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On the four-loop strong coupling beta-function in the SM

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Abstract. In the talk the leading four-loop contribution to the beta-function of the strong coupling in the SM is discussed. Some details of calculation techniques are provided. Special attention is paid to the ambiguity due to utilized γ_5 treatment and a particular prescription with anticommuting γ_5 is advocated. As a by-product of our computation the four-loop beta-function in QCD with "gluino" is also obtained.

The Standard Model (SM) of fundamental interactions being renormalizable can, in principle, by used to make predictions at scales far above the Z-boson mass $Q^2 \gg M_Z$. At such scales it is convenient to use "running", or scale-dependent, couplings $a(Q)$, which are obtained from a set of measurable quantities $\{O\}$ by means of the following two-step procedure:

$$\underbrace{\begin{array}{c}\text{PDG [1] 20XX}\\ \hline \{O\} = M_b, M_W, M_Z,\\ M_H, M_t, G_F\end{array}} \longrightarrow \begin{array}{c}\text{Fixed } \mu_0\\ \hline g_i(\mu_0), y_i(\mu_0), \lambda(\mu_0)\\ \text{in } \overline{\text{MS}} \text{ scheme}\end{array} \longrightarrow \begin{array}{c}\text{Evolve from } \mu_0\\ \text{to scale } \mu\end{array}$$

The first step is called *matching* and boils down to the extraction/fitting of the model parameters $a(\mu_0 \simeq M_Z)$ at the electroweak scale (in what follows, we employ $\overline{\text{MS}}$-scheme). The second step — "running" — allows one to utilize renormalization-group equations (RGEs) to re-summ potentially large logarithms $\log \mu^2/\mu_0^2$ contributing to finite-order relations between $a(\mu_0)$ and $a(\mu)$.

One of the most important applications of such a procedure is the vacuum stability analysis of the SM (see, e.g., [2, 3] and references therein). It turns out that for large values of Higgs field ϕ the effective potential can be approximated as

$$V_{\text{eff}}(\phi \gg v) \simeq \frac{\lambda(\mu = \phi)}{4}\phi^4, \tag{1}$$

where the scale dependence of self-coupling $\lambda(\mu)$ is governed by the following (one-loop) RGEs

$$(4\pi)^2 \frac{d\lambda}{d\ln\mu^2} = 12\lambda + 6y_t^2\lambda - 3y_t^4 + \ldots, \qquad (4\pi)^2 \frac{dy_t}{d\ln\mu^2} = \frac{9}{4}y_t^3 - 4{g_s}^2 y_t + \ldots, \tag{2}$$

in which the "de-stabilizing" contribution due to top-quark Yukawa coupling y_t is emphasized. The importance of the strong coupling g_s can be deduced from RGE for y_t - strong interactions tend to decrease the latter with μ.

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At present, the state-of-the-art analysis utilizes full two-loop matching [4] together with three-loop evolution via RGEs [5–7]. In this talk, we discuss one little step towards the full four-loop analysis — calculation of leading N^{3}LO corrections to β_{a_s}. The latter is defined here as (h counts powers of couplings)

$$\frac{d\ a_s}{d\ \log\mu^2} = \beta_{a_s} a_s = -a_s \sum_{i=0}^{3} \beta_i h^{i+2}. \tag{3}$$

For convenience, we introduce a set of SM parameters (with ξ being a gauge-fixing parameter)

$$(16\pi^2)a = \left\{g_s^2, y_t^2, \lambda, (16\pi^2)\xi\right\}. \tag{4}$$

Since we are interested in the leading corrections to β_3 (3), the electroweak gauge interactions are neglected together with Yukawa interactions of all SM fermions but the top-quark.

For completeness, let us mention here that the *matching* procedure for the strong coupling constant is different than that mentioned earlier. One usually considers five-flavor ($n_f = 5$) QCD as an effective theory obtained from a more fundamental one (e.g., QCD with "active" top quark) and find the relations of the form:

$$a_s^{(5)}(\mu) = a_s(\mu)\zeta_{a_s}(\mu, M),$$

where M corresponds to the mass of a heavy field. The ("threshold") corrections to the so-called decoupling constant ζ_{a_s} are known in pure QCD up to four loops [8–10], while two-loop electroweak contribution is considered in Ref. [11].

Before going to the result, let us discuss some technicalities and important issues encountered in our calculation. To simplify our life we made use of the background-field gauge (BFG) [12, 13]. The advantage of BFG lies in the QED-like relation between the gauge coupling renormalization constant Z_{a_s} and that of the background gluon field $Z_{\hat{G}}$:

$$Z_{a_s} = 1/Z_{\hat{G}}, \qquad Z_\xi = Z_{\tilde{G}}. \tag{5}$$

Obviously, this allows one to obtain the final result solely from massless propagator-type integrals. In (5), we also indicate the relation between the renormalization constants of quantum gluon field $\tilde{G}$ and gauge-fixing parameter. It is worth mentioning that, since in $\overline{\text{MS}}$-scheme beta-functions do not depend on masses, one can avoid any special infra-red rearrangement (IRR) [14] tricks.

For diagram generation we employ the package `DIANA` [15], which internally uses `QGRAF` [16]. The color [17] and Dirac algebra are carried out by means of `FORM`. All the generated two-point functions are mapped onto three auxiliary topologies, each containing 11 propagators and 3 irreducible numerators. The corresponding diagrams are evaluated by means of the `C++` version of the `FIRE` package [18], which performs integration-by-parts (IBP) [19] reduction based on the reduction rules prepared by the `LiteRed`[20] package. The IBP reduction leads to a small set of master integrals. The expressions for the latter are known in analytical form up to the finite parts [21].

Let us also note that as an independent cross-check of our setup, we prepared a simple QCD-like model with additional fermions in the adjoint representation of SU(3) color group ("gluino"). We calculated four-loop correction $\Delta\beta_3 \equiv \beta_3(n_f, n_g) - \beta_3(n_f)$ to the beta-function of the strong coupling

$$
\begin{aligned}
\Delta\beta_3 \; &= n_g\Big[\frac{d_A^{abcd}d_A^{abcd}}{N_A}\left(\frac{256}{9} - \frac{832}{3}\zeta_3\right) - C_A^4\left(\frac{68507}{243} - \frac{52}{9}\zeta_3\right)\Big] \\
&+ n_f n_g\Big[C_A^2 C_F T_F\left(\frac{23480}{243} - \frac{352}{9}\zeta_3\right) + C_A C_F^2 T_F\left(-\frac{152}{27} - \frac{64}{9}\zeta_3\right) + \\
&\qquad C_A^3 T_F\left(\frac{30998}{243} + \frac{128}{3}\zeta_3\right) + \frac{d_F^{abcd}d_A^{abcd}}{N_A}\left(-\frac{704}{9} + \frac{512}{3}\zeta_3\right)\Big] \\
&+ n_g^2\Big[C_A^4\left(\frac{26555}{486} - \frac{8}{9}\zeta_3\right) + \frac{d_A^{abcd}d_A^{abcd}}{N_A}\left(-\frac{176}{9} + \frac{128}{3}\zeta_3\right)\Big] \\
&+ n_g^2 n_f\Big[C_A^3 T_F\frac{934}{243} + C_A^2 C_F T_F\frac{308}{243}\Big] + C_A^4 n_g^3\frac{23}{27} \\
&+ n_f^2 n_g\Big[C_A^2 T_F^2\frac{1252}{243} + C_A C_F T_F^2\frac{1232}{243}\Big]
\end{aligned}
\tag{6}
$$

in terms of the SU(3) casimirs and $n_f(n_g)$ corresponding to the number of quarks(gluino). The beta-function for such a model at four loops was predicted by A.F. Pikelner [22] along the lines of Ref. [23] and can be used, e.g, in the derivation of $\{\beta\}$-expansions [24]. We found perfect agreement and, thus, both confirmed the prediction and verified our computer setup.

Let us now discuss an important obstacle – the ambiguities in the dimensionally regularized expressions due to γ_5. It is known that there is a clash between anticommutativity $\{\gamma_\mu, \gamma_5\} = 0$ and strictly four-dimensional relation

$$
\mathrm{tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5) = -4i\epsilon^{\mu\nu\rho\sigma} \tag{7}
$$

in $D \neq 4$ (see, e.g.,[25]). A self-consistent BMHV-algebra [26, 27] breaks D-dimensional Lorentz invariance and requires too much effort when applied to multi-loop problems involving chiral fermions. External axial currents in QCD can be conveniently treated within the prescription due to Larin [28]. Another approach [29] is based on anticommuting γ_5 but promote every fermionic trace "tr" to a non-cyclic linear functional, which depends on the choice of utilized *reading point/prescription*, i.e., the position, at which we start(end) reading the trace.

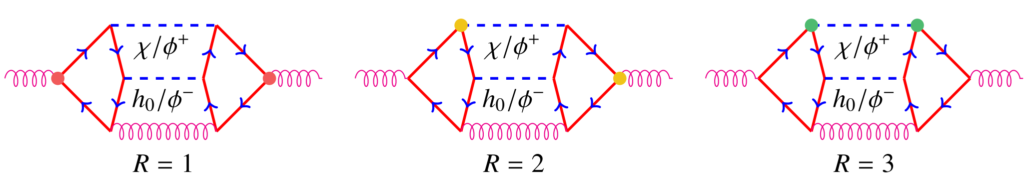

Figure 1. A typical diagram giving rise to a non-trivial contribution due to traces involving odd number of γ_5. Three non-equivalent reading prescriptions are indicated by dots. In our problem it does not matter, whether we start or end the traces at the indicated points. All internal "cut" points turn out to be equivalent.

Since the relevant diagrams (48 non-planar and 24 planar graphs, see, e.g., Fig. 1) involve only single poles in the regularization parameter $\epsilon \equiv (4 - D)/2$, we expected that there should be no ambiguity in β_3. We made a (incorrect) assumption that it is safe to read a trace from any position and use anticommuting γ_5, Eq. (7) and the contraction[1]

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta} = -\mathcal{T}^{[\mu\nu\rho\sigma]}_{[\alpha\beta\gamma\delta]}, \qquad \mathcal{T}^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta} = \delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}\delta^{\rho}_{\gamma}\delta^{\sigma}_{\delta}$$

to get a unique result. However, similar calculation was carried out by M. Zoller [30] and an agreement was found only in the "naive" part, in which contributions due to traces with odd number of γ_5 are neglected. The discrepancy triggered further investigation of the issue and it was found that, indeed, the results for the diagrams giving rise to non-trivial γ_5 contribution do depend on the choice of "cut" points, at which one breaks a closed Dirac trace.

The result for the $1/\epsilon$ part of the diagrams can be casted into

$$\frac{a_s^2 a_t^2 T_F^2}{\epsilon}(X_1 + X_2\zeta_3)\cdot R$$

and for non-planar ones we have $X_1 = -1/18$, $X_2 = 1/6$, while in the planar case $X_1 = 1/6$, $X_2 = 0$.

The coefficient R depends on the "cut" points and it turns out that there are three non-equivalent cases, indicated by dots in Fig.1. If both traces are cut at external gluon vertices, one has $R = 1$. If only one external vertex is chosen as a "cut" point, $R = 2$. Finally, for both traces terminated at internal vertices we have $R = 3$.

A natural question arises whether it is possible to single out a unique prescription. In our original paper [31] we advocate the choice $R = 3$. The main argument comes from the calculation of finite, $O(\epsilon^0)$, parts of the diagrams. It is known that IRR procedure (e.g.,of Ref. [30]), usually utilized to find RGEs in $\overline{\text{MS}}$, is only aimed to calculate the pole part of a diagram and does not guarantee that the $O(\epsilon^0)$ terms remain the same after its application. Since we effectively do not do any IRR tricks, we can safely calculate the finite parts and check, whether it is transverse in D-dimensions or not[2].

It turns out that the case with $R = 3$ leads to transverse gluon self-energy, while the case $R = 2$ gives rise to a correction to the longitudinal part, thus, explicitly breaking gauge invariance. In spite of the fact that the prescription $R = 2$ also produce zero upon multiplication by the product of external momenta $q_\mu q_\nu$, we exclude it by simple symmetry argument (we do not want to give preference to either external vertex).

At the end of the day we obtain the following gauge-parameter independent expression [31]:

$$\begin{aligned}\beta_3 &= \beta_3^{\text{QCD}}(n_f = 2n_G) + a_s^3 a_t\left[T_F C_F^2(6 - 144\zeta_3) + T_F C_A C_F\left(\frac{523}{9} - 72\zeta_3\right) + \frac{1970}{9}T_F C_A^2\right.\\ &- \left.\frac{1288}{9}T_F^2 C_F n_G - \frac{872}{9}T_F^2 C_A n_G\right] + a_s a_t^3 T_F\left(\frac{423}{2} + 12\zeta_3\right) + 60 a_s a_t^2 a_\lambda T_F - 72 a_s a_t a_\lambda^2 T_F\\ &- a_s^2 a_t^2\left[T_F^2\left(48 - 96\zeta_3 + \underbrace{R}_{3}\cdot\left[\frac{16}{3} + 32\zeta_3\right]\right) + T_F C_F(117 - 144\zeta_3) + 222 T_F C_A\right],\end{aligned} \quad (8)$$

where n_G corresponds to the number of SM families.

It is interesting to compare the relative sizes of different four-loop terms (8) and recent five-loop pure QCD contribution to β_4 [32]. From Fig. 2 one can see that a_s^4 ammounts for about 94% of $\beta_3 + \beta_4$ both at the top-mass and Planck scales. The mixed $a_s^3 a_t$ and $a_s^2 a_t^2$ terms have opposite signs and partially compensate each other. The contributions due to five loops [32] and that from γ_5 are also of different signs and are both less than a percent.

[1] Non-trivial contributions due to γ_5 can only appear when even number of such traces are present.
[2] There seems to be no problem with gauge-invariance in the pole part.

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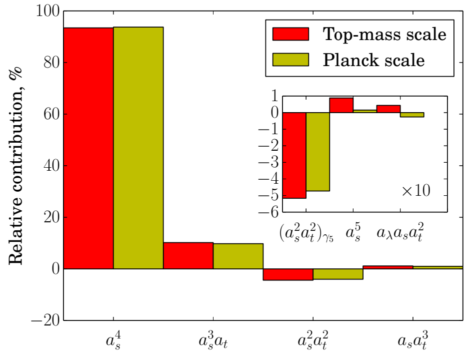

Figure 2. Relative size of the calculated four-loop contributions and the pure QCD five-loop term $O(a_s^5)$ with respect to the sum $\beta_3 + \beta_4$. Non-trivial part due to γ_5 is indicated.

To summarize, we calculated different four-loop corrections to beta-functions for α_s both in the SM and in hypothetical QCD with "gluino". The γ_5 ambiguities were studied and a reading prescription for "odd" fermion traces, consistent with gauge symmetry, was singled out. In our future studies, we plan to extend the result for β_3 to the full SM case and compute leading electroweak threshold corrections at three loops.

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