On the four-loop strong coupling beta-function in the SM

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Abstract. In the talk the leading four-loop contribution to the beta-function of the strong coupling in the SM is discussed. Some details of calculation techniques are provided. Special attention is paid to the ambiguity due to utilized $\gamma_5$ treatment and a particular prescription with anticommuting $\gamma_5$ is advocated. As a by-product of our computation the four-loop beta-function in QCD with “gluino” is also obtained.

The Standard Model (SM) of fundamental interactions being renormalizable can, in principle, by used to make predictions at scales far above the Z-boson mass $Q^2 \gg M_Z$. At such scales it is convenient to use “running”, or scale-dependent, couplings $a(Q)$, which are obtained from a set of measurable quantities $\{O\}$ by means of the following two-step procedure:

\[ \text{PDG [1] 20XX} \]
\[ \{O\} = M_P, M_W, M_Z, M_H, M_t, G_F \]
\[ \text{Fixed } \mu_0 \]
\[ g_i(\mu_0), y_i(\mu_0), \lambda(\mu_0) \]
\[ \text{in } \overline{\text{MS}} \text{ scheme} \]
\[ \text{Evolve from } \mu_0 \text{ to scale } \mu \]

The first step is called matching and boils down to the extraction/fitting of the model parameters $a(\mu_0 \approx M_Z)$ at the electroweak scale (in what follows, we employ $\overline{\text{MS}}$-scheme). The second step — “running” — allows one to utilize renormalization-group equations (RGEs) to re-summ potentially large logarithms $\log \mu^2/\mu_0^2$ contributing to finite-order relations between $a(\mu_0)$ and $a(\mu)$.

One of the most important applications of such a procedure is the vacuum stability analysis of the SM (see, e.g., [2, 3] and references therein). It turns out that for large values of Higgs field $\phi$ the effective potential can be approximated as

\[ V_{\text{eff}}(\phi \gg v) \approx \frac{\lambda(\mu = \phi)}{4} \phi^4, \]

where the scale dependence of self-coupling $\lambda(\mu)$ is governed by the following (one-loop) RGEs

\[ (4\pi)^2 \frac{d\lambda}{d \ln \mu^2} = 12\lambda + 6y_t^2\lambda - 3y_t^4 + \ldots, \]
\[ (4\pi)^2 \frac{dy_t}{d \ln \mu^2} = \frac{9}{4} y_t^3 - 4g_s^2 y_t + \ldots, \]

in which the “de-stabilizing” contribution due to top-quark Yukawa coupling $y_t$ is emphasized. The importance of the strong coupling $g_s$ can be deduced from RGE for $y_t$ - strong interactions tend to decrease the latter with $\mu$.

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At present, the state-of-the-art analysis utilizes full two-loop matching [4] together with three-loop evolution via RGEs [5–7]. In this talk, we discuss one little step towards the full four-loop analysis — calculation of leading N3LO corrections to $\beta_{as}$. The latter is defined here as ($h$ counts powers of couplings)

$$\frac{d a_s}{d \log \mu^2} = \beta_{as} a_s = -a_s \sum_{i=0}^{3} \beta_i h^{i+2}. \tag{3}$$

For convenience, we introduce a set of SM parameters (with $\xi$ being a gauge-fixing parameter)

$$(16\pi^2)a = \{g_2^2, g_1^2, \lambda, (16\pi^2)\xi\}. \tag{4}$$

Since we are interested in the leading corrections to $\beta_3$ (3), the electroweak gauge interactions are neglected together with Yukawa interactions of all SM fermions but the top-quark.

For completeness, let us mention here that the matching procedure for the strong coupling constant is different than that mentioned earlier. One usually considers five-flavor ($n_f = 5$) QCD as an effective theory obtained from a more fundamental one (e.g., QCD with “active” top quark) and find the relations of the form:

$$a_s(\mu) = a_s(\mu) \zeta_{as}(\mu, M), \tag{5}$$

where $M$ corresponds to the mass of a heavy field. The (“threshold”) corrections to the so-called decoupling constant $\zeta_{as}$ are known in pure QCD up to four loops [8–10], while two-loop electroweak contribution is considered in Ref. [11].

Before going to the result, let us discuss some technicalities and important issues encountered in our calculation. To simplify our life we made use of the background-field gauge (BFG) [12, 13]. The advantage of BFG lies in the QED-like relation between the gauge coupling renormalization constant $Z_{as}$ and that of the background gluon field $Z_{\hat{G}}$:

$$Z_{as} = 1/Z_{\hat{G}}, \quad Z_{\xi} = Z_{\hat{G}}. \tag{5}$$

Obviously, this allows one to obtain the final result solely from massless propagator-type integrals. In (5), we also indicate the relation between the renormalization constants of quantum gluon field $\hat{G}$ and gauge-fixing parameter. It is worth mentioning that, since in $\overline{\text{MS}}$-scheme beta-functions do not depend on masses, one can avoid any special infra-red rearrangement (IRR) [14] tricks.

For diagram generation we employ the package DIANA [15], which internally uses QGRAF [16]. The color [17] and Dirac algebra are carried out by means of FORM. All the generated two-point functions are mapped onto three auxiliary topologies, each containing 11 propagators and 3 irreducible numerators. The corresponding diagrams are evaluated by means of the C++ version of the FIRE package [18], which performs integration-by-parts (IBP) [19] reduction based on the reduction rules prepared by the LiteRed[20] package. The IBP reduction leads to a small set of master integrals. The expressions for the latter are known in analytical form up to the finite parts [21].

Let us also note that as an independent cross-check of our setup, we prepared a simple QCD-like model with additional fermions in the adjoint representation of SU(3) color group (“gluino”). We
calculated four-loop correction $\Delta\beta_3 \equiv \beta_3(n_f,n_g) - \beta_3(n_f)$ to the beta-function of the strong coupling

$$\Delta\beta_3 = n_f \left[ \frac{d^{abcd}_A d^{abcd}_A}{N_A} \left( \frac{256}{9} - \frac{832}{3} \zeta_3 \right) - C_A^4 \left( \frac{68507}{243} - \frac{52}{9} \zeta_3 \right) \right] +$$

$$+ n_f n_g \left[ C_A^2 C_F T_F \left( \frac{23480}{243} - \frac{352}{9} \zeta_3 \right) + C_A^2 T_F \left( -\frac{152}{27} - \frac{64}{9} \zeta_3 \right) + \right.$$

$$\left. C_A^2 T_F \left( \frac{30998}{243} + \frac{128}{3} \zeta_3 \right) + \right.$$

$$\left. \frac{d^{abcd}_A d^{abcd}_A}{N_A} \left( -\frac{704}{9} + \frac{512}{3} \zeta_3 \right) \right] +$$

$$+ n_f n_g \left[ C_A^4 \left( \frac{26555}{486} - \frac{8}{9} \zeta_3 \right) + \frac{d^{abcd}_A d^{abcd}_A}{N_A} \left( -\frac{176}{9} + \frac{128}{3} \zeta_3 \right) \right] +$$

$$+ n_g \left[ C_A^3 T_F \left( \frac{934}{243} + \frac{308}{243} \right) + C_A^2 n_g \frac{23}{27} \right] +$$

$$+ n_g^2 \left[ C_A^2 T_F \left( \frac{1252}{243} + C_A C_F T_F \frac{1232}{243} \right) \right]$$

(6)

in terms of the SU(3) casimirs and $n_f(n_g)$ corresponding to the number of quarks(gluino). The beta-function for such a model at four loops was predicted by A.F. Pikelner [22] along the lines of Ref. [23] and can be used, e.g. in the derivation of $\{\beta\}$-expansions [24]. We found perfect agreement and, thus, both confirmed the prediction and verified our computer setup.

Let us now discuss an important obstacle – the ambiguities in the dimensionally regularized expressions due to $\gamma_5$. It is known that there is a clash between anticommutativity $\{\gamma_\mu, \gamma_5\} = 0$ and strictly four-dimensional relation

$$\text{tr} (\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5) = -4i\epsilon^{\mu\nu\rho\sigma}$$

(7)

in $D \neq 4$ (see, e.g.,[25]). A self-consistent BMHV-algebra [26, 27] breaks $D$-dimensional Lorentz invariance and requires too much effort when applied to multi-loop problems involving chiral fermions. External axial currents in QCD can be conveniently treated within the prescription due to Larin [28]. Another approach [29] is based on anticommuting $\gamma_5$ but promote every fermionic trace “tr” to a non-cyclic linear functional, which depends on the choice of utilized reading point/prescription, i.e., the position, at which we start(end) reading the trace.

Since the relevant diagrams (48 non-planar and 24 planar graphs, see, e.g., Fig. 1) involve only single poles in the regularization parameter $\epsilon \equiv (4 - D)/2$, we expected that there should be no ambiguity in $\beta_3$. We made a (incorrect) assumption that it is safe to read a trace from any position and
use anticommuting $\gamma_5$, Eq. (7) and the contraction
\[ \epsilon^{\mu
u
\sigma
\rho} \epsilon_{\alpha
\beta
\gamma
\delta} = -\mathcal{T}[\mu\nu\rho]_{[\alpha\beta\gamma]} \]

\[ \tau^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta} = \delta^{\alpha}_{\mu} \delta^{\beta}_{\nu} \delta^{\gamma}_{\rho} \delta^{\delta}_{\sigma} \]
to get a unique result. However, similar calculation was carried out by M. Zoller [30] and an agreement was found only in the “naive” part, in which contributions due to traces with odd number of $\gamma_5$ are neglected. The discrepancy triggered further investigation of the issue and it was found that, indeed, the results for the diagrams giving rise to non-trivial $\gamma_5$ contribution do depend on the choice of “cut” points, at which one breaks a closed Dirac trace.

The result for the $1/\epsilon$ part of the diagrams can be casted into
\[ \frac{a_2^2 a_1^2 T_F^2}{\epsilon} (X_1 + X_2 \zeta_3) \cdot R \]

and for non-planar ones we have $X_1 = -1/18$, $X_2 = 1/6$, while in the planar case $X_1 = 1/6$, $X_2 = 0$.

The coefficient $R$ depends on the “cut” points and it turns out that there are three non-equivalent cases, indicated by dots in Fig.1. If both traces are cut at external gluon vertices, one has $R = 1$. If only one external vertex is chosen as a “cut” point, $R = 2$. Finally, for both traces terminated at internal vertices we have $R = 3$.

A natural question arises whether it is possible to single out a unique prescription. In our original paper [31] we advocate the choice $R = 3$. The main argument comes from the calculation of finite, $O(\epsilon^0)$, parts of the diagrams. It is known that IRR procedure (e.g., of Ref. [30]), usually utilized to find RGEs in $\overline{\text{MS}}$, is only aimed to calculate the pole part of a diagram and does not guarantee that the $O(\epsilon^0)$ terms remain the same after its application. Since we effectively do not do any IRR tricks, we can safely calculate the finite parts and check, whether it is transverse in $D$-dimensions or not.

It turns out that the case with $R = 3$ leads to transverse gluon self-energy, while the case $R = 2$ gives rise to a correction to the longitudinal part, thus, explicitly breaking gauge invariance. In spite of the fact that the prescription $R = 2$ also produce zero upon multiplication by the product of external momenta $q_{\mu} q_{\nu}$, we exclude it by simple symmetry argument (we do not want to give preference to either external vertex).

At the end of the day we obtain the following gauge-parameter independent expression [31]:

\[ \beta_3 = \beta_3^{QCD} (n_f = 2n_G) + a_1^3 a_2 \left[ T_F C_F^2 (6 - 144\zeta_3) + T_F C_A C_F \left( \frac{523}{9} - 72\zeta_3 \right) + \frac{1970}{9} T_F C_A^2 \right] 
- \frac{1288}{9} T_F^2 C_F n_G - \frac{872}{9} T_F^2 C_A n_G + a_1^3 a_2^3 T_F \left( \frac{423}{2} + 12\zeta_3 \right) + 60 a_1 a_3 a_4 T_F - 72 a_1 a_2 a_4^2 T_F 
- a_1^2 a_2^2 \left[ T_F^2 \left( 48 - 96\zeta_3 + \frac{R}{3} \right) + T_F C_F \left( 117 - 144\zeta_3 \right) + 222 T_F C_A \right], \tag{8} \]

where $n_G$ corresponds to the number of SM families.

It is interesting to compare the relative sizes of different four-loop terms (8) and recent five-loop pure QCD contribution to $\beta_4$ [32]. From Fig. 2 one can see that $a_4^4$ ammounts for about 94% of $\beta_3 + \beta_4$ both at the top-mass and Planck scales. The mixed $a_3^2 a_4$ and $a_2^2 a_4^2$ terms have opposite signs and partially compensate each other. The contributions due to five loops [32] and that from $\gamma_5$ are also of different signs and are both less than a percent.

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1 Non-trivial contributions due to $\gamma_5$ can only appear when even number of such traces are present.

2 There seems to be no problem with gauge-invariance in the pole part.
To summarize, we calculated different four-loop corrections to beta-functions for $\alpha_s$ both in the SM and in hypothetical QCD with "gluino". The $\gamma_5$ ambiguities were studied and a reading prescription for "odd" fermion traces, consistent with gauge symmetry, was singled out. In our future studies, we plan to extend the result for $\beta_3$ to the full SM case and compute leading electroweak threshold corrections at three loops.

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**References**