Few-Body Problem in Nuclear Reactions

Beyond the horizon of the three-body Faddeev equations

Shinsho Oryu[1,a], Yasuhisa Hiratsuka[1,b], and Takashi Watanabe[1,c]

[1] *Department of Physics, Faculty of Science and Technology, Tokyo University of Science, 2641 Yamazaki, Noda-city, Chiba 278-8510, Japan*

Abstract. Recent progress in few-body problem physics based on the three-body Faddeev equations is reviewed for three-related fields. The first field involves the description of light nuclear reactions in terms of multi-channel three-body Faddeev equations. The second field is the investigation of the two- and three-body threshold behaviors for the NNπ system using the three-body Faddeev equations, where the πD and the NN′ or N-(Nπ) scattering lengths are calculated, and also we show that the NN′ potential has a long range term of $1/r^2$ form. The third is a new Coulomb treatment in terms of a generalized screening range to describe on-shell Coulomb amplitudes which is useful in the three-body Faddeev equations. This procedure reproduces both of the Coulomb phase shift and the wave function from the electron-electron to the heavy-ion-heavy-ion systems.

1 Introduction

1.1 Nuclear reactions

In the numerical approach to the light nuclear reactions, the exchange dynamics of a few-nucleons or the light nuclear-clusters are dominant for large momentum transfer and the details of the nuclear force frequently appear. The present approach is different from the method used for heavy nuclei. Especially, the molecular structure with several alpha-clusters is well developed in light nuclei [1]. Beside the alpha-particle, realistic nuclei are constructed by the proton, neutron, d, t, ^{3}He, ^{12}C, ^{16}O, ^{20}Ne, ^{24}Mg, ^{28}Si, ^{32}S, and so on.

Those structures can be seen by cooling the A-nucleons with the anti-symmetrized molecular dynamics (AMD)-method [2], or by the Jacobi-coordinate based AMD (JAMD) [3]. However, the simplest three-nucleon system has been extensively investigated by the three-body Faddeev equations [4].

The three-body Faddeev equations can theoretically represent all the spectra in the three-body problem, Figure 1. Therefore, one can say that the A-body Faddeev equations are the most powerful method for light nuclear reactions, although the increase in numerical cost and the advances in hard-ware are always put in the balance. At this stage, the three-cluster Faddeev equations can represent A-nucleon reaction systems by using the coupled-channel method with the three-cluster potentials which will be presented in the section 2.

[a] e-mail:oryu@rs.noda.tus.ac.jp
[b] e-mail:yasuhisa1059@gmail.com
[c] e-mail:watanabe@ph.noda.tus.ac.jp

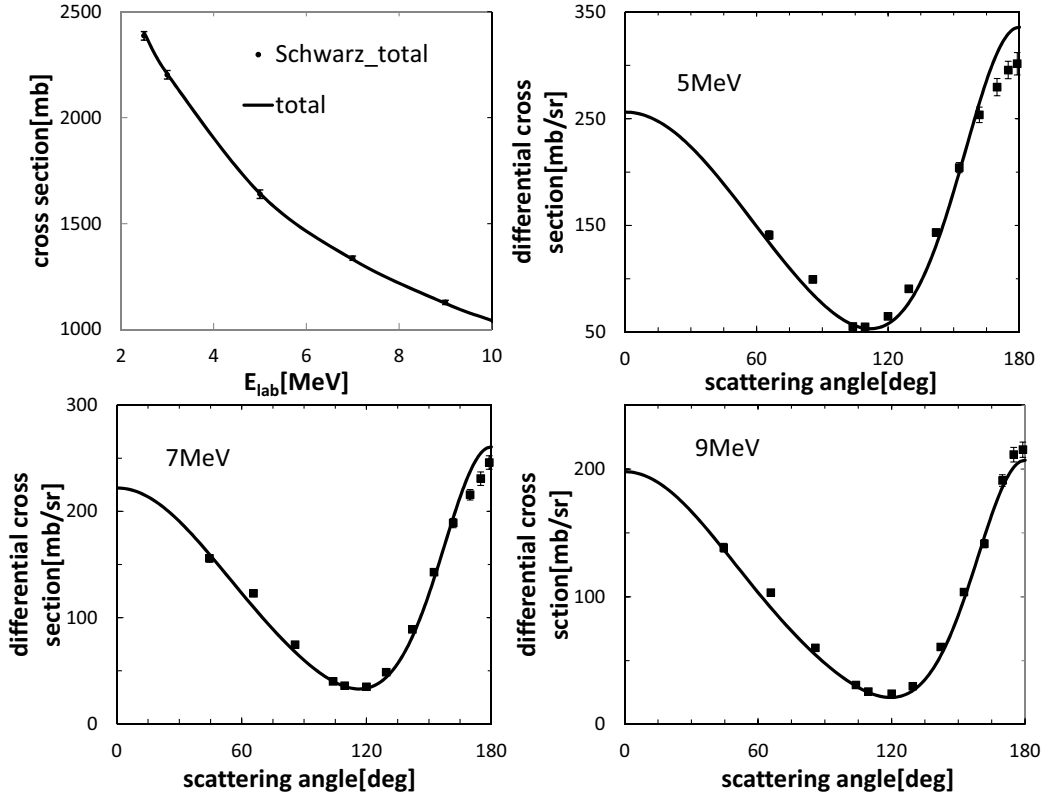

Figure 1. The *nd* total cross section and the differential cross sections for 5MeV, 7MeV and 9MeV. The calculations (solid-line) are done by Eq.(5) in our three-body program. The experimental data are taken from Ref.[5].

1.2 The threshold behaviors

In Lovelace's idea in the early 1960s, he pointed out that the three-body Faddeev equations become "a set of coupled Lippmann-Schwinger equations for the scattering of bound states and unstable particles. The potentials in these scattering equations for composite particles are nonlocal and energy-dependent". In other words, the Hamiltonian below the three-body threshold is different from that above the threshold. Therefore, the eigenvalues are also different. However, we require that the coupled Lippmann-Schwinger equations are analytically continued from the three-body Faddeev equations at the three-body break up threshold. This idea suggests that a composite two-body Hamiltonian is analogous to the three-body Hamiltonian at the threshold between the two-body and the three-body systems. We would like to show our recent progress in section 3.

1.3 The generalized screening range band in the Coulomb problem

There is a long history involving the Coulomb treatment in the few-body problems. Unfortunately, because we had no analytic solution of the three-body Faddeev equations, numerical calculations have been performed using a screened Coulomb potential. Such a screened Coulomb potential becomes a

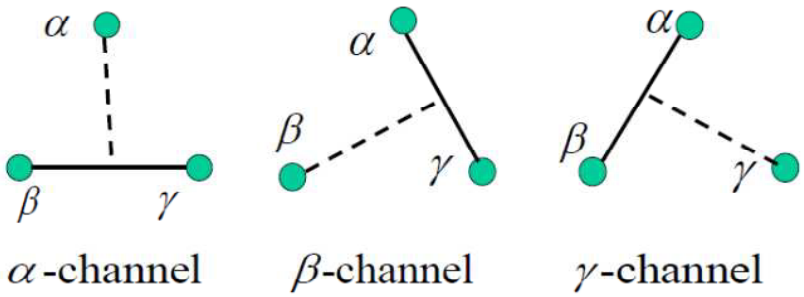

Figure 2. Three particle channels, and Jacobi coordinate.

genuine Coulomb potential with increasing a range, however, the phase shift and the wave function don't converge. It was claimed that in the proton-deuteron scattering calculation using the three-body Faddeev equation with a screened Coulomb potential, convergence of the procedure is found at moderate screening radii [16]; where the screening and a phase shift renormalization approach was employed for including the Coulomb interaction by an on-shell approximation. In this paper, we will show that the screening range is given by a unique band which is an energy-dependent function. Using this range, we obtain the two-body on- and off-shell Coulomb amplitudes for the three-body Faddeev equations in section 4.

2 Multi-channel Faddeev equations

2.1 Single-channel three-body Faddeev equations

Let us define the three *particle*-channels: the α-channel or $\alpha(\beta,\gamma)$-channel indicates (β,γ)-interacting-pair with a spectator α-particle, and the β-channel denotes $\beta(\gamma,\alpha)$, and the γ-channel is $\gamma(\alpha,\beta)$, respectively (see Figure 2). Then we have the three-body Faddeev equations,

$$T^{\alpha} = T_{\alpha} + \sum_{\gamma\neq\alpha} T_{\alpha} G_0 T^{\gamma}, \tag{1}$$

where T^{α} is the three-body T-matrix for the α-initial channel, and T_{α} denotes the two-body T-matrix in the three-body Hilbert space, and G_0 is the three-body free Green's function, respectively.

The equation can be reformulated for the three-body rearrangement amplitudes by putting

$$T^{\alpha} \equiv \sum_{\beta=1,2,3} T_{\alpha,\beta}, \tag{2}$$

$$U_{\alpha,\beta} \equiv T_{\alpha,\beta} - T_{\alpha}\delta_{\alpha\beta} = T_{\alpha} G_0 u_{\alpha,\beta} G_0 T_{\beta}, \tag{3}$$

$$X_{\alpha n,\beta m} \equiv \psi_{\alpha n} u_{\alpha,\beta} \psi_{\beta m} = g_{\alpha n} G_0 u_{\alpha,\beta} G_0 g_{\beta m}, \tag{4}$$

where $\psi_{\alpha n}$ and $\psi_{\beta m}$ are the two-body wave functions in the three-body Hilbert space, and $g_{\alpha n}$ and $g_{\beta m}$ are the corresponding two-body form factors. Therefore, $X_{\alpha n,\beta m}$ is the three-body rearrangement

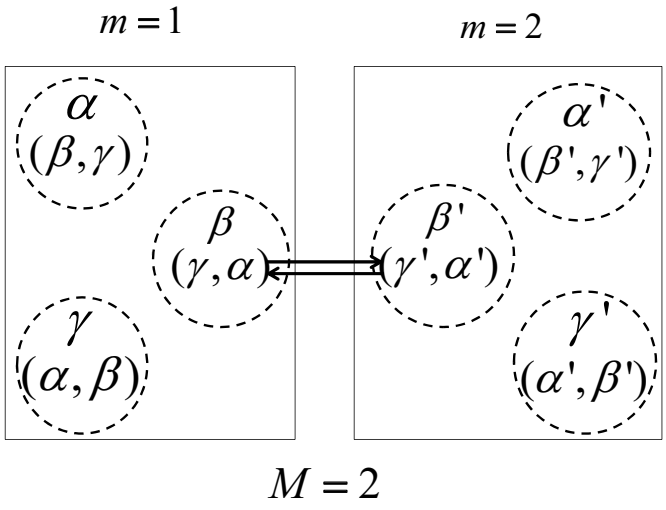

Figure 3. The mixture between two-different three-cluster systems: (α, β, γ) and $(\alpha', \beta', \gamma')$ is performed by the two-body inter-cluster amplitude: $(\gamma, \alpha) \Leftrightarrow (\gamma', \alpha')$ in the case $\beta = \beta'$. The figure shows the multiplicity number: $M = 2$.

amplitudes for the *physical*-states $n(\in \alpha)$, $m(\in \beta)$ which satisfies the following equations,

$$X_{\alpha n,\beta m} = Z_{\alpha n,\beta m} + \sum_{\gamma} \sum_{s,t} Z_{\alpha n,\gamma s} \tau^{\gamma}_{s,t} X_{\gamma t,\beta m}, \tag{5}$$

$$Z_{\alpha n,\beta m} = g_{\alpha n} G_0 g_{\beta m} (1 - \delta_{\alpha\beta}), \tag{6}$$

where $Z_{\alpha n,\beta m}$ gives the Born term of the so called AGS equations [6],[7]. The function $\tau^{\gamma}_{s,t}$ represents the two-body propagators.

2.2 Multi-channel three-body Faddeev [MC3F] equations

The three-cluster systems which consist of A-nucleons, are characterized by the *multiplicity*: M. Each three-cluster dynamics is described by the three-body Faddeev equations; however for $M \geq 2$, the equations are mixed via the two-cluster subsystem with a common spectator particle. Fig.3 illustrates the mixing process for the $M = 2$ case. Let us call the equations the "multi-channel three-body Faddeev" (MC3F) equations where only the two-body sub-amplitudes are input [8],[9]. The two-body sub-amplitudes are obtained by the two-cluster equation with the resonating group method (RGM) [10], which is constructed based on the NN potential where the spurious solutions are extracted by using the so called Orthogonality Condition Model(OCM) [11][12] or the Fish-bone optical potential method (FBOM)[13]. Therefore, our MC3F equations are given by [9],

$$X^{a,b}_{\alpha n,\beta m} = Z^{a,b}_{\alpha n,\beta m} + \sum_{c,d}^{M} \sum_{\gamma}^{3} \sum_{s,t}^{N} Z^{a,c}_{\alpha n,\gamma s} \tau^{c,d}_{\gamma,st} X^{d,b}_{\gamma' t,\beta m}, \tag{7}$$

$$Z^{a,b}_{\alpha n,\beta m} = g^{a}_{\alpha n} G_0 g^{b}_{\beta m} (1 - \delta_{\alpha\beta}) \delta_{ab}, \qquad (\alpha \in a,\ \beta \in b) \tag{8}$$

where the particle channels belong to the multiplicity numbers; i.e. $\alpha \in a$, $\beta \in b$, $\gamma \in c$ & d. Therefore, the equation becomes $3 \times N \times M$ multiple-coupled-integral equations. The dimension of the multiplicity is very large but the equations are manageable using modern computers.

Let us consider an example of Eq.(7) the scattering: ^{4}He+$p \rightarrow {}^4$He+p in the three-cluster systems for $M = 2$ case where the ^{3}He+n+p, and ^{3}H+p+p systems appear for a certain energy region. Since p ($\gamma = \gamma'$) is common in both systems, then the two-body subsystems (^{3}He+$n \leftrightarrow {}^3$H+p) can be mixed by $M = 2$, and then a, b, c, d=1, 2, and α-channel: ^{3}He(n, p), β-channel: $n(p,{}^3$He) and γ-channel: $p(^3$He,$n)$, while α'-channel: ^{3}H(p, p), β'-channel: $p(p,{}^3$H) and γ'-channel: $p(^3$H,$p)$. Therefore, the Born term is described only for the same multiplicity number, the mixture of the multiplicity numbers occurs by the two-body propagator: $\tau^{c,d}_{\gamma,st}$. Since, $\gamma \in c$, $\gamma' \in d$ and $\gamma = \gamma'$, then $\tau^{c,d}_{\gamma,st} \equiv \tau^{c,d}_{\gamma\gamma',st}$ [9]. In this method, the most important aspect is the investigation of the cluster-cluster interactions. If we obtain sufficient quality for the cluster-cluster amplitude, the MC3F equation is automatically solved. In alternative approach for such a system, the four-, and five-cluster Faddeev equations are available.

3 Two-body reduction of the three-body Faddeev equations

Let us consider the NNπ three-body system which is the simplest system related to the nuclear potential. The system was extensively studied for πd scattering by Afnan and Thomas [17]. Below the N$'$ $\rightarrow$Nπ break up threshold, the three-body NNπ system is reduced to the quasi two-body NN$'$ system which can be represented by the NN$'$ multi-channel Lippmann-Schwinger (MLS) equations, where N$'$ (or Nπ bound states) is described by the p_{11} state.

3.1 The two-body reduction in the three-body system

Following the Lovelace idea, the MLS equations are analytically continued from the three-body Faddeev equations at the three-body break up threshold $E^{(3)} \equiv E = 0$. The requirement is satisfied by adopting a "quasi two-body free Hamiltonian" $\overline{H}^{(3)}_0 \equiv H^{(3)}_0 + \epsilon_B$ instead of the "two-body free Hamiltonian" $H^{(2)}_0$. In other words, the zero limit of the binding energy: $\epsilon_B(> 0) \rightarrow 0$ shows that a quasi two-body set: $\{\overline{H}^{(3)}_0 \equiv H^{(3)}_0 + \epsilon_B, \overline{E} \equiv E + \epsilon_B\}$ becomes a three-body set: $\{H^{(3)}_0, E\}$, they are illustrated in Figure 4. In general, the analytic continuation is performed at every threshold between the A-body and $(A-1)$-body systems. Figure 4 shows the thresholds between the three-body and the two-body systems, and between the two-body and the one-body systems which are generally caused by decreasing the A-body system energy or cooling the A-body system's temperature down.

1) The analytic continuation between 3-body and 2-body is not given directly by the path "a": from the three-body free Hamiltonian and the system energy: $\{H^{(3)}_0, E^{(3)}\}$ to the two-body: $\{H^{(2)}_0, E^{(2)}\}$ with the two-body free Hamiltonian and the system energy, but the path "b" with the quasi two-body Hamiltonian and the system energy $\{\overline{H}^{(3)}_0, E^{(3)}_{\rm cm}\}$.

The Born term of the relativistic three-body Faddeev equations is given by N/D. The denominator is written in terms of the invariant energy S and the individual energies $\omega_j(q_j) \equiv \sqrt{q_j^2 + m_j^2}$ (with j=1, 2, 3),

$$
\begin{aligned}
D_{\rm Fadd} &\equiv \sqrt{S} - (\omega_1(q_1) + \omega_2(q_2) + \omega_3(q_3)) && (9)\\
&= (\sqrt{S} - m_1 - m_2 - m_3) - (\omega_1(q_1) + \omega_2(q_2) + \omega_3(q_3) - m_1 - m_2 - m_3) && (10)\\
&= E^{(3)} - H^{(3)}_0 && (11)\\
&= (E^{(3)} + \epsilon_B) - (H^{(3)}_0 + \epsilon_B) && (12)\\
&= \overline{E}^{(3)} - \overline{H}^{(3)}_0 \equiv E^{(3)}_{\rm cm} - \overline{H}^{(3)}_0 \equiv D_{\rm E2Q} && (13)
\end{aligned}
$$

where the $D_{\rm Fadd}$ function is defined for the original Faddeev equations, and $D_{\rm E2Q}$ is the denominator of the energy dependent two-body quasi (E2Q) potential. The nonrelativistic approximation for the $D_{\rm Fadd}$ function is given by using $\left(\sqrt{S} - m_1 - m_2 - m_3\right) + \epsilon_B \equiv E^{(3)} + \epsilon_B \equiv E + \epsilon_B \equiv E^{(3)}_{\rm cm}$, and from Eqs.(12), (13), we obtain

$$D_{\rm E2Q} \approx E^{(3)}_{\rm cm} - \Big(\frac{q_1^2}{2\mu_1} + \frac{p_1^2}{2\nu_1} + \epsilon_B\Big) \tag{14}$$

$$= E^{(3)}_{\rm cm} - \Big(\frac{\overline{q}_1^2}{2\mu_1} + \frac{\overline{p}_1^2}{2\nu_1}\Big) \tag{15}$$

with the reduced masses $\mu_1 = m_1(m_2 + m_3)/(m_1 + m_2 + m_3)$, $\nu_1 = m_2 m_3/(m_2 + m_3)$. If we define the relations between $\overline{q}_1$ and q_1 by,

$$\frac{\overline{q}_1^2}{2\mu_1} = \frac{q_1^2}{2\mu_1} + \epsilon_B, \tag{16}$$

and substitute Eq.(16) into Eq.(14), and compare with Eq.(15), we obtain

$$\frac{\overline{p}_1^2}{2\nu_1} = \frac{p_1^2}{2\nu_1} \quad \text{with} \quad \overline{\mathbf{p}}_1 = \frac{m_3\overline{\mathbf{q}}_2 - m_2\overline{\mathbf{q}}_3}{m_2 + m_3}, \quad \text{and} \quad \mathbf{p}_1 = \frac{m_3\mathbf{q}_2 - m_2\mathbf{q}_3}{m_2 + m_3}, \tag{17}$$

where the regions of the MLS equations and the Faddeev equations are defined by

$$0 \le \overline{q}_j < \infty, \quad \text{and} \quad 0 \le q_j < \infty. \tag{18}$$

The transformation from the Faddeev equations to the MLS equations is accomplished with the normalization path "c" by manipulating the Hamiltonian by taking $\overline{p}_1^2/2\nu_1 \to 0$ (or $\overline{\mathbf{p}}_1 \equiv m_3\overline{\mathbf{q}}_2 - m_2\overline{\mathbf{q}}_3 = 0$), and by putting $E^{(3)}_{\rm cm} \to E^{(2)}$ in Eq.(15). Therefore, we obtain

$$D_{\rm E2Q} = E^{(3)}_{\rm cm} - \Big(\frac{\overline{q}_1^2}{2\mu_1} + \frac{\overline{p}_1^2}{2\nu_1}\Big) \to E^{(2)} - \frac{\overline{q}_1^2}{2\mu_1} \equiv E^{(2)} - H_0^{(2)} \equiv D_{\rm MLS}. \tag{19}$$

2) The path a′: from the two-body free Hamiltonian and the system energy: $\{H_0^{(2)}, E^{(2)}\}$ to the one-body set: $\{H_0^{(1)}, E^{(1)}\}$ is replaced with the path b′ to ensure the analytic continuation to the set $\{\overline{H}_0^{(2)}, E^{(2)}_{\rm cm}\}$, where $H_0^{(1)}$ is the one-body kinetic operator for the center-of-mass motion, and finally $\{\overline{H}_0^{(2)}, E^{(2)}_{\rm cm}\}$ arrives at $\{H_0^{(1)}, E^{(1)}\}$ via the path c′. These paths are illustrated by,

$$D_{\rm MLS} = E^{(2)} - H_0^{(2)} = (E^{(2)} + \zeta) - (H_0^{(2)} + \zeta) \equiv \overline{E}^{(2)}_{\rm cm} - \overline{H}_0^{(2)} \to E^{(1)} - H_0^{(1)} \tag{20}$$

with $E^{(1)} = E^{(2)} + \zeta = E + \epsilon_B + \zeta$. Because the one-body Hamiltonian $H_0^{(1)}(\to H_0^{(1)})$ will vanish due to the zero relative energy and no "one-body potential", then the "eigenenergy" $E^{(1)}$ should be zero. Therefore, we obtain the three-body eigenenergy $E = -\epsilon_B - \zeta$, where ζ is the separation energy from the three-body bound state (see Figure 4).

This means that if the total wave function is written in separable form: $\Phi(q_j, p_j) = \psi_j(q_j)\phi_j(p_j)$ with the two-body wave function: $\phi_j(p_j)$, and the wave function between a particle j and a pair (ki): $\psi_j(q_j)$. However, such a separable form is not always correct. The Coulomb three-body problem, for example, cannot be written in such an expansion. Generally, the eigenequation with single argument is well defined. Hence the variables of the NNπ three-body Faddeev equations are $\{q, p\}$, the NNπ binding energy $E = -\epsilon_d - \epsilon_\pi$ will be obtained by our two-step method via the E2Q potential with single argument, where the integration region changes to $0 \le \overline{q} < \infty$ which starts from the NN′ branch point not from the NNπ threshold.

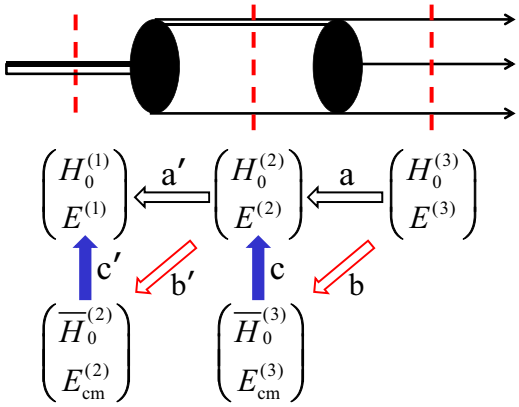

Figure 4. The threshold between the three-body system and the two-body system, and sequentially the threshold between the two-body system and the one-body system appear by cooling the temperature of the system. 1) The path "a": from the three-body free Hamiltonian and the system energy: $\{H_0^{(3)}, E^{(3)}\}$ to the two-body set: $\{H_0^{(2)}, E^{(2)}\}$ is replaced by the path "b" to ensure the analytic continuation with the quasi two-body Hamiltonian of the kinetic part and the system energy: $\{\overline{H}_0^{(3)}, E_{\rm cm}^{(3)}\}$. Finally, the usual procedure from the Faddeev equations to the coupled Lippmann-Schwinger equations is accomplished by the normalization path "c" by putting the two-body relative kinetic energy of the subsystem to zero. 2) The path a′: from the two-body free Hamiltonian and the system energy: $\{H_0^{(2)}, E^{(2)}\}$ to the one-body set: $\{H_0^{(1)}, E^{(1)}\}$ is replaced with the path b′ to ensure the analytic continuation with the set $\{\overline{H}_0^{(2)}, E_{\rm cm}^{(2)}\}$ and finally arrives at $\{H_0^{(1)}, E^{(1)}\}$ via the path c′ with zero kinetic energy.

3.2 The NNπ system below the three-body break up threshold

In the NNπ system, the denominator of the E2Q is given by $\omega_j(k_j) = \sqrt{k_j^2 + m_j^2}$ $(j = 1,\ 2,\ 3)$ with the pion mass $m_3 = m_\pi \equiv m$ and the nucleon mass $m_1 = m_2 \equiv M$, and the three-body total energy $\sqrt{S}$. Eqs.(11), (12) become

$$\begin{aligned} D_{\rm Fadd} &= E^{(3)} - H_0^{(3)} && (21)\\ &= (E^{(3)} + m) - (H_0^{(3)} + m) && (22)\\ &= \overline{E}^{(3)} - \overline{H}_0^{(3)} \equiv E_{\rm cm}^{(3)} - \overline{H}_0^{(3)} \equiv D_{\rm E2Q}. && (23) \end{aligned}$$

The non relativistic approximation of Eq.(23) gives for the process: $(N_2\pi)N_1 \to N_2(\pi N_1)$ [18],

$$D_{\rm E2Q} \approx (E + m) - \overline{q}_1^2/2m_1 - \overline{q}_2^2/2m_2 - (\overline{\mathbf{q}}_1 - \overline{\mathbf{q}}_2)^2/2m_3 \qquad (24)$$

with the energy relation $E^{(3)} \equiv E = \sqrt{S} - 2M - m$, the three-body free energy. The scaling of Eq.(16) is given by,

$$\frac{\overline{q}_{1,2}^2}{2\mu_{1,2}} = \frac{q_{1,2}^2}{2\mu_{1,2}} + m, \qquad (25)$$

Therefore, the non relativistic approximation of the Hamiltonian $H_0^{(3)}$ is given by,

$$\begin{aligned} H_0^{(3)} &\approx \sum_j \frac{\overline{q}_j^{\,2}}{2m_j} = \sum_j \frac{q_j^2}{2m_j} + m \\ &= \frac{\overline{q}_{1,2}^2}{\mu_{1,2}} + \frac{\overline{p}_{1,2}^2}{2\nu_{1,2}} = \frac{q_{1,2}^2}{2\mu_{1,2}} + \frac{p_{1,2}^2}{2\nu_{1,2}} + m = \frac{q_3^2}{2\mu_3} + \frac{p_3^2}{2\nu_3} + \alpha, \end{aligned} \tag{26}$$

Eq.(16) gives with the reduced masses for $\mu_{1,2}^{-1} = M^{-1} + (M+m)^{-1}$ and $\mu_3^{-1} = m^{-1} + (2M)^{-1}$

$$\frac{\overline{q}_{1,2}^2}{2m_{1,2}} = \frac{q_{1,2}^2}{2m_{1,2}} + \frac{\mu_{1,2}}{m_{1,2}} m = \frac{q_{1,2}^2}{2m_{1,2}} + \frac{(M+m)}{(2M+m)} m. \tag{27}$$

Using Eqs.(26) and (27), the total kinetic energy becomes

$$\sum_J \frac{\overline{q}_j^2}{2m_j} = \sum_J \frac{q_j^2}{2m_j} + \Big(\frac{\mu_1}{m_1} m + \frac{\mu_2}{m_2} m + \frac{\mu_3}{m_3}\alpha\Big) = \sum_J \frac{q_j^2}{2m_j} + m. \tag{28}$$

Therefore, we obtain α,

$$\alpha = \frac{-m_3}{(m_1+m_2)} m = -\frac{m^2}{2M}, \tag{29}$$

and also we have a relation for the pion momentum for Eq.(25),

$$\frac{\overline{q}_3^2}{2\mu_3} = \frac{q_3^2}{2\mu_3} - \frac{m^2}{2M}. \tag{30}$$

Finally, from Eq.(30) and the relation

$$\frac{\overline{q}_3^{\,2}}{2\mu_3} + \frac{\overline{p}_3^2}{2\nu_3} = \frac{q_3^2}{2\mu_3} + \frac{p_3^2}{2\nu_3} - \frac{m^2}{2M}, \tag{31}$$

we obtain

$$\frac{\overline{p}_3^2}{2\nu_3} = \frac{p_3^2}{2\nu_3}. \tag{32}$$

Eq.(25) shows that the nucleon momentum $\overline{q}_{1,2}$ is increased and the integral with respect to the $\overline{q}_{1,2}$ compensates for the two-body Nπ-information which is missed in the region $-2\mu_{1,2}m \le q_{1,2}^2 \le 0$ by the old treatment, while, Eq.(30) suggests that the pion momentum $\overline{q}_3$ is decreased to become the p_{11} bound state. Therefore, the integration region for $0 \le q_3^2 \le \mu_3 m^2/M$ in the old treatment is forbidden in utilizing the two-body NN-information. Now, let us call the new treatment the E2Q method [18].

Consequently, the MLS equations are described by choosing the same particle channels as the Faddeev equations: α=(Nπ)N, β=(πN)N, and γ=(NN)π, but omitting α, β

$$X_{n,m} = V_{n,m} + \sum_{s,t}^{3} V_{n,s} G_0^{st} X_{t,m}, \tag{33}$$

with $\tau_{s,t}^{\alpha} \equiv G_0^{st}$ and the effective potential is given by

$$V_{n,m} = Z_{n,m} + 2 \sum_{s,t} Z_{n,\gamma s} \tau_{s,t}^{\gamma} Z_{\gamma t,m}. \tag{34}$$

where the subscripts $n(\in \alpha)$, $m(\in \beta)$, $s,t(\in \gamma)$ (=1, 2, 3) in Eq.(5) denote the physical states with respect of the each particle channel. Therefore, Eq.(33) is the N+N$'$ $\rightarrow$ N+N$'$ scattering equations with the effective NN$'$ potential of Eq.(34).

3.3 Long range NN′ potential from the NNπ system

Recently, one of us (SO) found that the AGS Born term [7] or the potential of the multi-channel two-body LS equation assumes the well known Yukawa form below the Nπ break up threshold [18].

The leading term of the AGS Born term is given by $\chi = (\sigma^2 + q^2 + q'^2)/2qq'$ with $\sigma^2 = -2m_\pi(E + \epsilon_B)/\Lambda$, and $\Lambda = 1 + m_\pi/m_N \equiv 1 + \Delta$,

$$Z_{N\pi,N\pi}(-\vec{q},\vec{q}';E) = \frac{-2g_{N\pi}(\vec{p})m_\pi g_{N\pi}(\vec{p}')}{2qq'(\chi\Lambda - x)} \equiv \frac{C^\pi_{N\pi,N\pi}(\mathbf{q},\mathbf{q}')}{2qq'(\chi\Lambda - x)}, \tag{35}$$

where m_π, m_N, E and ϵ_B are the pion mass, the nucleon mass, the three-body energy, and the πN binding energy, respectively. Especially for our purpose, the $N\pi$-binding energy $\epsilon_B \equiv \epsilon_{N\pi}$. The form factor $C^\pi_{N\pi,N\pi}(\mathbf{q},\mathbf{q}') \equiv -2g_{N\pi}(\vec{p})m_\pi g_{N\pi}(\vec{p}')$ is defined for simplicity.
Here we expand the Green's function $\{2qq'(\chi\Lambda - x)\}^{-1}$ with respect to the mass factor $\Delta \equiv m_\pi/m_N$,

$$Z_{N\pi,N\pi}(-\mathbf{q},\mathbf{q}';E) = \frac{C^\pi_{N\pi,N\pi}(\mathbf{q},\mathbf{q}')}{2qq'}\sum_{j=0}^{\infty}\frac{(-\Delta)^j\chi^j}{(\chi - x)^{j+1}} = C^\pi_{N\pi,N\pi}(\mathbf{q},\mathbf{q}')\sum_{j=0}^{\infty}\frac{(-\Delta)^j(\sigma^2 + q^2 + q'^2)^j}{[\sigma^2 + (\mathbf{q} - \mathbf{q}')^2]^{j+1}}. \tag{36}$$

Putting $\mathbf{q}' - \mathbf{q} = 2\mathbf{K}$ and $\mathbf{q}' + \mathbf{q} = \mathbf{Q}$, and also $\sigma^2 + 2K^2 + Q^2/2 = [(\sigma^2 + 4K^2) + (\sigma^2 + Q^2)]/2$, gives, with $C^\pi_{N\pi,N\pi}(\mathbf{q},\mathbf{q}') \equiv C^\pi_{N\pi,N\pi}(\mathbf{K},\mathbf{Q})$,

$$\begin{aligned} Z_{N\pi,N\pi}(-\mathbf{q},\mathbf{q}';E) &= C^\pi_{N\pi,N\pi}(\mathbf{K},\mathbf{Q})\frac{1}{(\sigma^2 + 4K^2)}\sum_{j=0}^{\infty}(-\Delta/2)^j\Big[1 + \frac{\sigma^2 + Q^2}{\sigma^2 + 4K^2}\Big]^j \\ &= C^\pi_{N\pi,N\pi}(\mathbf{K},\mathbf{Q})\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}{}_{j+k}C_k\left(\frac{-\Delta}{2}\right)^{j+k}\frac{(\sigma^2 + Q^2)^k}{(\sigma^2 + 4K^2)^{k+1}}. \end{aligned} \tag{37}$$

Taking the sum with respect of j, it becomes

$$Z_{N\pi,N\pi}(-\mathbf{q},\mathbf{q}';E) = \frac{2C^\pi_{N\pi,N\pi}(\mathbf{K},\mathbf{Q})}{2 + \Delta}\sum_{k=0}^{\infty}\Big(\frac{-\Delta}{2 + \Delta}\Big)^k\frac{(\sigma^2 + Q^2)^k}{(\sigma^2 + 4K^2)^{k+1}}. \tag{38}$$

One can find that the last equation includes the Yukawa type potential for $k = 0$, and $\mathbf{Q} = \mathbf{0}$, i.e.,

$$Z_{N\pi,N\pi}(-\mathbf{q},\mathbf{q}';E) \approx \frac{2C^\pi_{N\pi,N\pi}(\mathbf{K},\mathbf{0})}{2 + \Delta}\frac{\sigma^2}{(\sigma^2 + 4K^2)}. \tag{39}$$

The form factor $C^\pi_{N\pi,N\pi}(\mathbf{q},\mathbf{q}')$ is a monotonic function; compared to the value of the Green's function at very low energy, one could make it a constant: $C^\pi_{N\pi,N\pi}(\mathbf{q},\mathbf{q}') \approx C_{N\pi,N\pi}$. Therefore, we obtain the Fourier transform of the AGS Born term as,

$$\mathcal{F}\{Z_{N\pi,N\pi}(-\mathbf{q},\mathbf{q}';E)\} \approx \frac{C_{N\pi,N\pi}}{8\pi(2 + \Delta)}\Big(\frac{e^{-\sigma r/2}}{r}\Big) \tag{40}$$

Let us examine the range σ,

$$\sigma^2 = -\frac{2m_\pi}{\Lambda}(E + \epsilon_B) = -2\frac{m_N m_\pi}{m_N + m_\pi}(E + \epsilon_B) > 0 \tag{41}$$

$$\Lambda = \frac{m_N + m_\pi}{m_N} \equiv 1 + \Delta, \tag{42}$$

Table 1. Deuteron binding energy sequence and corresponding rms radii for the $1/r^2$-type n-p triplet potential.

n	E_n [MeV]	E_n/E_{n+1}	$\sqrt{< r_n^2 >}$ [m]	$\sqrt{< r_{n+1}^2 >}/\sqrt{< r_n^2 >}$
1	2.226	171.0	1.97×10^{-15}	13.1
2	1.30×10^{-2}	171.0	2.58×10^{-14}	13.1
3	7.61×10^{-5}	171.0	3.37×10^{-13}	13.1
4	4.45×10^{-7}	171.0	4.40×10^{-12}	13.1
5	2.61×10^{-9}	171.0	5.76×10^{-11}	13.1
6	1.52×10^{-11}	171.0	7.53×10^{-10}	13.1
7	8.91×10^{-14}	171.0	9.85×10^{-9}	13.1

where $E + \epsilon_B \leq 0$ should be satisfied, and $E = -\epsilon_B$ generates the long range Coulomb-like potential. Here a special energy average gives the $1/r^2$ type potential [18].
Therefore, we can find a new phenomena at the two-body threshold in the three-body system which is only seen using the Faddeev approach. On the other hand, the three-body threshold satisfies $E = -\epsilon_B \to 0$ which leads to the Efimov effect [14], where $\epsilon_B = 0$ also suggests that the scattering length becomes $\pm\infty$. This means that our theory contains the Efimov effect automatically. Therefore, the theory involves not only the Efimov theory, but also a wider field near the different thresholds which has not been touched by the traditional three-body Faddeev treatment.
It is well known that the $1/r^2$ potential has the energy sequence $E_{n+1} = e^{-2\pi/\mu}E_n$ and the rms radii $r_{n+1} = e^{\pi/\mu}r_n$. If we fit the parameters $C_{N\pi,N\pi}$ and a_0 to the deuteron binding energy 2.226 MeV, then possible quantized binding energies are available only for $l = 0$, in which the quantum number for the modified Bessel function $K_{i\mu}(\kappa r)$ is $\mu = 1.2221$. Therefore, $e^{2\pi/\mu} = 170.98$ and $e^{\pi/\mu} = 13.076$ are obtained. As a consequence, we obtain the energy sequence and rms radii which are seen in the Table 1 [18].

Finally, the first excited state E_2 is around 13keV, however such an energy region has not been measured although the region with more than 50keV was measured. Here, we don't insist that the excited states of the deuteron exist, because the tensor force is not included in fitting the potential depth to obtain the deuteron binding energy. Another question is whether the energy average is unique, however, we confirmed that the calculation without the energy average has also a bound state.

3.4 NN′ and πD scattering lengths

The NN′ and πD scattering lengths are investigated in the NNπ system. Both are measured above the N-N′ and the π-D thresholds but below the NNπ three-body threshold. The N-N′ threshold energy is $E = -m_\pi$ in the original (or traditional) Faddeev calculation which has no singularity at that energy; however, in the E2Q theory $E_{\rm cm} = 0$ is the branch point of the scattering cut and singular. The π-D threshold energy is $E = -\epsilon_d$ of the deuteron binding energy which has no singularity in the kernel of the original Faddeev equations, $D_{\rm Fadd}$ of Eq.(21); however, in the E2Q, equation $E_{\rm cm} = 0$, which is the singularity in $D_{\rm E2Q}$ of Eq.(23). We show the results of the scattering lengths in Table 2. One can see that the discrepancy between the original Faddeev treatments and the E2Q method is significant, although the Nπ potential differences and the corresponding states are also sensitive [19][20].

Finally, we conclude that the three-body calculation below the break up threshold should be carried out using the E2Q method. One should note that the treatment *below* the break up threshold in the Faddeev method is completely different from that *above* the break up threshold, although three-

Table 2. The NN′ and πD scattering lengths are calculated using the original three-body Faddeev equation (Fadd-Org) and the E2Q Faddeev calculation (Fadd-E2Q). Type A is the result for the potential which is given by Thomas [17] and type B is the result for the potential by Fuda [21]. Fadd-E2Q is the present calculated result. Fadd-E2Q results show good agreement with the experimental data [22, 23].

Method	Scattering length [fm]	System/State
Fadd-Org (type A[17])	0.280	NN′ $^3S_1$
Fadd-Org (type B[21])	2.85	NN′ $^3S_1$
Fadd-E2Q (type B[21])	4.66	NN′ $^3S_1$
EXP [22]	5.424±0.004	NN′ $^3S_1$
Fadd-Org (type A[17])	0.033	πD
Fadd-Org (type B[21])	$-0.019 + 0.019i$	πD
Fadd-E2Q (type B[21])	$-0.023 + 0.019i$	πD
EXP [23]	$-0.038 + 0.009i$	πD

body systems, such as the three-nucleon system, with the negligibly small two-body binding energies compared to the pion mass, have been historically investigated without this kind of discussion.

4 Screened Coulomb Potential Method in Universal Variables

4.1 The Coulomb equivalent screening range

Recently we showed that the Lippmann-Schwinger (LS) equation with a screened Coulomb potential in momentum space cannot give the solution. If we assume that the actual solution in momentum space exists, then the on- and half-on-shell auxiliary amplitudes should be zero: $t_l^\phi(k,k;z) = t_l^\phi(p,k;z) = t_l^\phi(k,p';z) = 0$ which is satisfied for the auxiliary potential: $V_l^\phi(p,p') = V_l^C(p,p') - V_l^R(p,p')$ [24][25]. This is the *Lemma* which was introduced in ref. [24]. Therefore, the corresponding phase shift of the on-shell $t_l^\phi(k,k;z)$ suggests $\phi_l(k) \equiv \sigma_l(k) - \delta_l^R(k) = 0$.

In the following, we will define the screening range which can reproduce the Coulomb phase shift $\sigma_l(k) = \arg\Gamma(l+1+i\eta(k))$. Using this range, the auxiliary potential can satisfy the *Lemma*.

Using the two-potential theory, we obtain the off-shell Coulomb amplitude, i.e.,

$$t_l^C(p,p';z) = \int_0^\infty \int_0^\infty \overline{\omega}^\phi(p,p'';z) t_l^{R\phi}(p'',p''',z) \omega_l^\phi(p''',k) \frac{p''^2 p'''^2 dp dp'}{(2\pi^2)^2} + t_l^\phi(p,p';z) \quad (43)$$

with

$$t_l^{R\phi}(p,p';z) = V_l^R(p,p') + \int_0^\infty \int_0^\infty V_l^R(p,p'') G_l^\phi(p'',p''';z) t_l^{R\phi}(p''',p';z) \frac{p''^2 p'''^2 dp'' dp'''}{(2\pi^2)^2} \quad (44)$$

$$\overline{\omega}_l^\phi(p,p'';z) = \delta(p-p'') + t_l^\phi(p,p'';z) G_0(p'';z) \quad (45)$$

$$\omega_l^\phi(p''',p';z) = \delta(p'''-p') + G_0(p''';z) t_l^\phi(p''',p';z) \quad (46)$$

$$G_l^\phi(p'',p''';z) = \delta(p''-p''') G_0(p'';z) + G_0(p'';z) t_l^\phi(p'',p''';z) G_0(p''';z) \quad (47)$$

$$t_l^\phi(p,p';z) = V_l^\phi(p,p') + \frac{1}{2\pi^2} \int_0^\infty V_l^\phi(p,p'') G_0(p'';z) t_l^\phi(p'',p';z) p''^2 dp'' \quad (48)$$

where a nonsingular $\overline{V}_l^{\phi}$ potential is used instead of the original auxiliary potential V_l^{ϕ} which is equivalent to the pure $\overline{V}_l^{\phi}$ by the Fourier transformation, because $\overline{V}_l^{\phi}$ is well converged to the V_l^{ϕ} potential in the integral as an integrand that is seen in the form of Eq.(43). The modified auxiliary potential is defined by [25]

$$\overline{V}_l^{\phi}(p,p') \equiv \begin{cases} 0 & (p=p') \\ V_l^C(p,p') - V_l^R(p,p') & (p \neq p'). \end{cases} \tag{49}$$

Therefore, the two-potential theory with the Lemma will represent the fully off-shell Coulomb amplitude in momentum space.

4.2 Numerical result of the screening range

We search for the proper screening range parameter using the r-space method such as the Schrödinger equation and the phase shift differential equation with more than three digit accuracy [26], [27]. One may imagine that the Coulomb phase shift is obtained by increasing the distance of the screened Coulomb potential because the screened Coulomb potential converges to the Coulomb potential by increasing the distance. However, this idea is wrong, because the calculated phase shift, which is derived from the screened Coulomb potential not only with the Schrödinger equation but also with the LS equation, does not converge to the Coulomb phase shift by increasing the distance [16][25].

We found that five discrete screening range bands are necessary to reproduce the Coulomb phase shift for $\sigma_0 \leq 4\pi$.

1. The first (lowest) band (black circle in Figure 5) can represent the higher energy region better than the energy which satisfies $\sigma_0(k) = 0$, and the range should be $R = R_0(k) = 0$ fm at $\sigma_0(k) = 0$.
2. The second lower band (black nabla symbol) can be applied to the phase shift region: $0 \leq \sigma_0(k) \leq \pi$ and the range must be $R = R_1(k) = 0$ fm at $\sigma_0(k) = \pi$,
3. and the next band (black square symbol) gives the phase shift region: $\pi \leq \sigma_0(k) \leq 2\pi$, and the range $R = R_2(k) = 0$ fm at $\sigma_0(k) = 2\pi$,
4. and the next band (white circle symbol) applies to: $2\pi \leq \sigma_0(k) \leq 3\pi$, and the range $R = R_3(k) = 0$ fm at $\sigma_0(k) = 3\pi$, and
5. the highest band (black delta symbol) applies to: $3\pi \leq \sigma_0(k) \leq 4\pi$ and the range $R = R_4(k) = 0$ fm at $\sigma_0(k) = 4\pi$, respectively.

We find that $\sigma_0(k) = n\pi$ $(n = 0, 1, 2, \cdots)$ leads to $R_n(k) = 0$ which denotes $V^R(r) = 0$. However, the zero range screened Coulomb property seems to be unusual; then we adopt the finite range of the upper band instead (see Figure 5).

These bands are fitted by the N/D form

$$\mathcal{R} = \frac{a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 + \cdots}{b_0 + b_1\eta + b_2\eta^2 + b_3\eta^3 + \cdots} \tag{50}$$

where $\mathcal{R} \equiv kR$ is the so-called universal range, and $\eta(k)$ is the Sommerfeld parameter. The parameter a_0, a_1,$\cdots$ and b_0, b_1, $\cdots$ are shown in the Table 3. Therefore, the Coulomb phase shifts of the each system are obtained by taking the individual range $R(k) = \mathcal{R}(k)/k$ and incorporating the individual values of the Sommerfeld parameters.

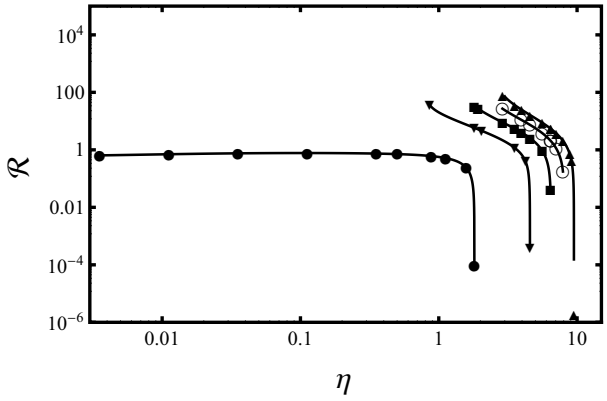

Figure 5. The universal ranges $\mathcal{R}(k)$ are denoted by five different symbols: black circle, black nabla, black square, white circle, and black delta. The universal ranges $\mathcal{R} = \mathcal{R}(k) = \mathcal{R}_n(k)$ $(n = 0, 1, 2, 3, 4)$ are fitted as functions of $\eta = \eta(k)$. 1) $\mathcal{R}_0 = 0$ at $\sigma_0(k) = 0$, 2) $\mathcal{R}_1 = 0$ at $\sigma_1(k) = \pi$, 3) $\mathcal{R}_2 = 0$ at $\sigma_2(k) = 2\pi$, 4) $\mathcal{R}_3 = 0$ at $\sigma_3(k) = 3\pi$, 5) $\mathcal{R}_4 = 0$ at $\sigma_0(k) = 4\pi$. All bands become zero at $\sigma_0(k) = n\pi$ $(n = 0,\ 1,\ 2,\ \cdots)$, however one could replace them by the finite range of the upper band. Each band makes it possible to derive the Coulomb phase shifts for the corresponding energy region.

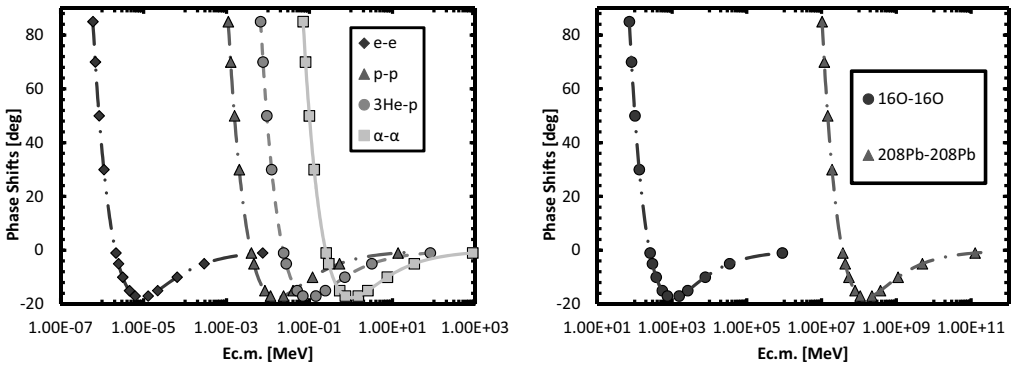

Figure 6. The Coulomb phase shifts in the S-wave are illustrated for several systems from e^--e^- to ^{208}Pb-^{208}Pb. These are calculated using the universal ranges which are given in Figure 5. The present screening range is only available for the Yukawa-type screened Coulomb potential which can reproduce the Coulomb phase shift not only for a wide energy range but also for any system.

Table 3. The parameters of $\mathcal{R} = \mathcal{R}(\eta)$. The parameter setting is defined by Eq. (50).

$\mathcal{R}_n$	k	0	1	2	3	4
$\mathcal{R}_0$	a_k	−0.122527	67.0542	33.4284	−33.6583	
$0.00353 \le \eta \le 1.58$	b_k	−0.14194	90.8834	15.8711	12.7285	
$\mathcal{R}_1$	a_k	5.57504	9.02951	12.9823	−3.20748	
$1.58 \le \eta \le 4.22$	b_k	−0.731152	−0.925015	1.87304	0.68854	
$\mathcal{R}_2$	a_k	1.04150	2.62713	5.58478	−0.64538	
$3.95 \le \eta \le 5.60$	b_k	2.11619	3.39619	3.91962	−2.57886	0.397671
$\mathcal{R}_3$	a_k	3.26102	12.2893	43.3765	−5.55534	
$5.60 \le \eta \le 7.90$	b_k	0.419669	−0.514822	−1.45874	0.924641	0.00556185
$\mathcal{R}_4$	a_k	1.1429	1.8394	3.89062	−0.245209	−0.0191509
$7.07 \le \eta \le 9.13$	b_k	2.12168	3.59564	−1.4752	0.186987	

In order to obtain the on-shell phase shift in the first step of this paper, a screened Coulomb potential is used instead of V^ϕ. Therefore, the same phase shifts and range bands are also obtained by the LS equation. However, the long range behavior should be treated by V^ϕ in the two potential theory from Eq.(43) to Eq.(48). The range parameters will be obtained only using the auxiliary potential. Therefore, the parameters in Table 3 are tentative.

5 Conclusion

We reviewed our recent works involving the Faddeev equations and related topics. First is the A-body nuclear system. If we imagine that the hardware is powerful enough to calculate the equations, what is the next problems? One of the most fundamental problems is whether the nuclear force is good or not. The second problem is the meson degrees of freedom. An effective many-body force can be written, but the overlaps with the nuclear force and the double counting problem should be carefully checked. Therefore, MC3F is the second best way to treat a realistic nuclear systems. In the analogy of the data analysis, MC3F is an interpolation method with some fitting of data for the inter-cluster amplitudes rather than an extrapolation method with a nuclear force from the A-body Faddeev equations. The inter-cluster amplitudes are given by the RGM, for example, in which the cluster-cluster interaction is constructed analytically from a simple nuclear potential and checked by comparison with the on-shell data. In other words, one can investigate the multi three-cluster system with the same accuracy for the three-nucleon system.

The MC3F method is an extension of the Faddeev equations to many-body systems. On the other hand, the reduction method for the three-body system to the two-body and the one-body systems is discussed in this paper. We found that the integral range is critically changed above and below the break up threshold. The momentum of residual particles after a pion absorption could be increased by releasing the binding energy. A careful investigation of the A-body Faddeev equations may bring new physics by crossing through A-number of thresholds. One example at the three-body break up threshold is the Efimov effect. Fortunately, the Efimov effect [14] was observed in the atomic system [15], although the effect has not been clearly observed in nuclear physics. We mentioned that several phenomena below the three-body break up threshold such as the scattering length in the two-body quasi system, and the bound state in the two-body quasi system as the three-body bound state, may be linked with the three-body force choice. Another interesting phenomenon near the quasi two-body break up threshold is that an energy average of the E2Q could produce the long range NN' interaction

where N′ stands for the pion absorbed nuclear state [18]. Furthermore, we showed that the NN′ and πD calculations by the E2Q method seem to give promising results for the scattering lengths, rather than the original three-body Faddeev calculations (see Table 2). Finally, it should be stressed that the new theory doesn't change the Faddeev theory, but reinterprets it.

Our Coulomb method is a completely new one. The results are useful for many applications.

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