EPJ Web of Conferences **125**, 01010 (2016) DOI: 10.1051/epjconf/201612501010
QUARKS-2016

Euler-Lagrange equations for high energy actions in QCD and in gravity ⋆

L. N. Lipatov[1,2,⋆⋆]

[1] *St. Petersburg State University*
[2] *Petersburg Nuclear Physics Institute*

Abstract. The high energy scattering in QCD and gravity can be described in terms of reggeized gluons and gravitons, respectively. At $N = 4$ SUSY the BFKL Pomeron is dual to the reggeized graviton living in the 10-dimensional anti-de-Sitter space. We discuss the corresponding effective actions for reggeized gluon and graviton interactions. The Euler-Lagrange equations for these effective theories are constructed with a variational approach and by using an invariance under the gauge and general coordinate transformations. We discuss their solutions and applications to the calculation of effective Reggeon vertices and trajectories.

1 Introduction

The hadron scattering amplitudes $A(s,t)$ at high energies $\sqrt{s}$ and fixed momentum transfers $q = \sqrt{-t}$ can be presented as sums of the amplitudes A^p with definite signatures $p = \pm 1$

$$A^p(s,t) = \int_{a-i\infty}^{a+i\infty} \frac{d\,j}{2\pi} \left((-s)^j + p\, s^j\right) f_j^p(t)\,. \tag{1}$$

The t-channel partial waves $f_j^p(t)$ have the poles with their positions depending on t

$$f_j^p(t) \sim \frac{1}{j - j(t)}\,. \tag{2}$$

They lead to the Regge behavior of the amplitudes

$$A^p(s,t) \sim (-s)^{j(t)} + p s^{j(t)}\,. \tag{3}$$

Pomeron is a special Regge pole with $p = 1$ and $j(0) \approx 1$, which is responsible for a slowly growing behavior of total cross sections σ_t at large energies and for the fulfilment of the Pomeranchuk theorem for the particle-particle and particle-anti-particle cross sections.

The exchange of two or more reggeons generates more complicated Mandelstam singularities of $f_j(t)$ in the j-plane [1]. To take into account all possible Pomeron contributions V.N. Gribov constructed the Reggeon calculus based on the 2+1 field theory of a complex scalar field [2].

⋆The investigation is supported by the St.Petersburg State University grant and RFBI grant 14-2200281.
⋆⋆e-mail: lipatov@thd.pnpi.spb.ru

In the Quantum Chromodynamics (QCD) the scattering amplitudes with the gluon quantum numbers in the t-channel have the Regge form in the so-called leading logarithmic approximation (LLA) [3]. Pomeron is a colorless composite state of the reggeized gluons. Its wave function in QCD and in other field models with the gauge group $SU(N_c)$ satisfies the BFKL equation [3]. It is remarkable, that the equations for the composite states of several reggeized gluons with the color singlet and octet quantum numbers are integrable at large N_c [4, 5]. Thus, it looks natural to reformulate QCD and other gauge models in terms of the effective degrees of freedom corresponding to the reggeized gluons. The gluon Regge trajectory and various reggeon couplings in upper orders of perturbation theory can be calculated from the effective action presented in Ref. [6, 7]. In the $N = 4$ SUSY the Pomeron is dual to the reggeized graviton in the anti-de-Sitter space due to AdS/CFT correspondence [8–10]. The effective action for the high energy scattering in gravity was derived in ref. [11]. Below we consider the Euler-Lagrange equations for the effective actions describing interactions of reggeized gluons and gravitons.

2 Effective action for high energy processes in QCD

The effective action in QCD is written for a cluster of usual quarks, gluons and reggeized gluons having their rapidities y in the interval η around its central value y_0.

Apart from the anti-hermitian matrix $N_c \times N_c$ for the gluon field v_μ we introduce also the fields $A_\pm = A_0 \pm A_3$ describing the production and annihilation of the reggeized gluons

$$v_\mu(x) = -iT^a v_\mu^a(x)\,,\ A_\pm(x) = -iT^a A_\pm^a(x)\,,\ [T^a, T^b] = i f^{abc} T_c\,. \tag{4}$$

The operators T^a are generators of the gauge group $SU(N_c)$ in the fundamental representation.

The fields $A^\mp = A_\pm$ are invariant under local gauge transformations of the gluon fields v_μ

$$\delta v_\mu = \frac{1}{g}\,[D_\mu, \chi(x)]\,,\ D_\mu = \partial_\mu + g v_\mu\,,\ \delta A_\pm = 0 \tag{5}$$

with the parameters $\chi(x)$ vanishing at $x \to \infty$. For the case of the global SU_n rotations with constant χ the Reggeon fields $A^\pm$ are transformed as usual gluon fields v_μ. They satisfy also the kinematical constraints

$$\partial_- A^- = \partial_+ A^+ = 0\,,\ \partial_\pm = \frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^3} \tag{6}$$

corresponding to the fact, that the Sudakov components α_i, β_i of the cluster momenta k_i are strongly ordered $\beta_i \gg \beta_{i+1},\ \alpha_i \ll \alpha_{i+1}$ in the multi-Regge kinematics.

The effective action for a cluster of real and virtual particles with their rapidities belonging to the small rapidity interval η has the form [6]

$$S_{eff} = \int d^4x \left(L_{QCD} + Tr\,\partial_\mu A^+ \partial_\mu A^-\right) + S_{ind}\,,\ S_{ind} = Tr \int d^4x\,(V_+ \partial_\mu^2 A^+ + V_- \partial_\mu^2 A^-)\,, \tag{7}$$

where L_{QCD} is the usual QCD lagrangian

$$L_{QCD} = i\bar{\psi}\hat{\partial}\psi + \frac{1}{2} Tr\, G_{\mu\nu}^2\,,\ G_{\mu\nu} = \frac{1}{g}\,[D_\mu, D_\nu]\,. \tag{8}$$

The anti-hermitian operators $V_\pm$ in the induced action S_{ind} are expressed in terms of eikonal amplitudes for a massless particle scattered off the external gluon field v. By chousing an appropriate normalization of the fields $A^\pm$ we can present $V_\pm$ in the form [6]

$$V_\pm = \frac{1}{g}\,\partial_\pm \frac{1}{D_\pm}\,\overleftarrow{\partial}_\pm \equiv -\frac{\partial_\pm}{g} + v_\pm - v_\pm \frac{g}{\partial_\pm} v_\pm + v_\pm \frac{g}{\partial_\pm} v_\pm \frac{g}{\partial_\pm} v_\pm + \ldots\,, \tag{9}$$

where by definition the derivative $\overleftarrow{\partial}$ acts on functions situated to the left from it. The contribution $-\partial_\pm/g$ is negligible in S_{int}, because the fields $A^\pm$ do not depend on $x^\pm$. It is natural to define the action of the integral operator $1/\partial_\pm$ in the symmetric form

$$\frac{1}{\partial_\pm}\, f(x^\pm) = \frac{1}{2}\left(\frac{1}{\partial_0} \mp \frac{1}{\partial_3}\right) f(x^\pm) = \frac{1}{4}\left(\int_{-\infty}^{x^\pm} dx'^\pm f(x'^\pm) - \int_{x^\pm}^{\infty} dx'^\pm f(x'^\pm)\right). \tag{10}$$

The invariance of the action under gauge transformations is a consequence of the relation

$$V'_\pm = \frac{1}{g}\,\partial_\pm\, e^{-\chi}\,\frac{1}{D_\pm}\,e^{\chi}\,\overleftarrow{\partial}_\pm \to V_\pm \tag{11}$$

valid after its integration over $x^\pm$ due to the vanishing of the gauge parameter $\chi(x)$ at $x \to \infty$.

The matrix $V_\pm$ can be written in one of two factorized forms

$$V_\pm = -\frac{\partial_\pm}{g}\, O(x^\pm) = O^+(x^\pm)\,\frac{\overleftarrow{\partial}_\pm}{g}\,, \tag{12}$$

where the operators $O(x^\pm)$ and $O^+(x^\pm)$ are defined below

$$O(x^\pm) \equiv -\frac{1}{D_\pm}\overleftarrow{\partial}_\pm\,,\;\; O^+(x^\pm) = \partial_\pm\,\frac{1}{D_\pm}. \tag{13}$$

The poles $1/D_\pm$ in $V_\pm$ appear from propagators of massless particles with other rapidities emitting gluons inside the given kinematical interval η. These particle virtualities $\sim k_\pm$ are large. Hence it looks natural to define the action of the operators $1/\partial_\pm$ as it was done above

$$\frac{1}{\partial_\pm}\, f(x^\pm) = \frac{1}{4}\int d\widetilde{x}^\pm\, \epsilon(x^\pm - \widetilde{x}^\pm)\, f(\widetilde{x}^\pm)\,, \tag{14}$$

but for all terms of the perturbative expansion of $1/D_\pm$. Such definition of $V_\pm$ is compatible with its anti-hermicity property and leads to the principal value prescription for corresponding poles $1/k_\pm$ in the momentum space.

Integrating S_{ind} over $x^\pm$ we can present it in terms of asymptotic values of functions $O_p(x^\pm)$

$$S_{ind} = -\frac{Tr}{g}\int d^2x_\perp\left(\int_{-\infty}^{\infty} dx^-\, O\,|_{x^+=-\infty}^{x^+=\infty}\,\partial_\sigma^2 A^+ + \int_{-\infty}^{\infty} dx^+ O\,|_{x^-=-\infty}^{x^-=\infty}\partial_\sigma^2 A^-\right). \tag{15}$$

The induced action is real due to the anti-hermicity relation for $O\,|_{x^\pm=-\infty}^{x^\pm}$. It can be written also in the four-dimensional form

$$S_{ind} = Tr\int d^4x\left(v_+(x)\, O(x^+)\,\partial_\sigma^2 A^+ + v_-(x)\, O(x^-)\,\partial_\sigma^2 A^-\right). \tag{16}$$

The product of the operators $O^+(x^\pm)$ and $O(x^\pm)$ does not depend on $x^\pm$, but generally $O(x^\pm)$ are not unitary operators. The absence of unitarity is related to our use of the principal value prescription $\theta(|k^\pm| - \epsilon)/k^\pm$ for particle propagators. The infrared cut-off ϵ can be chosen to be proportional to $e^{-\eta}$ where η is a low limit for the relative rapidity of neigbouring clusters described by the effective action.

We can use the representation of $1/D_\pm$ in terms of a modified P-exponents

$$\frac{1}{D_\pm} = P\,\frac{e^{-\frac{g}{4}\int_{-\infty}^{x_\pm} d\widetilde{x}^\pm v_\pm}}{e^{-\frac{g}{4}\int_{x_\pm}^{\infty} d\widetilde{x}^\pm v_\pm}}\,\frac{1}{\partial_\pm}\,\bar{P}\,\frac{e^{-\frac{g}{4}\int_{x_\pm}^{\infty} d\widetilde{x}^\pm v_\pm}}{e^{-\frac{g}{4}\int_{-\infty}^{x_\pm} d\widetilde{x}^\pm v_\pm}}\,. \tag{17}$$

It allows to write $O(x^\pm)$ in the form

$$O(x^\pm) = P\,\frac{e^{-\frac{g}{4}\int_{-\infty}^{x_\pm} d\widetilde{x^\mp} v_\pm}}{e^{-\frac{g}{4}\int_{x_\pm}^{\infty} d\widetilde{x^\mp} v_\pm}}\,\frac{1}{2}\left(Pe^{\frac{g}{4}\int_{-\infty}^{\infty} d\widetilde{x^\mp} v_\pm} + \bar{P}e^{-\frac{g}{4}\int_{-\infty}^{\infty} d\widetilde{x^\mp} v_\pm}\right). \tag{18}$$

The first factor here is an unitary matrix and second one is an hermitian operator. We obtain a simple representation for the difference of $O(x^\pm)$ at $x^\pm = \pm\infty$

$$O\,|_{x^\pm=-\infty}^{x^\pm=\infty} = \frac{1}{2}\left(Pe^{-\frac{g}{4}\int_{-\infty}^{\infty} d\widetilde{x^\mp} v_\pm} - \bar{P}e^{\frac{g}{4}\int_{-\infty}^{\infty} d\widetilde{x^\mp} v_\pm}\right)\left(Pe^{\frac{g}{4}\int_{-\infty}^{\infty} d\widetilde{x^\mp} v_\pm} + \bar{P}e^{-\frac{g}{4}\int_{-\infty}^{\infty} d\widetilde{x^\mp} v_\pm}\right), \tag{19}$$

which leads to an explicit expression for S_{ind} having the reality property.

Note, that effective vertices for the reggeized gluon with the indices ± 1, color index c and momentum q, annihilated to $r+1$ gluons with the polarization indices $\mu_0, \mu_1, ..., \mu_r$, color indices $a_0, a_1, ..., a_r$ and momenta $k_0, k_1, ..., k_r$, can be written in the form [6, 7]

$$\Delta^{\mu_0\mu_1...\mu_r\pm}_{a_0a_1...a_rc} = -q^2\prod_{i=0}^{r}\delta^{\mu_i}_{\pm}\Delta_{a_0a_1...a_rc}(k_0^\pm, k_1^\pm, ...k_r^\pm)\,,\;\; \Delta_{a_0c} = \delta_{a_0c}\,,\;\; \sum_{i=0}^{r} k_r^\pm = q^\pm = 0\,. \tag{20}$$

Here $\Delta_{a_0...a_r}$ has the Bose symmetry and satisfies the recurrent relation (the Ward identity) [6]

$$\Delta_{a_0a_1...a_r}(k_0^\pm, k_1^\pm, ...k_r^\pm) = \frac{1}{k_r^\pm}\sum_{t=0}^{r-1} i f_{aa_ta_r}\,\Delta_{a_0...a_{t-1}aa_{t+1}...a_r}(k_0^\pm, k_1^\pm, ...k_{t-1}^\pm, k_t^\pm + k_r^\pm, k_{t+1}^\pm, ...k_r^\pm)\,,$$

where f_{abc} is the structure constant of the gauge group. Thus, these vertices for the principal value prescription do not depend on the color representations of the matrices $v_\pm(x)$ in S_{ind}. In particular, one can use the adjoint representation for these matrices.

3 Classical equation for the effective action in QCD

The variation of the induced action S_{ind} over $v_\pm$ can be written as follows

$$\delta S_{ind} = -Tr\int d^4x\left(\partial_+\frac{1}{D_+}\,\delta v_+\,\frac{1}{D_+}\,\overleftarrow{\partial}_+\,\partial_\mu^2 A^+ + \partial_-\frac{1}{D_-}\,\delta v_-\,\frac{1}{D_-}\,\overleftarrow{\partial}_-\,\partial_\mu^2 A^-\right). \tag{21}$$

Using the relation

$$\partial_\pm\frac{1}{D_\pm}\,\delta v_\pm\,\frac{1}{D_\pm}\,\overleftarrow{\partial}_\pm = O^+(x^\pm)\,\delta v_\pm\,O(x^\pm)\,,$$

we obtain for the variation of S_{ind}

$$\delta S_{ind} = -Tr\int d^4x\left(\delta v_+\,O(x^+)\,\partial_\mu^2 A^+ O^+(x^+) + \delta v_-\,O(x^-)\,\partial_\mu^2 A^- O^+(x^-)\right). \tag{22}$$

Therefore the Euler-Lagrange equations for S_{eff} in a pure glue-dynamics have the form

$$[D_\mu, G^{\mu\nu}]^\perp = 0\,,\;\; [D_\mu, G^{\mu\pm}] = j^\pm_{ind}\,, \tag{23}$$

with the anti-hermitian currents

$$j^\pm_{ind} = O(x^\pm)(\partial_\sigma^2 A^\pm)O^+(x^\pm)\,. \tag{24}$$

They are conserved

$$[D_{\pm}, j^{\pm}_{ind}] = 0 \tag{25}$$

due to the relations $D_{\pm}\, O(x^{\pm}) = 0$.

In the quasi-elastic kinematics, where, for example, $A^{+} = 0$, after the gauge transformation, leading to the constraint $v'_{-} = 0$, one obtains

$$j'^{-}_{ind} = \lambda\, \partial^2_{\sigma} A^{-} \lambda^{+}\,, \;\; \lambda = \frac{1}{2}\left(P e^{\frac{g}{4}\int_{-\infty}^{\infty} d\widetilde{x^{-}} v'_{-}} + \bar{P} e^{-\frac{g}{4}\int_{-\infty}^{\infty} d\widetilde{x^{-}} v'_{-}}\right) = 1\,. \tag{26}$$

In this light cone gauge there is an explicit solution of the Euler-Lagrange equations

$$\widetilde{v^{\sigma}} = \delta^{\sigma}_{-} A^{-}. \tag{27}$$

Thus, the reggeon field A^{-} has the physical interpretation of a classical solution of the Euler-Lagrange equations for $A^{+} = 0$ in the light-cone gauge $v_{-} = 0$. This solution is a superposition of the shock waves proportional to $\delta(x^{\mp} - x_0^{\mp}) \ln |x^{\perp} - x_0^{\perp}|^2$.

Inserting the solution of the Euler-Lagrange equation for a general kinematics in the effective action, one obtains a generating function for the reggeized gluon vertices in the tree approximation [6, 7]. Note, that even for the quasi-elastic kinematics the Euler-Lagrange equations have other solutions corresponding to more complicated constraints at large times $t \to \pm\infty$. The asymptotic behavior of these contributions is fixed at $t = -\infty$ and $t = \infty$ in terms of two arbitrary functions $v_1(\vec{x})$ and $v_2(\vec{x})$, respectively. The effective action calculated on the general solution depending on $A^{\pm}$ and $v_{1,2}(\vec{x})$ gives a possibility to find a generating functional in a tree approximation for all possible scattering amplitudes and an arbitrary number of reggeized gluons [6]. Furthermore, by calculating the functional integral over the quantum fluctuations δv_{μ} around classical solutions we can find various effective vertices with loop corrections [12–14].

4 Effective action for the high energy gravity

According to J. Maldacena the $N = 4$ super-symmetric gauge theory is equivalent to the 10-dimensional super-string model living on the 10-dimensional anti-de-Sitter space [8–10]. As a result, here the BFKL Pomeron is dual to the reggeized graviton (see, for example, [15]) and the Gribov calculus for Pomerons should be generally covariant. Note, that the graviton Regge trajectory and its various couplings in a leading order were calculated many years ago [16].

The generally covariant effective action in gravity was constructed for a cluster of gravitons and reggeized gravitons having their rapidities in an interval around their central value [11]. Apart from the usual Einstein-Hilbert action and a kinetic term for the reggeon fields $A^{\pm\pm}$

$$S = -\frac{1}{2\kappa^2} \int d^4x \left(\sqrt{-g}\, R + \partial_{\sigma} A^{++} \partial_{\sigma} A^{--}\right) + S_{ind}\,, \tag{28}$$

it contains the induced term S_{ind}

$$S_{ind} = -\frac{1}{2\kappa^2} \int d^4x \left(\frac{j_{++}}{2}\, \partial^2_{\sigma} A^{++} + \frac{j_{--}}{2}\, \partial^2_{\sigma} A^{--}\right) \tag{29}$$

with the currents $j_{\pm\pm}$ being functionals of the metric tensor $g^{\mu\nu}$.

Fields $A^{\pm\pm}$ describing the production and annihilation of reggeized gravitons are invariant under the general coordinate transformations which are reduced to the Poincare group at large distances. They satisfy the kinematical constraints

$$\partial_{+} A^{++} = \partial_{-} A^{--} = 0\,, \tag{30}$$

corresponding to the strong ordering of the Sudakov components for momenta of produced clusters in the multi-Regge kinematics.

The gravity fields $h_{\mu\nu}$ are introduced as fluctuations of $g_{\mu\nu}$ around the Minkowsky metric tensor $\eta_{\mu\nu}$ having the diagonal structure $(1, -1, -1, -1)$. One can expand the Hilbert-Einstein lagrangian in h and obtain the solution of the effective Euler-Lagrange equation at small $A_{\pm\pm}$ in the form $\widetilde{h}_{\pm\pm} \approx A_{\pm\pm}$, providing that the currents $j_{\pm\pm}$ can be expanded in h as follows

$$j_{\pm\pm} = h_{\pm\pm} + O(h^2)\,. \tag{31}$$

The functional form of the current $j_{\pm\pm} = \partial_\pm j^\mp$ is fixed by the general covariance of the action. Due to this constraint the new current $j^\mp = 2x^\mp - \omega^\mp$ satisfies the Hamilton-Jacobi (HJ) equation [11]

$$g^{\rho\sigma}\,(\partial_\rho \omega^\mp)(\partial_\sigma \omega^\mp) = 0\,. \tag{32}$$

The function $\omega^\mp$ describes the light-front shock wave moving in the gravitational field.

If we search the solution of the HJ equation in the form $\omega^\mp = 2x^\mp - 2x^\mp(x^\pm, x^\perp)$, the quantities $x'^\mp = \omega^\mp/2$ can be considered as light-cone components of the coordinate transformation $x'(x)$ to the systems with the global light-cone times $x^\mp$. In these systems the light-cone component of $g^{\rho\sigma}$ is zero

$$g'^{\mp\mp} = g^{\mu\nu}\partial_\mu x'^\mp\,\partial_\nu x'^\mp = 0\,. \tag{33}$$

This HJ equation does not fix completely the global light-cone time systems. It is naturally to impose on $g'^{\rho\sigma}$ more restrictive constraints expressed in terms of the Minkowsky tensor

$$g'^{\rho\mp} = \eta^{\rho\mp}\,,\ \eta^{\mp\mp} = \eta_\perp^{\rho\mp} = 0\,,\ \eta^{\pm\mp} = 1\,. \tag{34}$$

They correspond to the global light-cone time inertial systems where we have in particular

$$g'^{\rho\mp}\,\frac{\partial}{\partial x'^\rho} = \frac{\partial}{\partial x'^\pm}\,. \tag{35}$$

Note, that $j^\mp$ in the effective action may be substituted by $-\omega^\mp$ because the term $2x^\mp$ gives a vanishing contribution. After its integration over $x^\pm$ the induced action can be expressed only in terms of the Hamilton-Jacobi functions $\omega^\mp$ at $x^\pm = \infty$ where we have the Minkowsky metric

$$S_{ind} = \frac{1}{2\kappa^2}\int d^2x^\perp \left(\int_{-\infty}^{\infty} \frac{dx^-}{4}\,\omega^-\,\partial^2_{\sigma\perp} A^{++}\,\Big|_{x^+=-\infty}^{x^+=\infty} + \int_{-\infty}^{\infty} \frac{dx^+}{4}\,\omega^+\,\partial^2_{\sigma\perp} A^{--}\Big|_{x^-=-\infty}^{x^-=\infty} \right). \tag{36}$$

5 Classical equations for the effective gravity

Variations $\delta g^{\mu\nu}$ and $\delta\omega^\mp$ along particle trajectories are not independent due to the HJ equation

$$\partial_\mu \omega^\mp\,\partial_\nu \omega^\mp\,\delta_\tau g^{\mu\nu} + 2\,g^{\rho\sigma}\,\partial_\rho \omega^\mp\,\delta_\tau \partial_\sigma \omega^\mp = 0 \tag{37}$$

and expressed through the corresponding infinitesimal shifts in the proper time τ

$$\delta_\tau g^{\mu\nu} = d\tau\,(\partial_\sigma g^{\mu\nu})\,\frac{d}{d\tau} x^\sigma\,,\ \ \delta_\tau \partial_\sigma \omega^\mp = d\tau \frac{d}{d\tau}\,\partial_\sigma \omega^\mp\,. \tag{38}$$

They can be calculated with the help of the Hamilton equations

$$\frac{d}{d\tau} x^\sigma = g^{\sigma\rho}\,\partial_\rho \omega^\mp\,,\ \ \frac{d}{d\tau}\,\partial_\sigma \omega^\mp = -\frac{1}{2}\,(\partial_\sigma g^{\mu\nu})\,\partial_\mu \omega^\mp \partial_\nu \omega^\mp\,. \tag{39}$$

Because $x'^{\mp} = \omega^{\mp}/2$ are light-cone components of $x'(x)$ in systems with $g'^{\mp\mp} = 0$ we have

$$2g^{\rho\sigma}\,\partial_\rho\omega^{\mp}\,\delta_\tau\partial_\sigma\omega^{\mp} = -2d\tau\,g'^{\chi\mp}\,\frac{\partial g^{\mu\nu}}{\partial x'^{\chi}}\,\partial_\mu\omega^{\mp}\partial_\nu\omega^{\mp} = -2d\tau\,\frac{\partial g^{\mu\nu}}{\partial x'^{\pm}}\,\partial_\mu\omega^{\mp}\partial_\nu\omega^{\mp} = \delta_\tau\frac{\partial\omega^{\mp}}{\partial x'^{\pm}}\,, \tag{40}$$

where the property $g'^{\chi\mp} = \eta^{\chi\mp}$ of the global light-cone time inertial systems was used. By integrating these equalities over the invariant phase space it is possible to verify the relations

$$\int d^4x\,\sqrt{-g}\,2g^{\rho\sigma}\,\partial_\rho\omega^{\mp}\,\delta_\tau(\partial_\sigma\omega^{\mp}) = \int d^4x'\,\sqrt{-g'}\,\delta_\tau\frac{\partial\omega^{\mp}}{\partial x'^{\pm}} = \int d^2x^{\perp}\int_{-\infty}^{\infty}\frac{dx^{\mp}}{4}\,\delta_\tau\omega^{\mp}\,|_{x^{\pm}=-\infty}^{x^{\pm}=\infty}\,,$$

where we took into account, that in the global light-cone inertial system g' does not depend on $x'^{\pm}$, which allows to integrate over this coordinate and neglect the factor $\sqrt{-g'}$ at $x'^{\pm} = \infty$.

The last expression enters in the variation of the induced action

$$\delta S_{ind} = \frac{1}{2\kappa^2}\int d^2x^{\perp}\left(\int_{-\infty}^{\infty}\frac{dx^-}{4}\delta\omega^-\,\partial^2_{\sigma\perp}A^{++}|_{x^+=-\infty}^{x^+=\infty} + \int_{-\infty}^{\infty}\frac{dx^+}{4}\delta\omega^+\,\partial^2_{\sigma\perp}A^{--}|_{x^-=-\infty}^{x^-=\infty}\right). \tag{41}$$

Therefore, taking into consideration also the HJ relation between $\delta_\tau g^{\mu\nu}$ and $\delta_\tau\omega^{\mp}$ on particle trajectories, one can present the variation of S_{ind} over $g^{\mu\nu}$ in a simple 4-dimensional form

$$\delta S_{ind} = -\frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\,\delta g^{\mu\nu}\left(\partial_\mu\omega^-\,\partial_\nu\omega^-\,\partial^2_{\sigma\perp}A^{++} + \partial_\mu\omega^+\,\partial_\nu\omega^+\partial^2_{\sigma\perp}\,A^{--}\right). \tag{42}$$

The integrand contains the factor $\sqrt{-g}$ as a result of our transformation from the special coordinate system with the global light-cone time $x'^{\mp}$ to a general system with coordinates x.

The above expression for δS_{ind} allows one to calculate easily the induced energy-momentum tensor $\theta_{\mu\nu}$ in the Euler-Lagrange equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\theta_{\mu\nu}\,,\;\; \theta_{\mu\nu} = \partial_\mu x'^-\partial_\nu x'^-\partial^2_\sigma A^{++} + \partial_\mu x'^+\partial_\nu x'^+\partial^2_\sigma A^{--}, \tag{43}$$

where $x'^{\pm} = \omega^{\pm}$ are coordinates in the corresponding light-cone global systems. This result can be found independently from considerations related to the general covariance of the Euler-Lagrange equations.

In a quasi-elastic kinematics, where $A^{++} = 0$, it is convenient to work in the inertial system with the metric tensor obeying the constraints $g'^{\rho+} = \eta^{\rho+}$ with the global light-cone time x'^+, because here the energy-momentum tensor is $T_{\mu\nu} \sim \delta^+_\mu\delta^+_\nu$. The classical equations in this system have the simple solution

$$g'^{\rho\sigma} = \eta^{\rho\sigma} + \delta^\rho_-\delta^\sigma_-A^{--}\,. \tag{44}$$

It is a superposition of the plane-wave solutions of Aichelburg and Sexl with the gravitation centers situated at $x^+ = z^+$, $x^{\perp} = z^{\perp}$ and distributed with the weight function $\partial^2_{\tau\perp}A^{--}(z^+, z^{\perp})$. The coordinate transformation $x' = x'(x)$ to this global light-cone time inertial system satisfies the equations

$$g^{\mu\nu}\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^+}{\partial x^\nu} = \eta^{\rho+} \tag{45}$$

and the tensors $\theta_{\mu\nu}$ and $T_{\rho\sigma}$ in these systems are related as follow

$$\theta_{\mu\nu}\,\frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma} = T_{\rho\sigma}\,. \tag{46}$$

Thus, the covariant energy-momentum tensor for the quasi-elastic kinematics is

$$\theta_{\mu\nu} = \partial_\mu x'^+ \partial_\nu x'^+ \partial^2_\sigma A^{--}\,, \tag{47}$$

where partial derivatives $\partial_\mu x'^+$ are found from the solution of the Hamilton-Jacobi equation

$$g^{\mu\nu}\,\partial_\mu x'^+ \partial_\nu x'^+ = 0\,. \tag{48}$$

The covariant energy-momentum tensor for a general kinematics is conserved

$$D_\mu\,\theta^{\mu\nu} = 0 \tag{49}$$

due to the kinematical constraints $\partial_\pm A^{\pm\pm} = 0$. As a consequence of the Hamilton-Jacobi equation for $x'^\pm$ the tensor $\theta_{\mu\nu}$ is traceless

$$g^{\mu\nu}\,\theta_{\mu\nu} = 0\,. \tag{50}$$

Thus, we constructed the generally covariant Euler-Lagrange equations for the effective action in the high energy gravity. In a quasi-elastic kinematics with $A^{++} = 0$ one of their solutions at an inertial coordinate system with the global light-cone time x'^+ has a simple form $g'^{--} = A^{--}$. There are also many solutions which are parameterized by their asymptotic behavior at $t \to -\infty$ and $t \to \infty$, respectively. The effective action calculated on these solutions allows to construct a generating functional for the effective multi-graviton scattering amplitudes in a tree approximation and to reproduce independently the known results [16]. The integration over fluctuations around the classical solutions gives a possibility to calculate loop corrections to reggeized graviton interactions and to effective graviton vertices. In particular, using the one-loop Regge trajectory obtained in Refs [11, 16], the graviton scattering amplitude in the double-logarithmic approximation was found [17]. A possible generalization of our approach to the super-gravity in the 10-dimensional AdS space will be interesting for the construction of the Gribov Pomeron calculus in N=4 SUSY.

References

[1] S. Mandelstam, *Nuovo Cim.* **30**, 1148 (1963).
[2] V. N. Gribov, *Sov. Phys. JETP* **26**, 414 (1968).
[3] L. N. Lipatov, *Sov. J. Nucl. Phys.* **23** 338 (1976);
V. S. Fadin, E. A. Kuraev, L. N. Lipatov, *Phys. Lett.* **B 60** 50 (1975);
E. A. Kuraev, L. N. Lipatov, V. S. Fadin, Sov. Phys. JETP **44** 443 (1976);
Ya. Ya. Balitsky, L. N. Lipatov, *Sov. J. Nucl. Phys.* **28** (1978) 822.
[4] L. N. Lipatov *High energy asymptotics of multi-colour QCD and exactly solvable lattice models*, hep-th/9311037, unpublished.
[5] L. N. Lipatov, *J. Phys.* **A 42**:304020 (2009).
[6] L. N. Lipatov, *Nucl. Phys.* **B 452**, 369 (1995); *Phys. Rept.* **286**, 131 (1997).
[7] E. N. Antonov, L. N. Lipatov, E. A. Kuraev, I. O. Cherednikov, *Nucl. Phys.* **B 721**, 111 (2005).
[8] J. M. Maldacena, *Adv. Theor. Math. Phys.* **2** 231 (1998).
[9] S. S. Gubser, I. R. Klebanov, A. M. Polyakov, *Phys. Lett.* **B 428** 105 (1998).
[10] E. Witten, *Adv. Theor. Math. Phys.* **2** 253 (1998).
[11] L. N. Lipatov, Phys. Part. Nucl. **44** 391 (2013), arXiv: 1105.31277 [hep-ph].
[12] V. S. Fadin, L. N. Lipatov, *Phys. Lett.* **B 429** 127 (1998);
M. Ciafaloni and G. Camici, *Phys. Lett.* **B 430** 349 (1998).

[13] A. V. Kotikov, L. N. Lipatov, *Nucl. Phys.* **B 582** 19 (2000).
[14] A. V. Kotikov, L. N. Lipatov, *Nucl. Phys.* **B 661** 19 (2003).
[15] A. V. Kotikov, L. N. Lipatov, Nucl. Phys. **B 874** 889 (2013).
[16] L. N. Lipatov, Phys. Lett. **116** (1982) 411.
[17] J. Bartels, L. N. Lipatov, A. Sabio Vera, J. High Energy Phys. **07**(2014) 056.