients that appear in the theory predictions are "wrong" at the $O(1\sigma)$ level; for some reason that is not fully understood, the shifts are correlated;
- the discrepancy is real and is explained by physics beyond the Standard Model.

Thanks to the new experiment at Fermilab, we will be able to check the first item whereas a careful re-analysis of the many Standard Model contributions, the continued measurements of the $e^+e^- \to$ hadron cross sections and the anticipated progress in lattice computations will most likely allow us to either confirm or disprove the second one. Hence, it is possible to imagine that during the next decade we will be able to find a resolution of the muon $g-2$ puzzle and understand the cause of the current discrepancy.

In what follows we will discuss the different contributions starting from the QED one in Section 2 and continuing with the hadronic vacuum polarization in Section 3 and hadronic light-by-light in Section 4. We conclude in Section 5.

2 QED

A glance at the first entry in Table 2 shows that the QED contribution to the muon magnetic anomaly is known with the precision that exceeds the needs of the current theory-experiment comparison. For this reason, it is tempting to say that the QED corrections to $g-2$ are well-established and require no further discussion. However, the QED corrections to the muon magnetic anomaly are large and one can easily imagine that a tiny inconsistency in their evaluation will have important consequences for the interpretation of the $g-2$ puzzle. Since the state of the art computations of the muon magnetic anomaly [5] are incomprehensible for most people, except for a tiny number of

QED	116584718.95(8)
Electroweak	154 ± 2
Hadronic vacuum polarization, LO	6949 ± 37 ± 21
Hadronic vacuum polarization, NLO	−98.4
Hadronic light-by-light	105 ± 26

Table 1. Contributions to the muon anomalous magnetic moment, in units of 10^{-11}. The results are taken from Ref. [2].

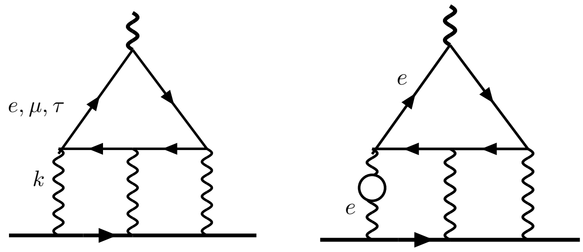

Figure 1. Enhanced three- and four-loop QED contirbutions to the muon $g-2$.

experts, it appears to be impossible to say how reliable these computations are without a completely independent confirmation of the final result. Unfortunately, such a confirmation has not yet fully happened although an impressive progress towards this goal occurred in recent years [6]. For this reason, we must take the results of the QED computations as they are currently reported and make our conclusions about the significance of the $g-2$ discrepancy based on this information. It is important to discuss why we believe that this is the sensible thing to do.

The answer to this question is actually quite simple: at the current level of discrepancy Δa_μ in Eq.(1) we do not need the results of the most advanced QED computations. In fact, all that is required are the three-loop QED corrections and the *leading* contributions at four-loops. Both the three-loop QED contributions to muon $g-2$ and the *leading* four-loop contributions are known since long ago and have been checked several times (see the discussion in Ref.[7] and references to the original papers therein). As we will see, these results *alone* allow us to predict the QED contribution to the muon magnetic anomaly with the precision that is better than 40×10^{-11} which is much smaller than Δa_μ.

The peculiarity of the QED contributions to the muon magnetic anomaly are the two enhancement parameters – the logarithm of the muon to electron mass ratio $\ln m_\mu/m_e \sim 5$ and $\pi^2 \sim 10$ that appear in certain cases. The logarithmic enhancement is a consequence of the renormalization-group running of either the QED coupling constant or effective operators that appear once diagrams that are sensitive to m_μ and m_e are written in such a way that contributions of disparate energy scales are separated. We note that it is somewhat unusual in perturbative computations to specifically address the enhancement by

π^2 factors[1] but in case of the muon magnetic anomaly there are cases when this enhancement can be sharply identified. This happens, for example, when π^2 and $\ln m_\mu/m_e$ appear in Feynman diagrams at the same time. Such situation occurs in the light-by-light scattering diagrams with the electron loop [8], see Fig. 1 The enhancement mechanism is very strong – at three loops, the light-by-light scattering diagrams with the electron loop provide 96% of the full result; all other diagrams give just about four percent.

At four loops, the situation is similar. We can not add another photon to connect the muon line and the electron loop since the resulting contribution vanishes due to Furry's theorem. Instead, we generate the logarithmic enhancement caused by the running of the coupling constant by inserting the electron vacuum polarization diagram into any of the photon propagators, see Fig. 1. This class of diagrams gives [9]

$$a_\mu^{(4),\mathrm{leading}} = 117.4 \left(\frac{\alpha}{\pi}\right)^4. \tag{3}$$

It is instructive to compare the approximate and the exact results

$$a_\mu^{(4),\mathrm{exact}} = (132.68 - 1.75) \left(\frac{\alpha}{\pi}\right)^4, \tag{4}$$

where the first term contains contributions from all diagrams enhanced by logarithms of m_μ/m_e ratio and the second represents contributions of all diagrams that are regular in the limit $m_e \to 0$. The difference between the full result and the leading contribution at four-loops translates into 40×10^{-11} shift at $g-2$; this number is much smaller than the current discrepancy in the muon magnetic anomaly. Moreover, other potentially enhanced contributions at the four-loop level were recently re-evaluated [6], confirming the results of the previous computations [10]. What remains unchecked at the four-loop level is the part of the computation that is regular in $m_e \to 0$ limit. We see from Eq.(4) that it only shifts the four-loop result by about 5×10^{-11} which is absolutely negligible at the current level of discrepancy. Finally, we note that the calculation of five-loop QED corrections to the magnetic anomaly was recently completed [5]. The result is consistent with a few earlier estimates [11] that utilized the possible enhancement mechanisms that we discussed earlier. The five-loop contribution shifts the muon anomalous magnetic moment by about 5×10^{-11} which is negligible compared to the current discrepancy.

Last year it was pointed out that the perturbative description of the QED corrections to the anomalous magnetic moment of an electron or a muon may be incomplete [12]. This discussion was motivated by an observation that contributions of QED bound states such as positronia, dimuonia etc. change the magnetic anomaly at the five-loop order, c.f. Fig. 2. However, since the bound states are non-perturbative in a sense that infinitely many Feynman diagrams need to be summed up to describe them, it seems hard to imagine a mechanism that makes these bound state contributions a part of conventional perturbative series. In fact such mechanism exists; it is known as "duality" in QCD. The point is that the magnetic anomaly is an Euclidean quantity that can be computed in QCD *after* the Wick rotation. If so, all the integrations over loop momenta can be performed over paths that are far from any of the physical singularities. For this reason, conventional perturbative expansion must be applicable. However, when one performs a computation *without* the Wick rotation and, in particular, uses dispersion representation for the photon vacuum polarization function, one has to integrate over sub-threshold positronia or dimuonia states. The non-perturbative $O(\alpha^5)$ correction to $g-2$ is then obtained. If one stops here, one concludes that there are contributions in QED that are not part of conventional perturbative series. However, such conclusion will be premature. Indeed, there is another non-perturbative contribution that appears in a form of a strong modification of the continuum scattering states *above* the e^+e^- or $\mu^+\mu^-$ production thresholds. These non-perturbative contributions in the continuum spectrum produce corrections to $g-2$ that *are equal and opposite in sign* to non-perturbative corrections produced by the bound states. When the sub- and above-threshold non-perturbative contributions are added, they cancel each other completely; the remaining part can be obtained within conventional perturbative expansion in QED [13, 14].[2] The reason for this is the Euclidean nature of the muon anomalous magnetic moment, as has been mentioned already.

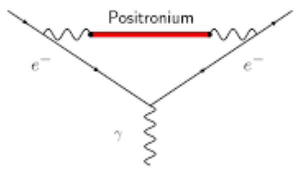

Figure 2. The contribution of a QED bound state to the anomalous magnetic moment of a lepton.

To summarize, I believe that QED corrections to the muon magnetic anomaly are understood so well that they can be discarded as a reason for the discrepancy in expected and measured values of the muon $g-2$. On the other hand, another measure of how well we need to control the QED corrections is provided by the expected precision of the new FNAL experiment, 15×10^{-11}. To match this precision, a re-calculation of the finite part of the four-loop muon $g-2$ probably becomes desirable. Given impressive recent developments in techniques for perturbative computations and their application to physics of the muon magnetic anomaly [6], there is no doubt that this task will be accomplished on the time scale of a few years.

[1] The usual argument against this is that π^2 may appear in many different ways in the final result and there are many known cases when large π^2 contributions are compensated by large rational contributions.

[2] Amusingly, this discussion was given in a more general context in full generality almost fifty years ago [15].

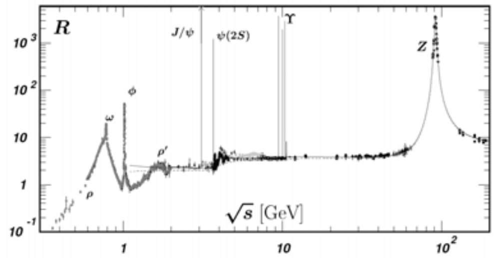

Figure 3. The spectral density $R(s)$ that appears in the calculation of the leading order hadronic vacuum polarization contribution to the muon magnetic anomaly.

3 Hadronic vacuum polarization contribution

The next largest correction to the muon anomalous magnetic moment is the hadronic vacuum polarization. The muon mass provides the energy scale for this contribution. This scale is smaller than the non-perturbative scale of strong interactions $\Lambda_{\rm QCD}$, so that perturbative QCD is not applicable and we have to resort to non-perturbative methods. The only viable option is to use the dispersion representation of the hadronic vacuum polarization which allows us to restore the full function from its imaginary part. Since the imaginary part of the hadronic contribution to photon vacuum polarization is proportional to $e^+e^- \to$ hadrons annihilation cross section, which is known thanks to many experimental studies, we can calculate the hadronic vacuum polarization non-perturbatively. Its contribution to the magnetic anomaly can be written as

$$a_\mu^{\rm hvp} = \frac{\alpha}{3\pi}\int\limits_{s_0}^{\infty}\frac{{\rm d}s}{s}R^{\rm hard}(s)\;a^{(1)}(s), \tag{5}$$

where $R(s) = \sigma_{e^+e^-\to{\rm hadr}}/\sigma_{e^+e^-\to\mu^+\mu^-}$ and $a_\mu^{(1)}$ is the one-loop anomalous magnetic moment that appears due to an exchange of a photon-like vector boson with the mass $\sqrt{s}$. Note that for the evaluation of the integral in Eq.(5) the low-energy region is of particular interest; this is so because the anomalous magnetic moment $a_\mu^{(1)}$ decreases for large vector boson masses $a_\mu^{(1)}(s) \sim m_\mu^2/s$ giving a stronger weight to the low-energy part of the integrand in Eq.(5).

Before I describe the results of the most recent computations, I would like to emphasize that it is possible to estimate $a_\mu^{\rm hvp}$ [7]. Such an estimate is interesting since, when contrasted with data-driven evaluations, it gives us an idea about how well such non-perturbative computations can be controlled. This is not very important for $a_\mu^{\rm hvp}$, where data is available at the first place, but it is crucial for the hadronic light-by-light scattering contribution $a_\mu^{\rm lbl}$, as we describe later.

The idea behind the estimate is quite simple. We try to model the essential features (see Fig.3) of the spectral density $R(s)$ at relatively low energies by accounting for three contributions:

- the chirally-enhanced two-pion threshold contribution $a_\mu^{\pi\pi}$ defined with an upper energy cut-off at $\sqrt{s} = m_\rho/2$;
- the contribution of ρ, ω, ϕ vector mesons $a_\mu^{\rho,\omega,\phi}$;
- the "continuum" contribution $a_\mu^{\rm cont}$ that starts above $\sqrt{s} \sim m_\phi$;

Numerically, we find $a_\mu^{\pi\pi} \sim 400\times10^{-11}$, $a_\mu^{\rho,\omega,\phi} = 5514\times10^{-11}$, $a_\mu^{\rm cont} = 1240\times10^{-11}$. Combining the three results we obtain the theoretical estimate of the leading order hadronic vacuum polarization contribution to the muon magnetic anomaly

$$a_\mu^{\rm hvp,th} = a_\mu^{\pi\pi} + a_\mu^{\rho,\omega,\phi} + a_\mu^{\rm cont} \approx 7160\times10^{-11}. \tag{6}$$

This theoretical estimate can be compared with one of the recent results of the data-driven evaluations [2]

$$a_\mu^{\rm hvp} = (6949 \pm 37.2 \pm 21.0)\times10^{-11}. \tag{7}$$

The proximity of the two results is obvious and gives us confidence that we understand the physics of the hadronic vacuum polarization quite well; of course, the uncertainty of the theoretical estimate is hard to access a'priori.

Let us discuss now to what extent the existent data-driven evaluations are satisfactory, given the physics goals of the current and forthcoming muon $g-2$ studies.[3] The important point to emphasize in this respect is that the hadronic vacuum polarization contribution to the magnetic anomaly is large and it needs to be know quite precisely. In fact, currently, it is the largest contributor to the uncertainty of the theoretical prediction of the muon magnetic anomaly. The two uncertainties in Eq.(7) have the following origin: the first one is related to the uncertainty of the experimental data and how they are combined; the second one reflects the poor understanding of how QED radiative corrections are applied to analyses of available data in exclusive hadronic channels. Both of these uncertainties are relatively small compared to the current discrepancy between theory and experiment but, taken together, they are not negligible. Moreover, these uncertainties become quite substantial when compared to the expected precision of the FNAL experiment, which suggests that further effort is required to improve the measurements of the $e^+e^- \to$ hadrons annihilation cross sections at low energies. It is re-assuring that the corresponding program of measurements exists both at BEPC and at Novosibirsk, so that substantial improvements in our understanding of $a_\mu^{\rm hvp}$ can be expected on a few years time scale [16, 17].

There are several important issues that are debated currently in the context of the calculation of the hadronic vacuum polarization. One is the compatibility of data sets obtained in measurements of $e^+e^- \to$ hadrons by CMD, SND, KLOE, BABAR and BESS III experiments. All the experiments measure the contributions of the kinematic region around the ρ-meson to $a_\mu^{\rm hvp}$ so that the results of different experiments can be compared directly. Such a comparison is shown in Fig. 4. In principle, the results of the different measurements are consistent but there are unwelcome systematic trends: KLOE results are smaller than the BABAR results and somewhat smaller than SND results.

[3]For a detailed discussion of this question, see Ref. [16].

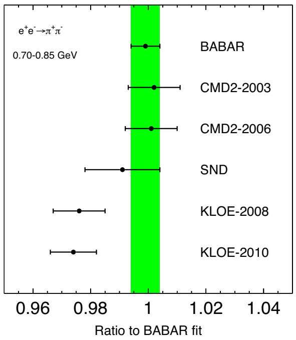

Figure 4. Comparison of recent evaluations of the ρ-meson contribution to $g-2$ that employ data from different experiments. The plot is taken from Ref. [18].

Figure 5. The so-called pion pole contribution to $a_\mu^{\rm hlbl}$ is the numerically dominant one.

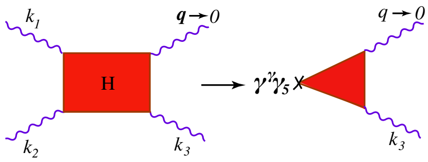

Figure 6. The OPE relation for the photon-photon scattering Green's function.

Also, the spectral densities $R(s)$ are somewhat tilted relative to each other in BABAR and KLOE measurements. The result of the very recent measurement by BESS III [19] is between the BABAR and KLOE results, but definitely closer to KLOE.

The other important topic that is being currently discussed is the question about the usefulness of τ data for the evaluation of the magnetic anomaly. The central issue is whether or not it is possible to control the isospin violating effects at the precision level that is required for the hadronic vacuum polarization to the muon magnetic anomaly. There were several talks about this issue during the Workshop; they seem to give an affirmative answer to this question [20, 21]. My personal opinion about this is that it is difficult to control the isospin violating effect at the level that is required to have meaningful description of the muon magnetic anomaly which implies that the use of τ data has the potential to create a lot of confusion. One thing, however, is clear. The recent work on the compatibility of the e^+e^- and the τ data seems to imply that, by and large, the data sets are quite compatible with each other [20, 21]; the situation therefore, is different compared to what was happening twelve years ago when the τ and e^+e^- data gave markedly different predictions for the muon $g-2$. Interestingly, one can check the compatibility of the τ data with the results of the $R(s)$ measurements by CMD, KLOE and BABAR; when this is done [21], the BABAR measurement appears to be disfavored.

4 Hadronic light-by-light scattering contribution

We will now discuss the current situation with the hadronic light-by-light scattering contribution to the muon anomalous magnetic moment. It is estimated to be $a_\mu^{\rm hlbl} = (105 \pm 26) \times 10^{-11}$ [22]. Clearly, $a_\mu^{\rm hlbl}$ is not large and its uncertainty, when compared to the current difference between theory and experiment, can be tolerated. However, given the persistent nature of the $g-2$ discrepancy, the volatile history of theoretical calculations of the hadronic light-by-light scattering contribution[4] and the fact that the calculation of $a_\mu^{\rm hlbl}$ only vaguely relies on the experimental data, there seems to be an uneasy feeling towards it. As the result, both the central value and the uncertainty estimate are being frequently questioned.

I begin by summarizing a few *solid facts* that we know about hadronic light-by-light scattering contribution. First of all, this contribution is non-perturbative since the energy scale set by the muon mass is small. Unfortunately, the use of robust non-perturbative methods such as e.g. the dispersion relations with experimentally-measured spectral densities, is very difficult in case of hadronic light-by-light.[5] As the consequence, most of the current computations of $a_\mu^{\rm hlbl}$ rely on models of low-energy hadron interactions. It is important to emphasize, however, that these models are constructed following *parametric* considerations. Indeed, to describe low-energy strong interactions physics, we can employ two parameters: 1) large number of colors N_c can be used to construct the $1/N_c$ expansion; 2) the apparent smallness of the pion mass m_π relative to the scale of strong interactions $\Lambda_{\rm QCD}$ allows us to treat a pion as a pseudo-Goldstone boson and uniquely predict its interactions with photons to leading order in the chiral expansion. The smallness of the pion mass suggests that loops of charged pions should provide the dominant contribution to the muon magnetic anomaly; the proximity of $N_c = 3$ to $N_c = \infty$ suggests that Green's functions relevant for the description of hadronic light-by-light scattering can be constructed as linear combination of contributions of non-interacting hadronic resonances. The existent studies of $a_\mu^{\rm hlbl}$ show that contributions that are enhanced by the large number of colors are (numerically) more important than the contributions that potentially exhibit the chiral enhancement.

Two statements can be made about hadronic light-by-light scattering contribution that are *exact* in the large-N_c

[4] The theory prediction for $a_\mu^{\rm hlbl}$ changed sign several times during its relatively short history.

[5] See Refs. [23–25] for the recent attempts to push forward with the data-driven evaluation of $a_\mu^{\rm hlbl}$.

approximation. First, we know that in the large-N_c approximation *and* in a situation where the mass gap between the pion and the ρ-meson is large, the hadronic light-by-light scattering contribution to the muon anomaly is given by the following formula [26]

$$a_\mu^{\rm lbl}[\pi^0] = \left(\frac{\alpha}{\pi}\right)^3 3\left(\frac{N_c}{3}\right)^2 \frac{m^2}{(4\pi F_\pi)^2}\ln^2\frac{m_\rho}{m_\pi} + ... \qquad (8)$$

The ellipses stand for terms that are enhanced by a single power of $\ln m_\rho/m_\pi$ and for terms that are regular in $m_\rho \to \infty$ limit. This contribution originates from the diagrams in Fig.5 where π_0 is exchanged between the photon pairs. In addition, there is a constraint on the behavior of the photon-photon scattering amplitude at large virtualities of the photons that follows from perturbative QCD [27]. It is based on the OPE relation for the time-ordered product of two vector currents

$$\begin{aligned} i\int {\rm d}^4x{\rm d}^4y e^{-iq_1x-iq_2y} T j_{\mu_1}(x) j_{\mu_2}(y) \\ = \int {\rm d}^4 z e^{-i(q_1+q_2)z}\frac{2i}{q^2}\epsilon_{\mu_1\mu_2\delta\rho}q^\delta j_5^\rho(z) + ... \qquad (9)\end{aligned}$$

Pictorially, this equation is illustrated in Fig. 6. The axial-vector current – that appears on the right-hand side of Eq.(10) – has a non-vanishing matrix element between the hadron vacuum and π_0; this allows to connect the short-distance computation of the photon-photon scattering amplitude where OPE is applicable and the long-distance description of the photon-photon scattering amplitude where π_0 exchange plays the primary role. The two constraints refer to opposite momenta scales, the very small and the very large; Eq.(10) shows that these constraints are connected and can be modeled by the π_0-exchange extrapolated to high-invariant masses in a consistent way [27].

Our best estimates of the large-N_c part of the hadronic light-by-light scattering contribution to the muon magnetic anomaly utilize models of the light-by-light scattering amplitudes based on pseudo-scalar (π_0, η, η') and pseudo-vector (a_1) exchanges , subject to two constraints mentioned above [27–31]. The current consensual result for the large-N_c part of the hadronic light-by-light scattering contribution to $g-2$ is [22]

$$a_\mu^{\rm hlbl, N_c} = (128 \pm 13)\times 10^{-11}. \qquad (10)$$

We can check our understanding of $a_\mu^{\rm hlbl,N_c}$ in a number of different ways. Since the hadronic light-by-light scattering contribution to $g-2$ is Euclidean, we never need to integrate over the resonance regions when evaluating $a_\mu^{\rm hlbl,N_c}$. It is then possible to think that there should be a duality between hadronic light-by-light scattering contribution to $g-2$ calculated using hadronic models and the constituent quarks. The only parameter that a constituent quark can have – its mass – can be estimated by requiring that the same theory works for the leading order hadronic vacuum polarization. This approach, pioneered in Refs.[32, 33], gives $a_\mu^{\rm hlbl} \approx 130\times 10^{-11}$.

One can check the stability of this result in a couple of ways. For example, one can include the “gluon” corrections to both light-by-light and hadronic vacuum polarization diagrams and check by how much the results for $a_\mu^{\rm hlbl}$ change if one changes the gluon-quark coupling [34]. Alternatively, one can combine the constituent quark loop with π_0 contribution to ensure that the chiral limit of the theory is correct [35]; in this case the quark loop contribution and the pion loop contribution must conspire to give the correct result for $a_\mu^{\rm hlbl}$. It turns out that whatever one does, the result comes out to be in in the range $a_\mu^{\rm hlbl,N_c} \approx (120-150)\times 10^{-11}$ which is perfectly consistent with the results of more detailed computations based on low-energy hadron models shown in Eq.(10).

Although further improvements in understanding the large-N_c part of the hadronic light-by-light scattering contributions are clearly warranted, the major current issue in the theory of $a_\mu^{\rm hlbl}$ is whether the sub-leading in N_c contributions are under control. The largest effect comes from loops of charged pions, which is the leading contribution in the chiral expansion. One finds $a_\mu^{\pi^+\pi^-} = -40\times 10^{-11}$. However, it is interesting to point out that this particular contribution is not robust and that the pion loop changes strongly if pion-photon interactions are modified to account for contributions of vector mesons. Another illustration of this fact is that introducing such modifications in two different ways gives two results that differ by a significant amount. The reason for this behavior was analyzed in Ref. [27]. It was found in that reference that the pion loop is sensitive to larger values of the loop momenta than one naively expects. Moreover, these values of the loop momenta are sufficiently high to be sensitive to hadronic modifications of the interaction vertices and propagators. One can also learn from these results that first non-trivial power correction to the chiral limit is insufficient and one needs quite a number of such terms to obtain the reliable result.

A separate, but related issue is the question by how much pion polarizability modifies the pion loop contribution [38, 39]. The pion polarizability is a modification of the pion-photon interaction due to the following higher-dimensional operator in the low-energy Lagrangian $\mathcal{L}_{\rm pol} \sim F_\pi^{-2}\pi^+\pi^- F_{\mu\nu}F^{\mu\nu}$. This term is clearly not part of the leading chiral Lagrangian and it is also not generated when effects of heavy mesons are modeled by using vector meson dominance or similar models. It is argued in Refs. [38, 39] that effects of the pion polarizability are large and very uncertain; it is estimated in Refs. [38, 39] that the pion loop contribution, including the polarizability operator, is in between -10×10^{-11} and -70×10^{-11}.

It is important to stress that, as the consequence of these issues, the sub-leading in N_c contribution to muon $g-2$ is, currently, the leading source of the uncertainty in the final theoretical estimate of $a_\mu^{\rm hlbl}$. Given this situation, it is interesting to ask to what extent our understanding of the sub-leading in N_c contributions can be improved in the future and we have seen some positive signs of the possible improvements during the Workshop. Indeed, it was argued in [40] that out of the two standard references for the pion loop contributions – the Hidden Local Symmetry (HLS) result -4×10^{-11} of Ref. [36, 37] and the vector meson dominance (VMD) result -19×10^{-11} of Ref.[29], only the VMD calculation satisfies the short-distance con-

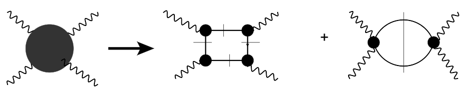

Figure 7. Decomposition of the chirally enhanced contributions to the photon-photon scattering amplitude in the dispersive approach of Refs.[23, 24].

straints on $\pi^+\pi^-\gamma\gamma$ vertex and leads, e.g. to the finite $\pi^+ - \pi^0$ mass difference. This fact would strongly favor the VMD result $a_\mu^{\pi^+\pi^-} = -19 \times 10^{-11}$ and would reduce the uncertainty since the previous uncertainty estimate was designed to cover both the VMD and the HLS model results. The analysis reported in Ref. [40] also seems to disfavor large effects of polarizability at least as long as one stays in the kinematic regime where the chiral expansion is well-behaved.

As we already mentioned, another interesting idea that was put forward recently is to use dispersion relations for the photon-photon scattering amplitude to make computations of hadronic light-by-light scattering less model-dependent [23–25]. For the subleading-N_c contributions, the decomposition of the photon-photon scattering amplitude shown in Fig. 7 is derived in Refs. [23, 24]. The first term on the right hand side, the four-pion cut, is simple and unique: it is chirally-enhanced pion box contribution modified by pion form factors on the external photon legs. The second term, the two-pion cut, is difficult; its analysis currently assumes the partial-wave expansion. The important point is that the four-pion cut is the same in VMD and HLS models and it is not affected by the pion polarizability issues. This means that all the "hot topics" in the current discussion of the hadronic light-by-light scattering contribution to $g-2$ reside in the two-pion cut contribution. This suggests that the dispersive approaches have a great chance to establish their credibility and usefulness by showing that the above issues (VMD vs. HLS and the polarizability) can be understood and clarified in a model-independent way. If anything, this is a simpler task than the complete model-independent analysis of hadronic light-by-light scattering contribution to the muon magnetic anomaly, but it is a very important one.

5 Conclusion

I would like to finish this discussion by re-iterating the following point. I believe that after more than ten years of searching for reasons for the discrepancy between the E821 result and the theoretical prediction for the muon magnetic anomaly, we can say with confidence that missed Standard Model effects as large as $\sim 260 \times 10^{-11}$ can be excluded as the reason for the discrepancy. The three logical explanation of the discrepancy are then 1) an experimental issue; 2) a coherent combination of small effects in theory and experiment that reduces the discrepancy to an "acceptable" level; 3) physics beyond the Standard Model.

The first item will be clarified by the new FNAL experiment. The second item will require some work on the theory side, new measurements of the $e^+e^- \to$ hadrons cross sections at Novosibirsk and Beijing and a better understanding of the hadronic light-by-light scattering contribution. The hope here is related to the new mode-independent approaches that are currently being developed, the improved measurements of photon transition form factors [41, 42] that appear to be possible at high-luminosity e^+e^- colliders and with continuous progress in applications of lattice QCD to hadronic light-by-light scattering that seems to be reaching the breakthrough moment [43].

The BSM contribution as an explanation of the discrepancy in muon $g-2$ is, arguably, the most exciting option. The likely candidate is still the supersymmetry with relatively small chargino and neutralino masses and relatively large value of $\tan\beta$. Here the interplay with direct measurements at the LHC is crucial but so far there is no contradiction between the LHC data and the masses of electroweakinos required to explain the muon magnetic anomaly [44].

Acknowledgements

I would like to thank the organizers for inviting me to this interesting Workshop and for creating nice and inspiring atmosphere.

References

[1] G. W. Bennett *et al.* [Muon g-2 Collaboration], Phys. Rev. D **73**, 072003 (2006).

[2] We take the results from K. Hagiwara, R. Liao, A.D. martin, D. Nomura and T. Teubner, J. Phys. **G38**, 085003 (2011).

[3] D. Hertzog, this conference, *Next Generation Muon $g-2$ Experiments*

[4] A. Czarnecki, B. Krause and W. J. Marciano, Phys. Rev. Lett. **76**, 3267 (1996); S. Peris, M. Perrottet and E. de Rafael, Phys. Lett. B **355**, 523 (1995); A. Czarnecki, B. Krause and W. Marciano, Phys. Rev. **D**52, 2619 (1995); A. Czarnecki, W. J. Marciano and A. Vainshtein, Phys. Rev. **D**67, 073006 (2003); Erratum-ibid. **D**73 (2006) 119901; C. Gnendiger, D. Stöckinger and H. Stöckinger-Kim, Phys. Rev. **D**88, 053005 (2013).

[5] T. Aoyama, M. Hayakawa, T. Kinoshita and M. Nio, Phys. Rev. Lett. **109**, 111808 (2012).

[6] A. Kurz, T. Liu, P. Marquard, A. V. Smirnov, V. A. Smirnov and M. Steinhauser, arXiv:1508.00901 [hep-ph].

[7] K. Melnikov and A. Vainshtein, "Theory of the Muon Anomalous Magnetic Moment", Springer Tracts in Modern Physics, 2006.

[8] A. Yelkhovsky, On the higher order QED contributions to the muon $g-2$ value, preprint INP-88-50, (1988); A. Milstein and A. Yelkhovsky, Phys. Lett. B **233**, 11 (1989).

[9] C. Chlouber and M. Samuel, Phys. Rev. D**16**, 3596 (1977).

[10] T. Aoyama, M. Hayakawa, T. Kinoshita and M. Nio, Phys. Rev. Lett. **109**, 11808 (2012).

[11] S. Karshenboim, Phys. Atom. Nucl. **56**, 857 (1993).
[12] G. Mishima, Bound State Effect on the Electron g-2, arXiv:1311.7109 [hep-ph].
[13] K. Melnikov, A. Vainshtein and M. Voloshin, Phys. Rev. D **90**, 017301 (2014).
[14] M. Fael and M. Passera, Phys. Rev. D **90**, no. 5, 056004 (2014).
[15] M. A. Braun, Zh. Eksp. Teor. Fiz. 54, 1220 (1968) [Sov. Phys. JETP 27, 652 (1968)].
[16] S. Eidelman, this conference, *Muon* $g-2$ *and hadronic vacuum polarization: recent developments.*
[17] T. Blum, A. Denig, I. Logashenko, E. de Rafael, B. Lee Roberts, T. Teubner and G. Venanzoni, arXiv:1311.2198 [hep-ph].
[18] M. Davier and B. Malaescu, Eur. Phys. J. C **73**, no. 11, 2597 (2013).
[19] B. Kloss, talk at CIPANP 2015, arXiv:1510.02293[hep-ph].
[20] F. Jegerlehner, this conference, *Leading order hadronic contribution to the electron and muon* $g-2$.
[21] M. Benayoun, this conference, *VMD-HLS approach to the muon* $g-2$*: a solution to the* $\tau - e^+e^-$ *puzzle.*
[22] J. Prades, E. de Rafael and A. Vainshtein, "Hadronic Light-by-Light Scattering Contribution to the Muon Anomalous Magnetic Moment," (Advanced series on directions in high energy physics. 20) [arXiv:0901.0306 [hep-ph]].
[23] G. Colangelo, M. Hoferichter, M. Procura and P. Stoffer, JHEP **1409**, 091 (2014).
[24] G. Colangelo, M. Hoferichter, B. Kubis, M. Procura and P. Stoffer, Phys. Lett. B **738**, 6 (2014).
[25] V. Pauk and M. Vanderhaeghen, Phys. Rev. D **90**, no. 11, 113012 (2014).
[26] M. Knecht, A. Nyffeler, M. Perrottet and E. de Rafael, Phys. Rev. Lett. **88**, 071802 (2002).
[27] K. Melnikov and A. Vainshtein, Phys. Rev. D **70**, 113006 (2004).
[28] J. Bijnens, E. Pallante and J. Prades, Phys. Rev. Lett. **75**, 1447 (1995) [(E) Phys. Rev. Lett. **75**, 3781 (1995)].
[29] J. Bijnens, E. Pallante and J. Prades, Nucl. Phys. B **474**, 379 (1996).
[30] J. Bijnens, E. Pallante and J. Prades, Nucl. Phys. B **626**, 410 (2002).
[31] M. Knecht and A. Nyffeler, Phys. Rev. D **65**, 073034 (2002).
[32] J. Erler and G. Toledo Sanchez, Phys. Rev. Lett. **97**, 161801 (2006).
[33] A. A. Pivovarov, Phys. Atom. Nucl. **66**, 902 (2003) [Yad. Fiz. **66**, 934 (2003)].
[34] R. Boughezal and K. Melnikov, Phys. Lett. B **704**, 193 (2014).
[35] D. Greynat and E. de Rafael, JHEP **1207**, 020 (2012).
[36] M. Hayakawa, T. Kinoshita and A. I. Sanda, Phys. Rev. Lett. **75**, 790 (1995).
[37] M. Hayakawa, T. Kinoshita and A. I. Sanda, Phys. Rev. D **54**, 3137 (1996).
[38] K. T. Engel, H. H. Patel and M. J. Ramsey-Musolf, Phys. Rev. D **86**, 037502 (2012).
[39] K. T. Engel and M. J. Ramsey-Musolf, Phys. Lett. B **738**, 123 (2014).
[40] J. Bijnens, this conference, *Hadronic light-by-light scattering contribution to* a_μ*: extended Nambu-Jona-Lasino, chrial quark models and chiral Lagrangians.*
[41] A. Nyffeler, this conference, *On the precision of a data-driven estimate of the pseudoscalar-pole contribution to hadronic light-by-light scattering in the muon g-2.*
[42] A. Kupsc, this conference, *Transition form factors of light mesons.*
[43] C. Lehner, this conference, *Hadronic light-by-light contribution to* $(g-2)_\mu$ *from lattice QCD.*
[44] D. Stöckinger, this conference, *Magnetic moment* $(g-2)_\mu$ *and Supersymmetry.*