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How to construct a consistent and physically relevant the Fock space of neutrino flavor states?

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Abstract. We propose a modification of the electroweak theory, where the fermions with the same electroweak quantum numbers are combined in multiplets and are treated as different quantum states of a single particle. Thereby, in describing the electroweak interactions it is possible to use four fundamental fermions only. In this model, the mixing and oscillations of the particles arise as a direct consequence of the general principles of quantum field theory. The developed approach enables one to calculate the probabilities of the processes taking place in the detector at long distances from the particle source. Calculations of higher-order processes including the computation of the contributions due to radiative corrections can be performed in the framework of perturbation theory using the regular diagram technique.

1 Introduction

The Standard Model of electroweak interactions based on the non-Abelian gauge symmetry of the interactions [1], the generation of particle masses due to the spontaneous symmetry breaking mechanism [2], and the philosophy of mixing of particle generations [3] is universally recognized. Its predictions obtained in the framework of perturbation theory are in a very good agreement with the experimental data, and there is no serious reason, at least at the energies available at present, for its main propositions to be revised.

However, in describing such an important and firmly experimentally established phenomenon as neutrino oscillations [4], an essentially phenomenological theory based on the pioneer works by B. Pontecorvo [5] and Z. Maki et al. [6] is used. This theory is well developed and is consistent with the experimental data.

The primary assumption of this theory is that the neutrinos are massive, and moreover, there are three neutrino types with different masses. It is also postulated that the neutrinos produced in reactions are in the states which are superpositions of states with fixed masses, the so-called mass states. These states form the so-called flavor basis. The transformation to this basis from the mass basis is given by a unitary mixing matrix. Initially, the mass basis elements are described by plane waves with the same (three-dimensional) momentum. The time evolution of the flavor states is described by the solution of the corresponding Cauchy problem.

For this reason it is taken for granted that the mass and flavor states can be connected by a unitary transformation. However, this supposition is not correct. This fact is well known. For example, it is

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discussed in paper [7]. So, it is impossible to construct the Fock space for flavor states using the conventional approach. And, as a consequence, it is impossible to calculate the transition probabilities for flavor states in the framework of the perturbation theory.

The purpose of this work is to construct in a consistent and physically relevant manner the Fock space for the Standard Model fermions (leptons and quarks) in such a way that the mixing of the particles and their oscillations would occur automatically.

The basis of the theoretical scheme is as follows. In relativistic quantum field theory a particle is usually associated with an irreducible representation of the Poincaré group [8, 9]. The eigenvalue in this representation of the Casimir operator constructed from the translation operators squared is identified with the observed mass of the particle.

The existence of states which are superpositions of states with different masses contradicts the relativistic invariance of the theory, if the canonical momentum operator is identified with the translation operator. This obviously follows from the fact that the metric is defined on the hyperboloid in the momentum space determined by the value of the particle mass.

To overcome this difficulty it seems natural to associate some set of particles (a multiplet) with an irreducible representation of a wider group. A number of theorems [10] indicates that the only reasonable extension of the symmetry group of the theory is the direct product of the Poincaré group and a group of internal symmetry. However, in this point we encounter the important theorem [11][1].

It seems that due to this theorem the result of extending the symmetry group of the theory will be trivial: the masses of the components of the multiplets will be equal. Just this circumstance is the most important reason of the failure of constructing the Fock space for flavor states. However, it is possible to circumvent this obstacle.

2 Wave function spaces

It is well known (see, for example, [12]) that the derivation algebra of the Poincaré algebra contains not only the translation operators P^μ and generators of the Lorentz group $M^{\mu\nu}$, but also the generator of dilatation D:

$$[P^\mu, D] = P^\mu, \quad [M^{\mu\nu}, D] = 0. \tag{1}$$

So it is possible to construct an external automorphism of Poincaré algebra, which leads to a scaling transformation of the translation generators. For irreducible representations the dilatation changes the value of the Casimir operator which defines the mass of a particle. Hence, such a transformation allows to consider particles with different masses even in the case when the masses were initially equal.

Such considerations conform to the spirit of the Standard Model, where the masses of all particles are generated due to the phenomenon of spontaneous symmetry breaking and are proportional to the vacuum expectation value of the Higgs field.

Let us choose the group $SU(3)$ as the group of internal symmetry. Such an assumption is quite reasonable because the experimental data indicate that there are three generations of fermions. And let us consider the direct product of the Poincaré group and the group $SU(3)$ as an extended symmetry group of the theory.

Let us consider an irreducible representation of this group which is constructed as the direct product of the Dirac representation of the Poincaré group and the fundamental representation of the $SU(3)$

[1] Let Γ be a continuous unitary representation of a finite-dimensional connected Lie group G in a Hilbert space $\mathcal{H}$. Let G contain the inhomogeneous Lorentz group as an analytic subgroup. Let finally the spectrum of the 4-momentum operator P_μ be contained in $\{0\} \bigcup V^+$, V^+ being the future cone in the Minkowski space M^4. If $m_1 > 0$ is an isolated eigenvalue of the mass operator $M = (P_\mu P^\mu)^{1/2}$ then the corresponding eigenspace $\mathcal{H}_1$ is invariant under $\Gamma(G)$.

group. The explicit form of the Lie algebra elements in the present case is obvious. We have the standard realization of the Poincaré group generators multiplied by 3×3 identity matrix $\mathbb{I}$

$$P_\mu = \mathrm{i}\partial_\mu \mathbb{I}, \quad M_{\mu\nu} = \mathrm{i}\Big((x_\mu \partial_\nu - x_\nu \partial_\mu) + (\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)/4\Big)\mathbb{I}, \tag{2}$$

and of the Hermitian generators $X_k, k = 1...8$ of the fundamental representation of the $SU(3)$ group defined by the Gell-Mann matrices

$$\begin{aligned}
X_1 &= \left(e^{(1)} \otimes e^{(2)}\right) + \left(e^{(2)} \otimes e^{(1)}\right), X_2 = \mathrm{i}\left(e^{(2)} \otimes e^{(1)}\right) - \mathrm{i}\left(e^{(1)} \otimes e^{(2)}\right),\\
X_3 &= \left(e^{(1)} \otimes e^{(1)}\right) - \left(e^{(2)} \otimes e^{(2)}\right), X_4 = \left(e^{(1)} \otimes e^{(3)}\right) + \left(e^{(3)} \otimes e^{(1)}\right),\\
X_5 &= \mathrm{i}\left(e^{(3)} \otimes e^{(1)}\right) - \mathrm{i}\left(e^{(1)} \otimes e^{(3)}\right), X_6 = \left(e^{(2)} \otimes e^{(3)}\right) + \left(e^{(3)} \otimes e^{(2)}\right),\\
X_7 &= \mathrm{i}\left(e^{(3)} \otimes e^{(2)}\right) - \mathrm{i}\left(e^{(2)} \otimes e^{(3)}\right), X_8 = \left(e^{(1)} \otimes e^{(1)}\right) + \left(e^{(2)} \otimes e^{(2)}\right) - 2\left(e^{(3)} \otimes e^{(3)}\right).
\end{aligned} \tag{3}$$

Here the vectors $e^{(l)}$ which, for definiteness, can be chosen as follows

$$e^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{4}$$

are a basis of a three-dimensional vector space over the field of complex numbers.

Let us consider the space $\mathcal{H}_{m,1/2}$ of solutions of the matrix Dirac equation

$$\left(\mathrm{i}\gamma^\mu \partial_\mu \mathbb{I} - m\mathbb{I}\right) \Psi(x) = 0, \tag{5}$$

which describes a multiplet containing three particles. Introduce in this space the scalar product

$$(\Psi, \Phi) = \sum_{s=1}^{3} \int d\mathbf{x}\, \Psi_s^\dagger(\mathbf{x}, t)\, \Phi_s(\mathbf{x}, t) \tag{6}$$

with summation over the coordinates of the vectors $e_s^{(l)}$. Then one can consider this space as the direct sum of the spaces of irreducible unitary representations of the direct product of the Poincaré group and the $SU(3)$ group corresponding to positive and negative frequencies. The action of this group in $\mathcal{H}_{m,1/2}$ is determined by the generators $P_\mu, M_{\mu\nu}, X_k$. Obviously, in these representations all the components of the multiplet have equal masses.

A natural basis of multiplet subspace of $\mathcal{H}_{m,1/2}$ with positive frequency can be defined as follows:

$$\Psi_{p,\zeta,l}(x) = \psi_{p,\zeta}(x)\, e^{(l)}. \tag{7}$$

Here $\psi_{p,\zeta}(x)$ are standard plane waves

$$\psi_{p,\zeta}(x) = \frac{1}{\sqrt{2p^0}} u_{p,\zeta} e^{-\mathrm{i}(px)}, \tag{8}$$

where $p^0 = \sqrt{\mathbf{p}^2 + m^2}$, and the spinors $u_{p,\zeta}$ satisfy the equation

$$(\gamma^\mu p_\mu - m)u_{p,\zeta} = 0 \tag{9}$$

and are normalized by the condition

$$\overline{u}_{p,\zeta} u_{p,\zeta'} = 2m\delta_{\zeta,\zeta'}, \quad \overline{u}_{p,\zeta} = u^\dagger_{p,\zeta}\gamma^0. \tag{10}$$

The indices $\zeta, \zeta' = \pm 1$ define the particle polarization. Similarly we can consider the subspace of $\mathcal{H}_{m,1/2}$ with negative frequency.

Let us construct previously mentioned external automorphisms. In the space $\mathcal{H}_{m,1/2}$ the dilatations are determined by the operator

$$D = x_\mu \partial^\mu. \tag{11}$$

The operator of the Dirac equation

$$\left(\mathrm{i}\gamma^\mu \partial_\mu \mathbb{I} - m\mathbb{I}\right) \Psi(x) = 0, \tag{12}$$

commutes with the generators of the extended group. However, for the generator of dilatation we have

$$\left[\left(\mathrm{i}\gamma^\mu \partial_\mu \mathbb{I} - m\mathbb{I}\right), D\right] = \mathrm{i}\gamma^\mu \partial_\mu \mathbb{I}. \tag{13}$$

So, the commutator is equal to zero on solutions for massless particles $m = 0$ only. Therefore, one can construct non-identical spaces $\mathcal{H}^{(i)}_{m,1/2}$, of irreducible representations for multiplets with non-zero masses.

It should be stressed that for massless fermions such a possibility is absent. However, in this case the theory possesses the scale invariance, and, as a consequence, the conformal invariance.

Let us associate the different representation spaces $\mathcal{H}^{(i)}_{m,1/2}$ for multiplets including the neutrinos ($i = \nu$), the charged leptons ($i = e$) as well as the down- ($i = d$) and up- ($i = u$) type quarks with the space $\mathcal{H}_{m,1/2}$ using a unitary intertwining operators $\mathcal{K}^{(i)}$. Then the elements $\Psi^{(i)}(x)$ of the space $\mathcal{H}^{(i)}_{m,1/2}$ and the elements $\Psi(x)$ of the space $\mathcal{H}_{m,1/2}$ become connected by the unitary (with respect to the introduced scalar product (6)) transformation

$$\Psi^{(i)}(x) = \mathcal{K}^{(i)} \Psi(x). \tag{14}$$

The intertwining operators can be constructed as follows. For each multiplet introduce new basis vectors $n^{(l)}(i) \equiv n^{(l)}$, acting by a unitary matrix $\mathbb{V}^{(i)}$ on the old basis vectors $e^{(l)}$:

$$n^{(l)}_s = \sum_{r=1}^{3} V^{(i)}_{sr} e^{(l)}_r. \tag{15}$$

The vectors $n^{(l)}$ are normalized by the conditions (asterisk denotes the complex conjugation)

$$\sum_{s=1}^{3} n^{(l)}_s \overset{*}{n}{}^{(k)}_s = \delta_{kl}, \qquad \sum_{l=1}^{3} n^{(l)}_s \overset{*}{n}{}^{(l)}_r = \delta_{sr}. \tag{16}$$

The operators

$$\mathbb{P}^{(i)}_{(l)} = n^{(l)} \otimes \overset{*}{n}{}^{(l)}, \; \mathbb{P}^{(i)}_{(l)} \mathbb{P}^{(i)}_{(k)} = \delta_{kl} \mathbb{P}^{(i)}_{(l)}, \; \sum_{l=1,2,3} \mathbb{P}^{(i)}_{(l)} = \mathbb{I}. \tag{17}$$

are orthogonal projectors.

We can write the non-trivial intertwining operators in the form

$$\mathcal{K}^{(i)} = \sum_{l=1}^{3} \mathcal{D}^{(i)}_{(l)} \left(n^{(l)} \otimes e^{(l)}\right), \tag{18}$$

where

$$\mathcal{D}^{(i)}_{(l)} = \exp\left((x_\nu \partial^\nu + 3/2) \ln \mu^{(i)}_l\right). \tag{19}$$

Here $\mu_l^{(i)}$ are positive numbers.

The Dirac equations, which are the conditions that the state spaces are irreducible, are now written as follows:

$$\left(\mathrm{i}\gamma^\mu \partial_\mu \mathbb{I} - \mathbb{M}^{(i)}\right) \Psi^{(i)}(x) = 0, \tag{20}$$

where

$$\mathbb{M}^{(i)} = \sum_{l=1}^{3} m_l^{(i)} \mathbb{P}_{(l)}^{(i)}, \quad m_l^{(i)} = m\mu_l^{(i)}. \tag{21}$$

We define bases in the spaces $\mathcal{H}_{m,1/2}^{(i)}$ in the form

$$\Psi^{(i)}_{q,\zeta,\mu_l^{(i)}}(x) = \mathcal{K}^{(i)} \Psi_{p,\zeta,l}(x) = \psi^{(i)}_{q,\zeta,\mu_l^{(i)}}(x) n^{(l)}, \tag{22}$$

where $\psi^{(i)}_{q,\zeta,\mu_l^{(i)}}(x)$ are the plane waves derived from the ordinary plane waves (8) by a dilatation of the coordinates:

$$\psi^{(i)}_{q,\zeta,\mu_l}(x) = \frac{(\mu_l^{(i)})^{3/2}}{\sqrt{2q^0}} u_{q,\zeta} e^{-\mathrm{i}\mu_l^{(i)}(qx)}. \tag{23}$$

Here $q^0 = \sqrt{\mathbf{q}^2 + m^2}$. The spinors $u_{q,\zeta}$ satisfy the equation

$$(\gamma^\mu q_\mu - m) u_{q,\zeta} = 0 \tag{24}$$

and are normalized by the condition

$$\bar{u}_{q,\zeta} u_{q,\zeta'} = 2m\delta_{\zeta,\zeta'}. \tag{25}$$

A direct calculation yields the explicit form of the elements of the Lie algebras of the considered representations. The action of the generators of the $SU(3)$ group

$$X_k^{(i)} = \mathcal{K}^{(i)} X_k \mathcal{K}^{(i)-1} \tag{26}$$

is reduced to the obvious permutations of the basis elements.

The generators of the Lorentz group do not change the form:

$$M_{\mu\nu}^{(i)} = \mathcal{K}^{(i)} M_{\mu\nu} \mathcal{K}^{(i)-1} = \mathrm{i}\Big((x_\mu \partial_\nu - x_\nu \partial_\mu) + (\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)/4\Big)\mathbb{I}, \tag{27}$$

since the dilatation operator commutes with the generators of rotations and boosts (see (1)). This fact is quite natural, because the results of the observations, in particular, of the study of particle oscillations cannot depend on the choice of the inertial reference frame used for measurement.

For the translation generators, we have

$$P_\mu^{(i)} = \mathcal{K}^{(i)} P_\mu \mathcal{K}^{(i)-1} = \mathrm{i}\partial_\mu \mathbb{N}^{(i)}, \tag{28}$$

where

$$\mathbb{N}^{(i)} = \sum_{l=1}^{3} \frac{1}{\mu_l^{(i)}} \mathbb{P}_l^{(i)} = \mathbb{M}^{(i)-1}/m. \tag{29}$$

This matrices are proportional to the inverse mass matrices.

We emphasize once again that the eigenvalues of the Casimir operators constructed from the new generators $P_\mu^{(i)}, M_{\mu\nu}^{(i)}, X_k^{(i)}$ take the same values on solutions of the modified Dirac equations as on elements of the initial representation space $\mathcal{H}_{m,1/2}$. In particular, $P^{(i)\mu} P_\mu^{(i)} \Psi^{(i)}(x) = m^2 \Psi^{(i)}(x)$, but now the parameter m is not the observed mass, it rather sets the scale of the multiplet masses. In other words, it represents the bare mass of the multiplet. The observed masses are determined by the action of the canonical momentum operator squared on the basis functions $\Psi^{(i)}_{q,\zeta,\mu_l^{(i)}}(x)$ and are equal to $m_l^{(i)} = m\mu_l^{(i)}$ for every component of the multiplet.

The general solutions of the Dirac equations (20) can be expanded in the basis functions that are eigenfunctions of any complete set of operators commuting with the operator of the equation. In contrast to the standard case of the Dirac equation, the complete set for these equations contains five operators.

We now find all complete sets of the operators for which the bases of the solution spaces of the modified Dirac equations (20) consist of plane waves. Let $\mathbb{L}^{(i)}$ be a non-degenerate Hermitian matrix which commutes with $\mathbb{M}^{(i)}$. This matrix can be written as follows:

$$\mathbb{L}^{(i)} = \sum_{l=1}^{3} \lambda_l^{(i)} \mathbb{P}^{(i)}_{(l)}, \quad \lambda_l^{(i)} \neq 0. \tag{30}$$

If we assume that the parameters $\lambda_l^{(i)}$ are pairwise distinct, then, in the space of the solutions of Eq. (20), this matrix defines the action of the operator $\mathcal{M}^{(i)}$ which is a linear combination of $X_3^{(i)}$ (the third projection of the isospin), $X_8^{(i)}$ (the hypercharge) and the Casimir operators of $SU(3)$. (Since the representation is irreducible, then the action of the Casimir operators is given by the matrices that are multiple of the identity matrix). Obviously $\mathcal{M}^{(i)}$ can be chosen as one of the operators of the complete set. This operator is used to isolate the orthogonal subspaces determined by the projectors $\mathbb{P}^{(i)}_{(l)}$. Therefore, the numerical values of the parameters $\lambda_l^{(i)}$ can be arbitrary. It is natural to set $\lambda_l^{(i)} = m_l^{(i)}$, that is, to assume that the action of the operator $\mathcal{M}^{(i)}$ is defined by the matrix $\mathbb{M}^{(i)}$.

For a basis to be a plane-wave one, the complete set must include three operators with continuous spectrum. If we choose the space translation generators, that is, the spatial components of the 4-vector $P_\mu^{(i)}$, as such operators, then any standard spin projector multiplied by $\mathbb{N}^{(i)}$ will determine the spin projection. In this case the complete orthonormal system of solutions of Eq. (20) corresponding to a fixed (positive) frequency delivers the basis $\Psi^{(i)}_{q,\zeta,\mu_l^{(i)}}(x)$ (see Eq. (22)) considered previously.

However, we can take the spatial components of the 4-vector

$$\mathbb{L}^{(i)} P_\mu^{(i)} = \mathrm{i}\partial_\mu \mathbb{L}^{(i)} \mathbb{N}^{(i)} \tag{31}$$

to play the role of the operators with continuous spectrum, where $\mathbb{L}^{(i)}$ is any matrix which satisfies (30). In particular, we can set $\mathbb{L}^{(i)} = (\mathbb{N}^{(i)})^{-1}$. Then the basis functions are the eigenfunctions of the spatial components of the canonical momentum operator $i\partial_\mu$, and

$$\Psi^{(i)}_{p,\zeta,m_l^{(i)}}(x) = \psi^{(i)}_{p,\zeta,m_l}(x) n^{(l)}, \tag{32}$$

where $\psi^{(i)}_{p,\zeta,m_l^{(i)}}(x)$ are the plane waves, describing particles with masses $m_l^{(i)}$:

$$\psi^{(i)}_{p,\zeta,m_l^{(i)}}(x) = \frac{1}{\sqrt{2p_l^0}} u_{p,\zeta}^{(m_l^{(i)})} e^{-\mathrm{i}(px)}. \tag{33}$$

Here $p_l^0 = \sqrt{\mathbf{p}^2 + (m_l^{(i)})^2}$, the spinors $u_{p,\zeta}^{(m_l^{(i)})}$ satisfy the equations

$$(\gamma^\mu p_\mu - m_l^{(i)}) u_{p,\zeta}^{(m_l^{(i)})} = 0 \tag{34}$$

and are normalized by the condition

$$\overline{u}^{(m_l^{(i)})}_{p,\zeta} u^{(m_l^{(i)})}_{p,\zeta'} = 2m_l^{(i)} \delta_{\zeta,\zeta'}. \tag{35}$$

Thus, the matrix $\mathbb{M}^{(i)}$ can be interpreted as the mass matrix of the multiplet, and the parameters $m_l^{(i)} = \mu_l^{(i)} m$ are the observed masses of the particles. The states that are the eigenfunctions of $\mathcal{M}^{(i)}$, can be naturally called the mass states for any choice of other operators of the complete set.

However, we can choose a basis in the representation space $\mathcal{H}^{(i)}_{m,1/2}$ in the form of a superposition of the mass states. Consider an arbitrary unitary matrix $\mathbb{U}$. The functions

$$\Psi^{(i)}_{p,\zeta,\alpha}(x) = \sum_{l=1}^{3} U_{\alpha l} \Psi^{(i)}_{p,\zeta,l}(x) \tag{36}$$

make up a complete orthonormal set in $\mathcal{H}^{(i)}_{m,1/2}$ with the scalar product (6). Here $\Psi^{(\nu)}_{p,\zeta,l}(x)$ are the wave functions of arbitrary mass states. The basis (36) can be obtained as a result of the unitary transformation $\mathcal{U}$ of the space $\mathcal{H}^{(i)}_{m,1/2}$ onto itself:

$$\Psi^{(i)}_{p,\zeta,\alpha}(x) = \mathcal{U}\, \Psi^{(i)}_{p,\zeta,l}(x) = \int \widetilde{K}(x,y)\, \Psi^{(i)}_{p,\zeta,l}(y)\, d\mathbf{y}. \tag{37}$$

The kernel of the transformation is determined by the formula

$$\widetilde{K}(x,y) = \frac{1}{(2\pi)^3} \sum_{\alpha=1}^{3} \sum_{l=1}^{3} \sum_{\zeta=\pm 1} \int d\mathbf{q}\, \delta_{\alpha l} \Psi^{(i)}_{p,\zeta,\alpha}(x) \otimes \Psi^{(i)\dagger}_{p,\zeta,l}(y). \tag{38}$$

The elements of this basis no longer are eigenfunctions of the operator $\mathcal{M}^{(i)}$ defined by the mass matrix. The fifth operator from the complete set (denote it as $\mathcal{F}^{(i)}$) is defined now as follows:

$$\mathcal{F}^{(i)} = \mathcal{U}\mathcal{M}^{(i)}\mathcal{U}^{-1}. \tag{39}$$

This operator can explicitly depend on the coordinates of the event space.

It should be emphasized that the form of the causal Green function for the new Dirac equation does not depend on the chosen basis:

$$S_c^{(i)}(x) = \frac{1}{(2\pi)^4} \sum_{l=1}^{3} \mathbb{P}_l^{(i)} \int \frac{(\gamma_\mu p^\mu + m_l^{(i)})\, e^{-\mathrm{i}(px)}}{(m_l^{(i)})^2 - p^2 - \mathrm{i}\epsilon}\, d^4 p. \tag{40}$$

3 Modified model of electroweak interaction

Now we are able to write the electroweak Lagrangian. Once the interaction is carried by the $SU(2) \times U(1)$-gauge fields, the Lagrangian is that of the Standard Model supplemented by the singlets of the right-handed neutrinos.

The Lagrangian for the physical fermion fields in our model is written as follows:

$$\mathcal{L}_f = \mathcal{L}_0 + \mathcal{L}_{int}, \tag{41}$$

where

$$\mathcal{L}_0 = \sum_{i=\nu,e,u,d} \frac{\mathrm{i}}{2} \left[\left(\bar{\Psi}^{(i)} \gamma^\mu (\partial_\mu \Psi^{(i)}) \right) - (\partial_\mu \bar{\Psi}^{(i)}) \gamma^\mu \Psi^{(i)} \right] - \bar{\Psi}^{(i)} \mathbb{M}^{(i)} \Psi^{(i)} \tag{42}$$

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is the Lagrangian of free fields and

$$\begin{aligned} \mathcal{L}_{int} = &-\sum_{i=\nu,e,u,d} \bar{\Psi}^{(i)} \mathbb{M}^{(i)}(H/v) \Psi^{(i)} - e \sum_{i=e,u,d} Q^{(i)} \bar{\Psi}^{(i)} \gamma^{\mu} \Psi^{(i)} A_{\mu} \\ &-\frac{g}{2\sqrt{2}} \left(\bar{\Psi}^{(e)} \gamma^{\mu} (1+\gamma^5) \Psi^{(\nu)} W_{\mu}^{-} + \bar{\Psi}^{(\nu)} \gamma^{\mu} (1+\gamma^5) \Psi^{(e)} W_{\mu}^{+} \right) \\ &-\frac{g}{2\sqrt{2}} \left(\bar{\Psi}^{(d)} \gamma^{\mu} (1+\gamma^5) \Psi^{(u)} W_{\mu}^{-} + \bar{\Psi}^{(u)} \gamma^{\mu} (1+\gamma^5) \Psi^{(d)} W_{\mu}^{+} \right) \\ &-\frac{g}{2\cos\theta_{\mathrm{W}}} \sum_{i=\nu,e,u,d} \bar{\Psi}^{(i)} \gamma^{\mu} \left(T^{(i)} - 2Q^{(i)} \sin^2\theta_{\mathrm{W}} + T^{(i)} \gamma^5 \right) \Psi^{(i)} Z_{\mu}. \end{aligned} \tag{43}$$

is the interaction Lagrangian between the fermion fields, the vector boson fields $W_{\mu}^{\pm}, Z_{\mu}, A_{\mu}$, and the Higgs field H. Here θ_{W} is the Weinberg angle, $e = g\sin\theta_{\mathrm{W}}$ is the positron electric charge, $T^{(i)}$ is the weak isospin ($T^{(\nu)} = T^{(u)} = 1/2$, $T^{(e)} = T^{(d)} = -1/2$), $Q^{(i)}$ is the electric charge of the multiplet in the units of e. The value v is the vacuum expectation of the Higgs field.

Thus, this Lagrangian formally coincides with the Lagrangian of the Standard Model. However, the wave functions $\Psi^{(i)}$ describe not the individual particles, but the multiplets as a whole. It is possible, so for each multiplet the action defined by the Lagrangian of free fields (42) is explicitly invariant with respect to $SU(3)$ transformations generated by $X_k^{(i)}$ (see (26)).

Therefore, when quantizing the model, the multiplet can be considered as a single particle. The one-particle states in the Fock space are defined as usual, the creation and annihilation operators satisfy the canonical commutation relations. However, these operators carry an additional discrete quantum number that is associated with the mass of the state. The multiplet can be either in one of the three mass states, or in a pure quantum state that is a superposition of the states with fixed masses. In a certain sense we may say that there exist only four fundamental fermions in this model.

We will discuss all this in detail, using the approach described in [13]. Consider the basis (22). We can write the components of the field functions as

$$\begin{aligned} \Psi_s^{(i)}(x) &= \frac{1}{(2\pi)^{3/2}} \sum_{l=1}^{3} \sum_{\zeta=\pm1} \int \frac{d\mathbf{q}}{\sqrt{2q^0}} (\mu_l^{(i)})^{3/2} n_s^{(l)} \left[e^{-\mathrm{i}\mu_l^{(i)}(qx)} u_{q,\zeta} \mathfrak{a}_{l,\zeta,(i)}^{-}(\mathbf{q}) + e^{\mathrm{i}\mu_l^{(i)}(qx)} u_{-q,\zeta} \mathfrak{a}_{l,\zeta,(i)}^{+}(\mathbf{q}) \right], \\ \bar{\Psi}_s^{(i)}(x) &= \frac{1}{(2\pi)^{3/2}} \sum_{l=1}^{3} \sum_{\zeta=\pm1} \int \frac{d\mathbf{q}}{\sqrt{2q^0}} (\mu_l^{(i)})^{3/2} \overset{*}{n}{}_s^{(l)} \left[e^{-\mathrm{i}\mu_l^{(i)}(qx)} \bar{u}_{-q,\zeta} \overset{*}{\mathfrak{a}}{}^{-}_{l,\zeta,(i)}(\mathbf{q}) + e^{\mathrm{i}\mu_l^{(i)}(qx)} \bar{u}_{q,\zeta} \overset{*}{\mathfrak{a}}{}^{+}_{l,\zeta,(i)}(\mathbf{q}) \right]. \end{aligned} \tag{44}$$

Using Noether's theorem we can find the integrals of motion for the fields. The analogue of the energy-momentum tensor associated with the translational symmetry is determined by the relation

$$T_{\alpha\beta}^{(i)} = \frac{1}{2} \sum_{s=1}^{3} \left[\bar{\Psi}_s^{(i)}(x) \gamma_{\alpha} \left(P_{\beta}^{(i)} \Psi_s^{(i)}(x) \right) + \left(P_{\beta}^{(i)*} \bar{\Psi}_s^{(i)}(x) \right) \gamma_{\alpha} \Psi_s^{(i)}(x) \right]. \tag{45}$$

It ensures the existence of the conserved vector

$$\mathcal{P}_{\beta}^{(i)} = \sum_{l=1}^{3} \sum_{\zeta=\pm1} \int q_{\beta} d\mathbf{q} \left[\overset{*}{\mathfrak{a}}{}^{+}_{l,\zeta,(i)}(\mathbf{q}) \mathfrak{a}_{l,\zeta,(i)}^{-}(\mathbf{q}) - \overset{*}{\mathfrak{a}}{}^{-}_{l,\zeta,(i)}(\mathbf{q}) \mathfrak{a}_{l,\zeta,(i)}^{+}(\mathbf{q}) \right]. \tag{46}$$

The current vector associated with the global gauge symmetry of the Lagrangian is defined by the relation

$$J_{\alpha}^{(i)} = \sum_{s=1}^{3} \bar{\Psi}_s^{(i)}(x) \gamma_{\alpha} \Psi_s^{(i)}(x). \tag{47}$$

It ensures the conservation of the total field charge:

$$\mathcal{Q}^{(i)} = \sum_{l=1}^{3} \sum_{\zeta=\pm 1} \int d\mathbf{q} \Big[\overset{*}{\mathfrak{a}}{}^{+}_{l,\zeta,(i)}(\mathbf{q}) \mathfrak{a}^{-}_{l,\zeta,(i)}(\mathbf{q}) + \overset{*}{\mathfrak{a}}{}^{-}_{l,\zeta,(i)}(\mathbf{q}) \mathfrak{a}^{+}_{l,\zeta,(i)}(\mathbf{q}) \Big]. \tag{48}$$

The tensor associated with the $SU(3)$-symmetry of the fields is determined by the relation

$$S^{(i)}_{k\alpha} = \frac{1}{2} \sum_{s=1}^{3} \left[\overline{\Psi}^{(i)}_s(x) \gamma_\alpha (X^{(i)}_k \Psi^{(i)}(x))_s + (\overline{\Psi}^{(i)}(x) X^{(i)}_k)_s \gamma_\alpha \Psi^{(i)}_s(x) \right], \quad k = 1...8. \tag{49}$$

It ensures the existence of eight integrals of motion. Using these integrals of motion and the total field charge it is possible to construct nine linear combinations

$$\mathcal{X}^{(i)}_{lk} = \sum_{\zeta=\pm 1} \int d\mathbf{q} \Big[\overset{*}{\mathfrak{a}}{}^{+}_{l,\zeta,(i)}(\mathbf{q}) \mathfrak{a}^{-}_{k,\zeta,(i)}(\mathbf{q}) + \overset{*}{\mathfrak{a}}{}^{-}_{l,\zeta,(i)}(\mathbf{q}) \mathfrak{a}^{+}_{k,\zeta,(i)}(\mathbf{q}) \Big], \quad k, l = 1, 2, 3, \tag{50}$$

and three of them are diagonal in the indices l, k.

The integrals of motion for the fields are associated with the generators of the Lie algebra of the symmetry group of the theory as follows:

$$Z \Psi^{(i)}(x) = \left[\Psi^{(i)}(x), \mathcal{Z} \right]. \tag{51}$$

Here $Z = \left\{ P^{(i)}_\beta, Q^{(i)}, X^{(i)}_k ... \right\}$ and $\mathcal{Z} = \left\{ \mathcal{P}^{(i)}_\beta, \mathcal{Q}^{(i)}, \mathcal{X}^{(i)}_k ... \right\}$. These relations enable one to interpret $\overset{*}{\mathfrak{a}}{}^{+}_{l,\zeta,(i)}(\mathbf{q})$ and $\mathfrak{a}^{-}_{l,\zeta,(i)}(\mathbf{q})$ as the operators of particle creation and annihilation in the state l with the kinetic momentum $\mathbf{q}$ and the polarization ζ. Accordingly, $\mathfrak{a}^{+}_{l,\zeta,(i)}(\mathbf{q})$ and $\overset{*}{\mathfrak{a}}{}^{-}_{l,\zeta,(i)}(\mathbf{q})$ are the operators of antiparticle creation and annihilation in the state l with the kinetic momentum $\mathbf{q}$ and the polarization ζ. Equation (51) for $\mathcal{P}^{(i)}_\beta$ and the invariance under the change of particles to antiparticles yields the canonical commutation relations

$$\left[\mathfrak{a}^{-}_{l,\zeta,(i)}(\mathbf{q}), \overset{*}{\mathfrak{a}}{}^{+}_{k,\zeta',(i)}(\mathbf{q}') \right]_{+} = \delta_{lk} \delta_{\zeta\zeta'} \delta(\mathbf{q} - \mathbf{q}'), \quad \left[\overset{*}{\mathfrak{a}}{}^{-}_{l,\zeta,(i)}(\mathbf{q}), \mathfrak{a}^{+}_{k,\zeta',(i)}(\mathbf{q}') \right]_{+} = \delta_{lk} \delta_{\zeta\zeta'} \delta(\mathbf{q} - \mathbf{q}'). \tag{52}$$

Let us now consider linear combinations of these operators:

$$\mathfrak{a}^{\pm}_{\alpha,\zeta,(i)}(\mathbf{q}) = \sum_{l=1}^{3} U_{\alpha l} \mathfrak{a}^{\pm}_{l,\zeta,(i)}(\mathbf{q}), \quad \overset{*}{\mathfrak{a}}{}^{\pm}_{\alpha,\zeta,(i)}(\mathbf{q}) = \sum_{l=1}^{3} U^{*}_{\alpha l} \overset{*}{\mathfrak{a}}{}^{\pm}_{l,\zeta,(i)}(\mathbf{q}). \tag{53}$$

Commutation relations for $\mathfrak{a}^{\pm}_{\alpha,\zeta,(i)}(\mathbf{q}), \overset{*}{\mathfrak{a}}{}^{\pm}_{\alpha,\zeta,(i)}(\mathbf{q})$ are canonical. When expressed in terms of these operators, only $\mathcal{P}^{(i)}_\beta$ and $\mathcal{Q}^{(i)}$ are diagonal. However, using $\mathcal{Q}^{(i)}$ and the remaining integrals of motion, we can always construct three linear combinations of the form

$$\mathcal{X}'^{(i)}_{\alpha\alpha} = \int d\mathbf{q} \Big[\overset{*}{\mathfrak{a}}{}^{+}_{\alpha,\zeta,(i)}(\mathbf{q}) \mathfrak{a}^{-}_{\alpha,\zeta,(i)}(\mathbf{q}) + \overset{*}{\mathfrak{a}}{}^{-}_{\alpha,\zeta,(i)}(\mathbf{q}) \mathfrak{a}^{+}_{\alpha,\zeta,(i)}(\mathbf{q}) \Big], \quad \alpha = 1, 2, 3. \tag{54}$$

So, the operators $\mathfrak{a}^{\pm}_{\alpha,\zeta,(i)}(\mathbf{q}), \overset{*}{\mathfrak{a}}{}^{\pm}_{\alpha,\zeta,(i)}(\mathbf{q})$ lead to well-defined states in the Fock space. The difference between the number of particles and the number of antiparticles of each type α with the same kinetic momentum $\mathbf{q}$ is an integral of motion.

Consider now the basis (32). We write the components of the field functions as

$$\begin{aligned}\Psi_s^{(i)}(x) &= \frac{1}{(2\pi)^{3/2}}\sum_{l=1}^{3}\sum_{\zeta=\pm1}\int\frac{d\mathbf{p}}{\sqrt{2p_l^0}}n_s^{(l)}\left[e^{-\mathrm{i}(px)}u_{p,\zeta}^{(m_l^{(i)})}\mathfrak{a}_{l,\zeta,(i)}^{-}(\mathbf{p}) + e^{\mathrm{i}(px)}u_{-p,\zeta}^{(m_l^{(i)})}\mathfrak{a}_{l,\zeta,(i)}^{+}(\mathbf{p})\right],\\ \bar{\Psi}_s^{(i)}(x) &= \frac{1}{(2\pi)^{3/2}}\sum_{l=1}^{3}\sum_{\zeta=\pm1}\int\frac{d\mathbf{p}}{\sqrt{2p_l^0}}\mathring{n}_s^{(l)}\left[e^{-\mathrm{i}(px)}\bar{u}_{-p,\zeta}^{(m_l^{(i)})}\mathring{\mathfrak{a}}_{l,\zeta,(i)}^{-}(\mathbf{p}) + e^{\mathrm{i}(px)}\bar{u}_{p,\zeta}^{(m_l^{(i)})}\mathring{\mathfrak{a}}_{l,\zeta,(i)}^{+}(\mathbf{p})\right].\end{aligned} \tag{55}$$

The operators $\mathfrak{a}_{l,\zeta,(i)}^{\pm}(\mathbf{p}), \mathring{\mathfrak{a}}_{l,\zeta,(i)}^{\pm}(\mathbf{p})$ arise as a result of scaling transformation

$$\mathfrak{a}_{l,\zeta,(i)}^{\pm}(\mathbf{p}) = (\mu_l^{(i)})^{-3/2}\mathfrak{a}_{l,\zeta,(i)}^{\pm}(\mathbf{q}),\ \mathring{\mathfrak{a}}_{l,\zeta,(i)}^{\pm}(\mathbf{p}) = (\mu_l^{(i)})^{-3/2}\mathring{\mathfrak{a}}_{l,\zeta}^{\pm}(\mathbf{q});\ \mathbf{q} = \mathbf{p}/\mu_l^{(i)}. \tag{56}$$

Therefore, using Eq. (51) for $\mathcal{P}_\beta^{(i)}$ and the invariance condition with respect to the change of particles to antiparticles we get that these operators as well as their linear combinations

$$\mathfrak{a}_{\alpha,\zeta,(i)}^{\pm}(\mathbf{p}) = \sum_{l=1}^{3}U_{\alpha l}\mathfrak{a}_{l,\zeta,(i)}^{\pm}(\mathbf{p}),\quad \mathring{\mathfrak{a}}_{\alpha,\zeta,(i)}^{\pm}(\mathbf{p}) = \sum_{l=1}^{3}U_{\alpha l}^{*}\mathring{\mathfrak{a}}_{l,\zeta,(i)}^{\pm}(\mathbf{p}) \tag{57}$$

satisfy the canonical commutation relations (52).

A similar reasoning shows that the state described by the operators $\mathfrak{a}_{l,\zeta,(i)}^{\pm}(\mathbf{p}), \mathring{\mathfrak{a}}_{l,\zeta,(i)}^{\pm}(\mathbf{p})$ and $\mathfrak{a}_{\alpha,\zeta,(i)}^{\pm}(\mathbf{p}), \mathring{\mathfrak{a}}_{\alpha,\zeta,(i)}^{\pm}(\mathbf{p})$ are well-defined in the Fock space too. However, there is an important difference. The integral of motion $\mathcal{P}_\beta^{(i)}$ expressed in terms of the operators $\mathfrak{a}_{\alpha,\zeta,(i)}^{\pm}(\mathbf{p}), \mathring{\mathfrak{a}}_{\alpha,\zeta,(i)}^{\pm}(\mathbf{p})$ is non-diagonal. This situation is quite expected. The integral of motion $\mathcal{P}_\beta^{(i)}$ is not the canonical momentum of the field, but the "kinetic momentum". Since the superpositions of the mass states are non-stationary states, the non-diagonal form of $\mathcal{P}_\beta^{(i)}$ reflects the fact of a possible energy transfer from one state to another.

The $SU(3)$-symmetry of the action defined by the interaction Lagrangian (43) is broken even if we will admit that the eigenvectors of the matrices $\mathbb{V}^{(i)}$ are equal. Therefore, the processes of the type $\mu^\pm \to e^\pm + \gamma$ are forbidden.

If the eigenvectors of the matrices $\mathbb{V}^{(i)}$ are distinct, the terms describing the charged currents automatically generate a phenomenon, which is known as the mixing of generations. The matrix of the mixing coefficients for quarks is an analogue to the Cabibbo–Kobayashi–Maskawa (CKM) matrix

$$\mathbb{U}^{\mathrm{CKM}} = \mathbb{V}^{(u)\dagger}\mathbb{V}^{(d)}, \tag{58}$$

and for leptons it is an analogue to the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix

$$\mathbb{U}^{\mathrm{PMNS}} = \mathbb{V}^{(\nu)\dagger}\mathbb{V}^{(e)}. \tag{59}$$

In experiment, as it is well known, the generation mixing occurs for quarks, while the transfer of energy from one neutrino state to another is seen as the oscillation phenomenon.

4 Conclusion

We put forward a modification of the electroweak interaction theory, in which the fermions with the same electroweak quantum numbers are placed in fermion multiplets and are treated as different quantum states of a single particle. That is, in describing the electroweak interactions it is possible to use four fundamental fermions only. In this model, the mixing and oscillations of the particles arise as a direct consequence of the general principles of quantum field theory.

The developed approach enables one to calculate the probabilities of the processes taking place in the detector at long distances from the particle source. Calculations of higher order processes including the computation of the contributions due to radiative corrections can be performed in the framework of perturbation theory using the regular diagram technique.

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References

[1] S. L. Glashow, Nucl. Phys. **22**, 579 (1961); S. Weinberg, Phys. Rev. Lett. **19**, 1264 (1967); A. Salam, *Elementary Particle Theory*, ed. N. Svartholm, (Almqvist and Wiksells, Stockholm, 1968) p. 137

[2] F. Englert, R. Brout, Phys. Rev. Lett. **13**, 321 (1964); P. W. Higgs, Phys. Rev. Lett. **13**, 508 (1964); G. S. Guralnik, C. R. Hagen, T. W. B. Kibble, Phys. Rev. Lett. **13**, 585 (1964)

[3] N. Cabibbo, Phys. Rev. Lett. **10**, 531 (1963); M. Kobayashi, T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973)

[4] K. Olive *et al.* (Particle Data Group), Chin. Phys. C **38**, 090001 (2014)

[5] B. Pontecorvo, Zh. Eksp. Teor. Fiz. **33**, 549(1957) [Sov. Phys. JETP **6**, 429(1958)]

[6] Z. Maki, M. Nakagawa, S. Sakata, Prog. Theor. Phys. **28**, 870 (1962)

[7] Chiu Man Ho, JHEP **12**, 022 (2012).

[8] E. P. Wigner, Ann. Math. (2) **40**, 149 (1939)

[9] N.N. Bogolubov, A.A. Logunov, A.I. Oksak, I.T. Todorov, *General Principlels of Quantum Field Theory* (Kluwer Academic Publishers, Dordrecht, 1990)

[10] L. Michel, Phys. Rev. **137**, B405 (1965); S. Coleman, Phys. Rev. **138**, B1262 (1965); S. Weinberg, Phys. Rev. **139**, B597 (1965)

[11] R. Jost, Helv. Phys. Acta **39**, 369 (1966); L. O'Raifeartaigh, Phys. Rev. Lett. **14**, 575 (1965); Phys. Rev. **161**, 1571 (1967)

[12] A.O. Barut, R. Raczka, *Theory of Group Representations and Applications* (World Scientific, Singapore, 1986)

[13] N.N. Bogoliubov, D.V. Shirkov, *Introduction to Theory of Quantized Fields* (John Wiley, New York, 1979)