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Universal effective coupling constant ratios of 3D scalar ϕ^4 field theory and pseudo-ϵ expansion

A. I. Sokolov[1,⋆], *M. A.* Nikitina[1,2], and *A.* Kudlis[1,2,⋆⋆]

[1]*Department of Quantum Mechanics, Saint Petersburg State University, Ulyanovskaya 1, Petergof, Saint Petersburg 198504, Russia*
[2]*ITMO University, Kronverkskii ave 49, Saint Petersburg 197101, Russia*

Abstract. The ratios $R_{2k} = g_{2k}/g_4^{k-1}$ of renormalized coupling constants g_{2k} entering the small-field equation of state approach universal values R_{2k}^* at criticality. They are calculated for the three-dimensional $\lambda\phi^4$ field theory within the pseudo-ϵ expansion approach. Pseudo-ϵ expansions for R_6^*, R_8^* and R_{10}^* are derived in the five-loop approximation, numerical estimates are obtained with a help of the Padé–Borel–Leroy resummation technique. Its use gives $R_6^* = 1.6488$, the number which perfectly agrees with the most recent lattice result $R_6^* = 1.649$. For the octic coupling the pseudo-ϵ expansion is less favorable numerically. Nevertheless the Padé–Borel–Leroy resummation leads to the estimate $R_8^* = 0.890$ close to the values $R_8^* = 0.871$, $R_8^* = 0.857$ extracted from the lattice and field-theoretical calculations. The pseudo-ϵ expansion for R_{10}^* turns out to have big and rapidly increasing coefficients. This makes correspondent estimates strongly dependent on the Borel–Leroy shift parameter b and prevents proper evaluation of R_{10}^*.

1 Introduction

The critical behavior of the systems undergoing continuous phase transitions is characterized by a set of universal parameters including, apart from critical exponents, renormalized effective coupling constants g_{2k} and the ratios $R_{2k} = g_{2k}/g_4^{k-1}$. These ratios enter the scaling equation of state via the small-magnetization expansion of the free energy:

$$F(z,m) - F(0,m) = \frac{m^3}{g_4}\left(\frac{z^2}{2} + z^4 + R_6 z^6 + R_8 z^8 + R_{10} z^{10} ...\right), \quad (1)$$

where $z = M\sqrt{g_4/m^{1+\eta}}$ is dimensionless magnetization, g_4 is the renormalized quartic coupling constant, $m \sim (T - T_c)^\nu$ being proportional to the inverse correlation length. The ratios R_{2k} along with the renormalized quartic coupling constant g_4 determine also the nonlinear susceptibilities of the system.

The scaling equation of state of the Ising model which is described by the three-dimensional (3D) Euclidean scalar field theory with $\lambda\phi^4$ self-interaction was intensively studied theoretically. In particular, higher-order renormalized coupling constants g_6, g_8,... and the ratios R_{2k} were evaluated

⋆e-mail: ais2002@mail.ru
⋆⋆e-mail: andrewkudlis@gmail.com

by a number of analytical and numerical methods [1–27]. Within the renormalization-group approach the numerical estimates for R_6, R_8 and R_{10} were obtained from the five-loop perturbative expansions of the higher-order renormalized couplings calculated in physical dimensions [10]. Corresponding 3D RG expansions for the universal ratios are as follows

$$R_6 = \frac{9}{\pi} g_4 \Big(1 - \frac{3}{\pi} g_4 + 1.38996295 g_4^2 - 2.50173246 g_4^3 + 5.275903\ g_4^4\Big), \tag{2}$$

$$R_8 = -\frac{81}{2\pi} g_4 \Big(1 - \frac{65}{6\pi} g_4 + 7.77500131 g_4^2 - 18.5837685 g_4^3 + 48.16781 g_4^4\Big), \tag{3}$$

$$R_{10} = \frac{243}{\pi} g_4 \Big(1 - \frac{20}{\pi} g_4 + 23.1841758 g_4^2 - 74.2747105 g_4^3 + 238.6138 g_4^4\Big). \tag{4}$$

These series are seen to have big and rapidly growing coefficients what makes them, along with a big value of the Wilson fixed point coordinate $g_4^* \approx 0.99$, rather inconvenient for getting numerical estimates.

The main aim of this work is to improve the situation converting the RG expansions (2), (3), (4) into alternative series looking more friendly from the numerical point of view. To do this we address the pseudo-ϵ expansion approach which proved to be very efficient numerically when used to evaluate the critical exponents and other universal parameters of various 3D and 2D systems [27–40].

2 Pseudo-ϵ expansions and resummation

The Hamiltonian of the Euclidean field theory describing the critical behavior of the 3D Ising-like systems reads:

$$H = \frac{1}{2} \int d^3x \left(m_0^2 \varphi^2 + \nabla\varphi^2 + \frac{\lambda}{12}\varphi^4 \right), \tag{5}$$

where φ is a real scalar field, bare mass squared m_0^2 being proportional to $T - T_c^{(0)}$, and $T_c^{(0)}$ – mean field transition temperature. The β function and critical exponents of the model (5) has been calculated [41] in the six-loop approximation within the massive theory with the propagator, quartic vertex and φ^2 insertion normalized in a conventional way at zero external momenta. Later, R. Guida and J.Zinn-Justin have found the RG expansions for the higher-order effective coupling constants of this model up to the five-loop terms [10].

To derive the pseudo-ϵ expansions for the universal coupling constant ratios one needs the pseudo-ϵ expansion for the value of renormalized quartic coupling g_4 at criticality, i. e. for the Wilson fixed point location g_4^*. Such an expansion may be found by an insertion of the formal small parameter τ into the first (linear) term of the perturbative series for β function and by solving the equation for the fixed point coordinate $\beta(g_4, \tau) = 0$ iteratively in τ. The pseudo-ϵ expansion for g_4^* thus obtained was reported in [27]. Substituting it into RG series (2), (3), (4) and reexpanding corresponding expressions in powers of τ we arrive to

$$R_6^* = 2\tau(1 - 0.244170096\tau + 0.120059430\tau^2 - 0.1075143\tau^3 + 0.1289821\tau^4). \tag{6}$$

$$R_8^* = -9\tau(1 - 1.98491084\tau + 1.76113570\tau^2 - 1.9665851\tau^3 + 2.741546\tau^4). \tag{7}$$

$$R_{10}^* = 54\tau(1 - 4.02194787\tau + 7.55009811\tau^2 - 11.784685\tau^3 + 20.05363\tau^4). \tag{8}$$

Since these expansions originate from divergent RG series and their coefficients do not monotonically decrease, proper numerical estimates may be extracted from them by means of some resummation procedure. Below we apply the Padé–Borel–Leroy resummation technique which proved to be rather efficient when used to evaluate critical exponents on the base of corresponding pseudo-ϵ expansions [28, 35, 38–40]. This technique employs the Borel–Leroy transformation of the original series

$$f(x) = \sum_{i=0}^{\infty} c_i x^i = \int_0^{\infty} e^{-t} t^b F(xt) dt, \qquad F(y) = \sum_{i=0}^{\infty} \frac{c_i}{(i+b)!} y^i \ , \tag{9}$$

with subsequent analytical continuation of the Borel–Leroy transform $F(y)$ with a help of Padé approximants. The shift parameter b is commonly used for optimization of the resummation procedure.

3 Numerical results

3.1 Sextic universal ratio

It is easy to see that the series (6) should demonstrate good approximating properties because of the smallness of its coefficients. So, one can expect that use of Padé–Borel–Leroy resummation technique will lead to high-precision numerical results in this case. To get most comprehensive information about the numerical value R_6^* and its accuracy we employ all the non-trivial Padé approximants except for the approximant [1/4] which turned out to be spoilt by a positive axis pole for the relevant values of b. The Padé–Borel–Leroy estimates of R_6^* given by approximants [4/1], [3/2], and [2/3] under various b are listed in Table 1. As is well known, diagonal and near-diagonal Padé approximants possess the best approximating properties. That is why the results obtained with a help of approximants [3/2] and [2/3] are expected to be the most reliable ones. Indeed, for various values of b the numbers given by mentioned near-diagonal approximants are very close. Moreover, under $b = 4$ these numbers coincide: as seen from figure 1 curves $R_6^*(b)_{[3/2]}$ and $R_6^*(b)_{[2/3]}$ touch (do not cross) each other at this point. Then it is natural to take the value $R_6^* = 1.6488$ corresponding to the touch point as a final pseudo-ϵ expansion estimate for the sextic coupling constant ratio at criticality.

Table 1. Padé–Borel–Leroy estimates of R_6^* under various values of the shift parameter b.

b	0	1	2	3	4
[4/1]	1.6485	1.6494	1.6500	1.6505	1.6509
[3/2]	1.64786	1.64822	1.64847	1.64867	1.6488294
[2/3]	1.64819	1.64832	1.64851	1.64868	1.6488294
b	5	7	10	15	20
[4/1]	1.6512	1.6516	1.6521	1.6526	1.65287
[3/2]	1.648958	1.64915	1.64936	1.64957	1.64970
[2/3]	1.648961	1.64917	1.64940	1.64964	1.64980

What may be referred to as an accuracy or, more precisely, an apparent accuracy of the numerical value thus obtained? It looks reasonable to accept as an error bar a range of variation of R_6^* values given by the most stable – [2/3] – Padé–Borel–Leroy approximant under b varying within the whole interval $[0, \infty)$. For b running from 0 to infinity $R_6^*(b)_{[2/3]}$ grows from 1.6482 to 1.6502. Hence, we adopt

$$R_6^* = 1.6488 \pm 0.0014. \tag{10}$$

This number is consistent with the five-loop 3D RG estimate $R_6^* = 1.644 \pm 0.006$ [10] and is in a brilliant agreement with the value $R_6^* = 1.649 \pm 0.002$ extracted from advanced lattice calculations [25].

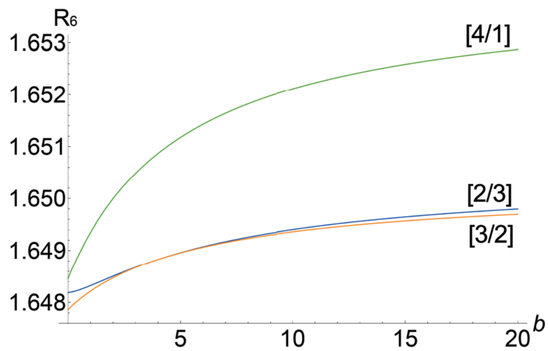

Figure 1. The values of R_6^* as functions of the parameter b obtained on the base of various Padé approximants.

3.2 Octic universal ratio

In the case of the eighth-order coupling we have pseudo-ϵ expansion with obviously less favorable structure. As seen from Eq. (7), the coefficients of the series in brackets are not small and grow what complicates obtaining of numerical estimates under the physical value of the expansion parameter $\tau = 1$. To overcome this difficulty we use the Padé–Borel–Leroy resummation technique addressing as before the most reliable approximants. Since approximant [3/2] suffers from the dangerous (positive-axis) pole, approximants [4/1] and [2/3] are taken as working ones. The numerical values of R_8^* given by these approximants are presented in Table 2. One can see that the estimates given by working

Table 2. Padé–Borel–Leroy estimates of R_8^* as functions of the shift parameter b.

b	0	1	2	3	4
[4/1]	0.89064	0.81273	0.75638	0.71374	0.68036
[2/3]	0.89163	0.82032	0.74242	0.67275	0.61365
b	5	7	10	15	20
[4/1]	0.65352	0.61304	0.57230	0.53131	0.50661
[2/3]	0.56408	0.48711	0.40829	0.32883	0.28125

approximants practically coincide under $b = 0$. The number corresponding to this value of the shift parameter is equal to 0.89. It may be referred to as a final pseudo-ϵ expansion estimate for R_8^*.

In contrast to the previous case, Padé–Borel–Leroy estimates for R_8^* are sensitive to the shift parameter. Figure 2 clearly demonstrates this fact. In principle, it does not look surprising since there is a big difference between the lowest-order ($R_8^* = -9$, see Eq. (7)) and final estimates. Since the higher-order perturbative contributions turn out to decrease the modulo of R_8^* by an order of magnitude and change its sign, the final estimate arises as a small difference of big numbers and its accuracy can not be high. Despite of this, our final result $R_8^* = 0.89$ differs only slightly from to the values $R_8^* = 0.871$, $R_8^* = 0.857$ given by the lattice [25] and field-theoretical [10] calculations.

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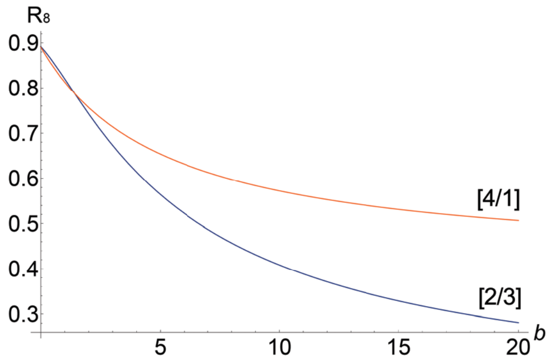

Figure 2. The numerical values of R_8^* given by two working approximants as functions of parameter b.

3.3 Universal ratio of tenth order

For the ratio R_{10} the situation is really dramatic. As can be seen from (8) the coefficients of corresponding pseudo-ϵ expansion rapidly grow up. On the other hand, the rate of their growth is much smaller than that of the original RG series (4). That is why addressing the Padé–Borel–Leroy resummation procedure in this case a priori does not look meaningless. We processed the series (8) with a help of the technique mentioned using all non-trivial Padé approximants – [4/1], [3/2], [2/3] and [1/4]. Obtained estimates as functions of the shift parameter b are listed in Table 3. One can see from this Table that the numbers given by various Padé approximants are strongly scattered. Moreover, all the Padé–Borel–Leroy estimates are very sensitive to the parameter b. In such a situation the way to determine the optimal value of b we used above could not lead to proper results.

Table 3. Padé–Borel–Leroy estimates of R_{10}^* as functions of the shift parameter b.

b	0	1	2	3	4
[4/1]	−9.2984	−6.3268	−4.1835	−2.5655	−1.3010
[3/2]	−1.6574	−1.7327	−1.4839	−1.0656	−0.5518
[2/3]	−10.1211	−9.0282	−8.4022	−7.9887	−7.6891
[1/4]	4.4362	3.0367	2.3928	2.0355	1.8126
b	5	7	10	15	20
[4/1]	−0.2858	1.2428	2.7778	4.3188	5.2460
[3/2]	0.0180	1.2432	3.14965	6.2645	9.1808
[2/3]	−7.4582	−7.1195	−6.7817	−6.4372	−6.2248
[1/4]	1.6621	1.4739	1.3222	1.1993	1.1369

For instance, although under $b = 7$ the numbers given by approximants [4/1] and [3/2] almost coincide and are close to that given by approximant [1/4], corresponding value $R_{10}^* = 1.24$ is in a marked disagreement with the alternative field-theoretical estimates: $R_{10}^* = -1.1 \pm 0.1$ [10], $R_{10}^* = -2.1 \pm 1.3$ [10], $R_{10}^* = -1.8 \pm 1.4$ [16], $R_{10}^* = -2.3 \pm 1.6$ [13], $R_{10}^* = -1.1 \pm 0.1$ [13], with the results of lattice calculations: $R_{10}^* = -0.75 \pm 0.38$ [12], $R_{10}^* = -1.2 \pm 0.4$ [18, 23], $R_{10}^* = -1.39 \pm 0.4$ [25], and with the "exact RG" prediction $R_{10}^* = -1.7 \pm 0.4$ [11]. So, the obvious improvement of the original RG series for R_{10}^* the pseudo-ϵ expansion technique provides turns out to be insufficient to make τ-series suitable for getting numerical estimates, at least within the Padé–Borel–Leroy resummation ideology.

At the same time, it is worthy to note that if one takes some other criterium for fixing the optimal value of b, a meaningful estimate for R^*_{10} can be found. Namely, as seen from Figure 3, the near-diagonal approximant [3/2] as function of b has a local minimum forming the "domain of stability" (plateau) around $b = 1$. The value $R^*_{10} = -1.73$ corresponding to this minimum is in agreement with many of the field-theoretical and lattice estimates mentioned above. Of course, this fact can not be considered as a serious argument in favor of using Padé–Borel–Leroy resummed pseudo-ϵ expansion (8) for evaluation of R^*_{10}.

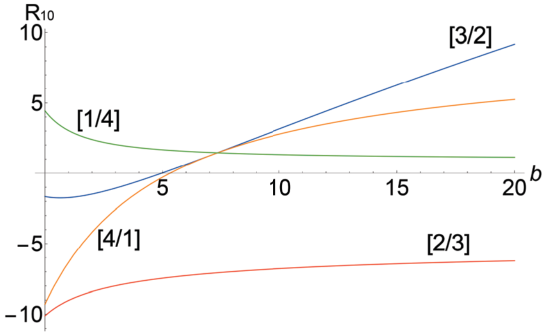

Figure 3. The dependence of the numerical values of R^*_{10} given by different approximants on parameter b.

4 Conclusion

To summarize, we have calculated the five-loop pseudo-ϵ expansions of the universal ratios R^*_6, R^*_8, R^*_{10} for 3D Euclidean scalar φ^4 field theory. To find numerical estimates of these quantities the Padé–Borel–Leroy resummation technique has been applied. The pseudo-ϵ expansion for R^*_6 has proven to be very convenient for getting accurate numerical estimates. For the octic coupling the situation is much less favorable but it is still possible to arrive to proper numerical results. The coefficients of the pseudo-ϵ expansion for R^*_{10} turned out to be rapidly increasing what leads to strong scattering of corresponding numerical estimates and makes the Padé-Borel-Leroy resummation technique practically useless in this case. We plan to use alternative, more advanced resummation procedures for evaluation of the quantities of interest in near future.

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