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Harmonic superspaces for $\mathcal{N} = (1,1), 6D$ SYM theory

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Abstract. This is a short account of the off-shell $\mathcal{N} = (1,0)$ and on-shell $\mathcal{N} = (1,1), 6D$ harmonic superspace formalism and its applications for the analysis of higher-dimension invariants in $\mathcal{N} = (1,1)$ SYM theory.

1 Motivations and contents

For last years, there is a permanent interest in the maximally extended (with 16 supercharges) supersymmetric gauge theories in diverse dimensions (see, e.g., [1]),

$$\mathcal{N} = 4, 4D \implies \mathcal{N} = (1,1), 6D \implies \mathcal{N} = (1,0), 10D\,.$$

The famous $\mathcal{N} = 4, 4D$ SYM theory was the first example of an UV *finite* theory. Perhaps, it is also *completely integrable* [2]. The $\mathcal{N} = (1,1), 6D$ SYM is not renormalizable by formal counting (the coupling constant is dimensionful) but it is also expected to possess unique properties. In particular, it respects the so called "dual conformal symmetry", like its $4D$ counterpart [3]. It provides the effective theory descriptions of some particular low energy sectors of string theory, such as D5-brane dynamics. The full effective action of D5-brane, generalizing the $\mathcal{N} = (1,1)$ SYM action, was conjectured to be of non-abelian Born-Infeld type [4, 5]. The $\mathcal{N} = (1,1)$ SYM is anomaly free, as distinct from $\mathcal{N} = (1,0)$ SYM.

The $\mathcal{N} = (1,1)$ and $\mathcal{N} = (1,0)$ SYM theories can be viewed as a laboratory for studying $\mathcal{N} = 8$ supergravity and its some lower $\mathcal{N}$ analogs, which are also non-renormalizable.

The recent perturbative calculations in $\mathcal{N} = (1,1)$ SYM show a lot of unexpected cancelations of the UV divergencies. The theory is UV finite up to 2 loops, while at 3 loops only a single-trace (planar) counterterm of canonical dim 10 is required. The allowed double-trace (non-planar) counterterms do not appear [6] - [8]. Various arguments to explain this were put forward [9] - [12], though the complete understanding is still lacking. Some new non-renormalization theorems could be expected in this connection.

The maximal off-shell supersymmetry one can gain in $6D$ is $\mathcal{N} = (1,0)$ supersymmetry. The natural off-shell formulation of $\mathcal{N} = (1,0)$ SYM theory is achieved in harmonic $\mathcal{N} = (1,0), 6D$ superspace [13, 14] as a generalization of the harmonic $\mathcal{N} = 2, 4D$ one [15, 16]. This harmonic $6D$ formalism was further developed in [17] - [20] and [21]. The $\mathcal{N} = (1,1)$ SYM theory in the harmonic formulation is a hybrid of two $\mathcal{N} = (1,0)$ theories, [$\mathcal{N} = (1,1)$ SYM] = [$\mathcal{N} = (1,0)$ SYM] + [$6D$ hypermultiplets], with the second hidden on-shell $\mathcal{N} = (0,1)$ supersymmetry. How to construct higher-dimension $\mathcal{N} = (1,1)$ invariants in the $\mathcal{N} = (1,0)$ superfield approach?

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One way is the "brute-force" method. One starts with the appropriate dimension $\mathcal{N} = (1,0)$ SYM invariant and then completes it to $\mathcal{N} = (1,1)$ invariant by adding the proper hypermultiplet terms. This approach is very complicated technically.

The things are simplified due to the fact that for finding superfield counterterms it is enough to stay on the mass shell. In a recent paper [21] a new approach to constructing higher-dimension $\mathcal{N} = (1,1)$ invariants was developed. It uses the concept of the *on-shell* $\mathcal{N} = (1,1)$ harmonic superspace with the double set of the harmonic variables $u^\pm_i, u^{\hat\pm}_A, i = 1,2; A = 1,2$ [22]. The novel point of the construction in [21] is solving the $\mathcal{N} = (1,1)$ SYM constraints [23, 24] through $\mathcal{N} = (1,0)$ superfields. The $\underline{d = 8}$ and $\underline{d = 10}$ invariants were built in a simple way and an essential difference between the single- and double-trace $\underline{d = 10}$ invariants was found. The present contribution is a brief account of the 6D harmonic methods, with the focus on their recent uses in [21].

2 6D superspaces and superfields

2.1 6D superspaces

- The standard $\mathcal{N} = (1,0), 6D$ superspace is parametrized by the coordinates:

$$z = (x^M, \theta^a_i), \quad M = 0,\ldots,5, \ a = 1,\ldots,4, \ i = 1,2, \tag{2.1}$$

- The harmonic $\mathcal{N} = (1,0)$ superspace is obtained by adding $SU(2)$ harmonics to (2.1):

$$Z := (z,u) = (x^M, \theta^a_i, u^{\pm i}), \quad u^-_i = (u^+_i)^*, u^{+i}u^-_i = 1, \ u^{\pm i} \in SU(2)_R/U(1). \tag{2.2}$$

- The *analytic* $\mathcal{N} = (1,0)$ superspace is an invariant subspace of (2.2):

$$\zeta := (x^M_{(\mathrm{an})}, \theta^{+a}, u^{\pm i}) \subset Z, \quad x^M_{(\mathrm{an})} = x^M + \frac{i}{2}\theta^a_k \gamma^M_{ab} \theta^b_l u^{+k} u^{-l}, \quad \theta^{\pm a} = \theta^a_i u^{\pm i}. \tag{2.3}$$

The differential operators in the analytic basis $Z_A := (x^M_{(\mathrm{an})}, \theta^{+a}, u^{\pm i}, \theta^{-a})$ are defined as

$$\begin{aligned}
&D^+_a = \partial_{-a}\,, \ D^-_a = -\partial_{+a} - 2i\theta^{-b}\partial_{ab}\,, \ D^0 = u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}} + \theta^{+a}\partial_{+a} - \theta^{-a}\partial_{-a} \\
&D^{++} = \partial^{++} + i\theta^{+a}\theta^{+b}\partial_{ab} + \theta^{+a}\partial_{-a}\,, \ D^{--} = \partial^{--} + i\theta^{-a}\theta^{-b}\partial_{ab} + \theta^{-a}\partial_{+a}\,, \\
&\partial_{\pm a}\theta^{\pm b} = \delta^b_a\,, \quad \partial^{++} = u^{+i}\frac{\partial}{\partial u^{-i}}, \quad \partial^{--} = u^{-i}\frac{\partial}{\partial u^{+i}}\,.
\end{aligned} \tag{2.4}$$

2.2 Basic superfields

The basic object of $\mathcal{N} = (1,0)$ SYM theory is the analytic gauge connection $V^{++}(\zeta)$

$$\nabla^{++} = D^{++} + V^{++}\,, \quad \delta V^{++} = -\nabla^{++}\Lambda\,, \quad \Lambda = \Lambda(\zeta)\,. \tag{2.5}$$

The second harmonic (non-analytic) connection $V^{--}(Z)$,

$$\nabla^{--} = D^{--} + V^{--}\,, \quad \delta V^{--} = -\nabla^{--}\Lambda\,,$$

is related to V^{++} by the harmonic flatness condition

$$\begin{aligned}
&[\nabla^{++}, \nabla^{--}] = D^0 \ \Leftrightarrow \ D^{++}V^{--} - D^{--}V^{++} + [V^{++}, V^{--}] = 0 \\
&\Rightarrow \ V^{--} = V^{--}(V^{++}, u^\pm)\,.
\end{aligned} \tag{2.6}$$

The off-shell content of $\mathcal{N}=(1,0)$ SYM theory is revealed in the Wess-Zumino gauge:

$$V^{++} \;=\; \theta^{+a}\theta^{+b}A_{ab} + 2(\theta^{+})^{3}_{a}\lambda^{ai}u^{-}_{i} - 3(\theta^{+})^{4}\mathcal{D}^{ik}u^{-}_{i}u^{-}_{k}\,. \tag{2.7}$$

Here A_{ab} is the gauge field, λ^{ai} is the gaugino and $\mathcal{D}^{ik}=\mathcal{D}^{ki}$ are the auxiliary fields.

The $\mathcal{N}=(1,0)$ SYM covariant derivatives are given by the expressions

$$\begin{aligned}
&\nabla^{-}_{a} = [\nabla^{--}, D^{+}_{a}] = D^{-}_{a} + \mathcal{A}^{-}_{a}, \qquad \nabla_{ab} = \frac{1}{2i}[D^{+}_{a}, \nabla^{-}_{b}] = \partial_{ab} + \mathcal{A}_{ab}\,,\\
&\mathcal{A}^{-}_{a}(V) = -D^{+}_{a}V^{--}, \quad \mathcal{A}_{ab}(V) = \frac{i}{2}D^{+}_{a}D^{+}_{b}V^{--},\\
&[\nabla^{++}, \nabla^{-}_{a}] = D^{+}_{a}\,, \quad [\nabla^{++}, D^{+}_{a}] = [\nabla^{--}, \nabla^{-}_{a}] = [\nabla^{\pm\pm}, \nabla_{ab}] = 0\,.
\end{aligned} \tag{2.8}$$

The covariant superfield strengths are defined as

$$\begin{aligned}
&[D^{+}_{a}, \nabla_{bc}] \;=\; \frac{i}{2}\varepsilon_{abcd}W^{+d}\,, \qquad [\nabla^{-}_{a}, \nabla_{bc}] \;=\; \frac{i}{2}\varepsilon_{abcd}W^{-d}\,,\\
&W^{+a} = -\frac{1}{6}\varepsilon^{abcd}D^{+}_{b}D^{+}_{c}D^{+}_{d}V^{--}\,, \quad W^{-a} := \nabla^{--}W^{+a}\,,\\
&\nabla^{++}W^{+a} = \nabla^{--}W^{-a} \;=\; 0\,, \quad \nabla^{++}W^{-a} = W^{+a}\,,\\
&D^{+}_{b}W^{+a} = \delta^{a}_{b}F^{++}\,, \quad F^{++} = \frac{1}{4}D^{+}_{a}W^{+a} = (D^{+})^{4}V^{--}\,,\\
&\nabla^{++}F^{++} = 0\,, \quad D^{+}_{a}F^{++} = 0\,.
\end{aligned} \tag{2.9}$$

The hypermultiplet is accommodated by the analytic superfield $q^{+A}(\zeta)\,, (A=1,2)$:

$$q^{+A}(\zeta) = q^{iA}(x)u^{+}_{i} - \theta^{+a}\psi^{A}_{a}(x) + \text{ An infinite tail of auxiliary fields}\,. \tag{2.10}$$

2.3 $\mathcal{N}=(1,0)$ superfield actions

The $\mathcal{N}=(1,0)$ SYM action was constructed by Boris Zupnik [14]:

$$\begin{aligned}
&S^{SYM} = \frac{1}{f^{2}}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\mathrm{Tr}\int d^{6}x\,d^{8}\theta\,du_{1}\ldots du_{n}\frac{V^{++}(z,u_{1})\ldots V^{++}(z,u_{n})}{(u^{+}_{1}u^{+}_{2})\ldots(u^{+}_{n}u^{+}_{1})}\,,\\
&\delta S^{SYM} = 0 \;\Rightarrow\; F^{++} = 0\,.
\end{aligned} \tag{2.11}$$

Here, $(u^{+}_{1}u^{+}_{2})^{-1}, \ldots (u^{+}_{n}u^{+}_{1})^{-1}$ are harmonic distributions [16].

The hypermultiplet action, with q^{+A} in adjoint of the gauge group, is written as

$$\begin{aligned}
&S^{q} = -\frac{1}{2f^{2}}\mathrm{Tr}\int d\zeta^{-4}q^{+A}\nabla^{++}q^{+}_{A}\,, \quad \nabla^{++}q^{+}_{A} = D^{++}q^{+}_{A} + [V^{++}, q^{+}_{A}]\,,\\
&\delta S^{q} = 0 \;\Rightarrow\; \nabla^{++}q^{+A} = 0\,.
\end{aligned} \tag{2.12}$$

The $\mathcal{N}=(1,0)$ superfield form of the $\mathcal{N}=(1,1)$ SYM action ia a sum of the two superfield actions given above:

$$\begin{aligned}
&S^{(V+q)} = S^{SYM} + S^{q} = \frac{1}{f^{2}}\left(\int dZ\mathcal{L}^{\mathrm{SYM}} - \frac{1}{2}\mathrm{Tr}\int d\zeta^{-4}q^{+A}\nabla^{++}q^{+}_{A}\right),\\
&\delta S^{(V+q)} = 0 \;\Rightarrow\; F^{++} + \frac{1}{2}[q^{+A}, q^{+}_{A}] = 0\,, \quad \nabla^{++}q^{+A} = 0\,.
\end{aligned} \tag{2.13}$$

It is invariant under the second hidden $\mathcal{N}=(0,1)$ supersymmetry:

$$\delta V^{++} = \epsilon^{+A}q^{+}_{A}\,, \quad \delta q^{+A} = -(D^{+})^{4}(\epsilon^{-}_{A}V^{--})\,, \quad \epsilon^{\pm}_{A} = \epsilon_{aA}\theta^{\pm a}\,. \tag{2.14}$$

These transformations have the correct closure only on shell.

3 Higher-dimensional invariants

3.1 Dimension $\underline{d=6}$

In the pure $\mathcal{N}=(1,0)$ SYM the $\underline{d=6}$ invariant is defined uniquely [17]:

$$S^{(6)}_{SYM} = \frac{1}{2g^2}\mathrm{Tr}\int d\zeta^{-4}du\,(F^{++})^2 \;\sim\; \mathrm{Tr}\int d^6x[(\nabla^M F_{ML})^2+\ldots]. \tag{3.1}$$

It vanishes on shell, when $F^{++}=0$. Using the results of [18], its $\mathcal{N}=(1,1)$ completion is defined up to a real parameter

$$\mathcal{L}^{d=6} = \frac{1}{2g^2}\mathrm{Tr}\int dud\zeta^{-4}\left(F^{++}+\frac{1}{2}[q^{+A},q^+_A]\right)\left(F^{++}+2\beta[q^{+A},q^+_A]\right). \tag{3.2}$$

But it vanishes on the full $\mathcal{N}=(1,1)$ SYM mass shell! This proves the *one-loop finiteness* of $\mathcal{N}=(1,1)$ SYM theory.

3.2 Dimension $\underline{d=8}$

All superfield operators of the canonical dimension $\underline{d=8}$ in the $\mathcal{N}=(1,0)$ SYM theory vanish on shell, in accord with ref. [24]. Can adding the hypermultiplet terms change something? We have found that there exist no $\mathcal{N}=(1,0)$ off-shell invariants of the dimension $\underline{d=8}$ which would respect the on-shell $\mathcal{N}=(1,1)$ invariance.

Surprisingly, the $\underline{d=8}$ superfield expression which is non-vanishing on shell and respects the on-shell $\mathcal{N}=(1,1)$ supersymmetry can be constructed by *giving up* the requirement of *off-shell* $\mathcal{N}=(1,0)$ supersymmetry.

An example of such an invariant in $\mathcal{N}=(1,0)$ SYM theory is very simple

$$\tilde{S}^{(8)}_1 \sim \mathrm{Tr}\int d\zeta^{-4}\,\varepsilon_{abcd}W^{+a}W^{+b}W^{+c}W^{+d}. \tag{3.3}$$

Indeed, $D^+_a W^{+b} = \delta^b_a F^{++}$, which vanishes on shell, with $F^{++}=0$. Thus, W^{+a} is on-shell analytic, and so the above action respects $\mathcal{N}=(1,0)$ supersymmetry on shell. Also, a double-trace on-shell invariant exists.

These invariants admit $\mathcal{N}=(1,1)$ completions. For (3.3) such a completion reads

$$\begin{aligned}\mathcal{L}^{+4}_{(1,1)} = {}& \mathrm{Tr}_{(S)}\Big\{\frac{1}{4}\varepsilon_{abcd}W^{+a}W^{+b}W^{+c}W^{+d} + 3iq^{+A}\nabla_{ab}q^+_A W^{+a}W^{+b}\\ &- q^{+A}\nabla_{ab}q^+_A\, q^{+B}\nabla^{ab}q^+_B - W^{+a}[D^+_a q^-_A, q^+_B]q^{+A}q^{+B}\\ &- \frac{1}{2}[q^{+C},q^+_C][q^-_A,q^+_B]q^{+A}q^{+B}\Big\}.\end{aligned} \tag{3.4}$$

Here, $\mathrm{Tr}_{(S)}$ stands for the *symmetrized* trace. This Lagrangian is analytic, $D^+_a\mathcal{L}^{+4}_{(1,1)}=0$, on the full mass shell $F^{++}+\frac{1}{2}[q^{+A},q^+_A]=0$, $\nabla^{++}q^{+A}=0$, and so it is on-shell $\mathcal{N}=(1,1)$ supersymmetric.

Though the nontrivial on-shell $\underline{d=8}$ invariants exist, the perturbative expansion for the amplitudes in the $\mathcal{N}=(1,1)$ SYM theory does not involve divergences at the two-loop level. The reason is that these $\underline{d=8}$ invariants do *not* possess the full off-shell $\mathcal{N}=(1,0)$ supersymmetry which the physically relevant counterterms should obey.

4 $\mathcal{N} = (1,1)$ on-shell harmonic superspace

Despite the fact that the $d = 8$ terms mentioned above cannot appear as counterterms in $\mathcal{N} = (1,1)$ SYM theory, they can come out, e.g., as quantum corrections to the effective Wilsonian action. For the pure $\mathcal{N} = (1,0)$ SYM theory this was recently observed in [20]. It was desirable to work out some systematic way of constructing such higher-order $\mathcal{N} = (1,1)$ invariants. This becomes possible within the on-shell harmonic $\mathcal{N} = (1,1)$ superspace.

As the first step, extend the $\mathcal{N} = (1,0)$ superspace to the $\mathcal{N} = (1,1)$ one,

$$z = (x^{ab}, \theta^a_i) \;\Rightarrow\; \hat{z} = (x^{ab}, \theta^a_i, \hat{\theta}^A_a)\,. \tag{4.1}$$

Then we define the covariant spinor derivatives,

$$\nabla^i_a = \frac{\partial}{\partial \theta^a_i} - i\theta^{bi}\partial_{ab} + \mathcal{A}^i_a\,, \quad \hat{\nabla}^{aA} = \frac{\partial}{\partial \hat{\theta}_{Aa}} - i\hat{\theta}^A_b\partial^{ab} + \hat{\mathcal{A}}^{aA}\,. \tag{4.2}$$

The constraints of the $\mathcal{N} = (1,1)$ SYM theory can now be written as follows [23, 24]:

$$\begin{aligned}
&\{\nabla^{(i}_a, \nabla^{j)}_b\} = \{\hat{\nabla}^{a(A}, \hat{\nabla}^{bB)}\} = 0\,, \quad \{\nabla^i_a, \hat{\nabla}^{bA}\} = \delta^b_a \phi^{iA} \\
&\Rightarrow \quad \nabla^{(i}_a \phi^{j)A} = \hat{\nabla}^{a(A}\phi^{B)i} = 0 \quad \text{(By Bianchis)}\,.
\end{aligned} \tag{4.3}$$

As the next step, we introduce the $\mathcal{N} = (1,1)$ harmonic superspace [22],

$$Z = (x^{ab}, \theta^a_i, u^\pm_k) \;\Rightarrow\; \hat{Z} = (x^{ab}, \theta^a_i, \hat{\theta}^A_b, u^\pm_k, u^{\hat\pm}_A)\,, \tag{4.4}$$

pass to the analytic basis and choose the "hatted" spinor derivatives short, $\nabla^{\hat{+}a} = D^{\hat{+}a} = \frac{\partial}{\partial \theta^{\hat{-}}_a}$. The set of constraints (4.3) is rewritten as

$$\begin{aligned}
&\{\nabla^+_a, \nabla^+_b\} = 0\,, \quad \{D^{\hat{+}a}, D^{\hat{+}b}\} = 0\,, \quad \{\nabla^+_a, D^{\hat{+}b}\} = \delta^b_a \phi^{+\hat{+}}\,, \\
&[\nabla^{\hat{+}\hat{+}}, \nabla^+_a] = 0\,,\; [\tilde{\nabla}^{++}, \nabla^+_a] = 0\,,\; [\nabla^{\hat{+}\hat{+}}, D^{a\hat{+}}] = 0\,,\; [\tilde{\nabla}^{++}, D^{a\hat{+}}] = 0\,, \\
&[\tilde{\nabla}^{++}, \nabla^{\hat{+}\hat{+}}] = 0\,,
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
&\nabla^+_a = D^+_a + \mathcal{A}^+_a(\hat{Z})\,,\; \tilde{\nabla}^{++} = D^{++} + \tilde{V}^{++}(\hat{\zeta})\,, \quad \nabla^{\hat{+}\hat{+}} = D^{\hat{+}\hat{+}} + V^{\hat{+}\hat{+}}(\hat{\zeta})\,, \\
&\hat{\zeta} = (x^{ab}_{\text{an}}, \theta^{\pm a}, \theta^{\hat{+}}_c, u^\pm_i, u^{\hat\pm}_A)\,.
\end{aligned} \tag{4.6}$$

5 Solving $\mathcal{N} = (1,1)$ SYM constraints through $\mathcal{N} = (1,0)$ superfields

The starting point of our analysis in [21] was the WZ gauge for the extra connection $V^{\hat{+}\hat{+}}(\hat{\zeta})$

$$V^{\hat{+}\hat{+}} = i\theta^{\hat{+}}_a\theta^{\hat{+}}_b\hat{\mathcal{A}}^{ab} + \varepsilon^{abcd}\theta^{\hat{+}}_a\theta^{\hat{+}}_b\theta^{\hat{+}}_c\varphi^A_d u^{\hat{-}}_A + \varepsilon^{abcd}\theta^{\hat{+}}_a\theta^{\hat{+}}_b\theta^{\hat{+}}_c\theta^{\hat{+}}_d\mathcal{D}^{AB}u^{\hat{-}}_A u^{\hat{-}}_B\,, \tag{5.1}$$

where $\hat{\mathcal{A}}^{ab}, \varphi^A_d$ and $\mathcal{D}^{(AB)}$ are some $\mathcal{N} = (1,0)$ harmonic superfields, still arbitrary.

Then the above constraints are reduced to some sets of harmonic equations. We have solved them and, as the eventual result, obtained that the first harmonic connection V^{++} coincides with the previous $\mathcal{N} = (1,0)$ one, $V^{++} = V^{++}(\zeta)$, while the dependence of all other geometric $\mathcal{N} = (1,1)$ objects on the "hatted" variables is strictly fixed

$$\begin{aligned}
&V^{\hat{+}\hat{+}} = i\theta^{\hat{+}}_a\theta^{\hat{+}}_b\mathcal{A}^{ab} - \frac{1}{3}\epsilon^{abcd}\theta^{\hat{+}}_a\theta^{\hat{+}}_b\theta^{\hat{+}}_c D^+_d q^{-\hat{-}} + \frac{1}{8}\epsilon^{abcd}\theta^{\hat{+}}_a\theta^{\hat{+}}_b\theta^{\hat{+}}_c\theta^{\hat{+}}_d[q^{+\hat{-}}, q^{-\hat{-}}] \\
&\phi^{+\hat{+}} = q^{+\hat{+}} - \theta^{\hat{+}}_a W^{+a} - i\theta^{\hat{+}}_a\theta^{\hat{+}}_b\nabla^{ab}q^{+\hat{-}} + \frac{1}{6}\varepsilon^{abcd}\theta^{\hat{+}}_a\theta^{\hat{+}}_b\theta^{\hat{+}}_c[D^+_d q^{-\hat{-}}, q^{+\hat{-}}] \\
&+ \frac{1}{24}\varepsilon^{abcd}\theta^{\hat{+}}_a\theta^{\hat{+}}_b\theta^{\hat{+}}_c\theta^{\hat{+}}_d[q^{+\hat{-}}, [q^{+\hat{-}}, q^{-\hat{-}}]]\,.
\end{aligned} \tag{5.2}$$

Here, $q^{+\hat{\pm}} = q^{+A}(\zeta)u^{\hat{\pm}}_A$, $q^{-\hat{\pm}} = q^{-A}(\zeta)u^{\hat{\pm}}_A$ and $W^{+a}, q^{\pm A}$ are just the $\mathcal{N} = (1,0)$ superfields explored previously. In the course of solving the constraints, there naturally appear the superfield equations of motion

$$\nabla^{++}q^{+A} = 0\,, \quad F^{++} = \frac{1}{4}D^+_a W^{+a} = -\frac{1}{2}[q^{+A}, q^+_A]\,. \tag{5.3}$$

Also, the structure of the spinor covariant derivatives is fully fixed

$$\begin{aligned} \nabla^+_a &= D^+_a - \theta^{\hat{+}}_a q^{+\hat{-}} + \theta^{\hat{-}}_a \phi^{+\hat{+}}\,, \\ \nabla^-_a &= D^-_a - D^+_a V^{--} - \theta^{\hat{+}}_a q^{-\hat{-}} + \theta^{\hat{-}}_a \phi^{-\hat{+}}\,, \quad \phi^{-\hat{+}} = \nabla^{--}\phi^{+\hat{+}}\,. \end{aligned} \tag{5.4}$$

The advantage of using the constrained $\mathcal{N} = (1,1)$ strengths $\phi^{\pm\hat{+}}$ for constructing invariants is the simple transformation rules of $\phi^{\pm\hat{+}}$ under the hidden $\mathcal{N} = (0,1)$ supersymmetry

$$\delta\phi^{\pm\hat{+}} = -\epsilon^{\hat{+}}_a \frac{\partial}{\partial\theta^{\hat{+}}_a}\phi^{\pm\hat{+}} - 2i\epsilon^{\hat{-}}_a\theta^{\hat{+}}_b\partial^{ab}\phi^{\pm\hat{+}} - [\Lambda^{(comp)}, \phi^{\pm\hat{+}}]\,, \tag{5.5}$$

where $\Lambda^{(comp)}$ is some composite gauge parameter which does not contribute under trace.

6 Invariants in $\mathcal{N} = (1,1)$ superspace

The single-trace $\underline{d = 8}$ invariant (3.4) admits a simple rewriting in $\mathcal{N} = (1,1)$ superspace

$$S_{(1,1)} = \int d\zeta^{-4}\mathcal{L}^{+4}_{(1,1)}\,, \ \mathcal{L}^{+4}_{(1,1)} = -\mathrm{Tr}\,\frac{1}{4}\int d\hat{\zeta}^{-4}d\hat{u}\,(\phi^{+\hat{+}})^4, \quad d\hat{\zeta}^{-4} \sim (D^{\hat{-}})^4 \tag{6.1}$$

$$\delta\mathcal{L}^{+4}_{(1,1)} = -2i\partial^{ab}\mathrm{Tr}\int d\hat{\zeta}^{-4}d\hat{u}\left[\epsilon^{\hat{-}}_a\theta^{\hat{+}}_b\frac{1}{4}(\phi^{+\hat{+}})^4\right].$$

Analogously, the double-trace $\underline{d = 8}$ invariant is given by

$$\hat{\mathcal{L}}^{+4}_{(1,1)} = -\frac{1}{4}\int d\hat{\zeta}^{-4}d\hat{u}\,\mathrm{Tr}\,(\phi^{+\hat{+}})^2\,\mathrm{Tr}\,(\phi^{+\hat{+}})^2. \tag{6.2}$$

Now it is easy to construct the single- and double-trace $\underline{d = 10}$ invariants

$$\begin{aligned} S^{(10)}_1 &= \mathrm{Tr}\int dZd\hat{\zeta}^{-4}d\hat{u}\,(\phi^{+\hat{+}})^2(\phi^{-\hat{+}})^2, \quad \phi^{-\hat{+}} = \nabla^{--}\phi^{+\hat{+}}\,, \\ S^{(10)}_2 &= -\int dZd\hat{\zeta}^{-4}d\hat{u}\,\mathrm{Tr}\big(\phi^{+\hat{+}}\phi^{-\hat{+}}\big)\,\mathrm{Tr}\big(\phi^{+\hat{+}}\phi^{-\hat{+}}\big)\,. \end{aligned} \tag{6.3}$$

It is notable that the single-trace $\underline{d = 10}$ invariant admits a representation as an integral over the *full* $\mathcal{N} = (1,1)$ superspace

$$S^{(10)}_1 \sim \mathrm{Tr}\int dZd\hat{Z}d\hat{u}\,\phi^{+\hat{+}}\phi^{-\hat{-}}\,, \quad \phi^{-\hat{-}} = \nabla^{\hat{-}\hat{-}}\phi^{-\hat{+}}\,, \tag{6.4}$$

with $d\hat{Z} \sim (D^{\hat{-}})^4(D^{\hat{+}})^4$. On the other hand, the double-trace $\underline{d = 10}$ invariant *cannot* be written as the full integral and so looks as being UV *protected*.

This could explain why in the perturbative calculations of the amplitudes in the $\mathcal{N} = (1,1)$ SYM single-trace 3-loop divergence is seen, while no double-trace structures at the same order were observed [6], [7], [8]. However, this does not seem to be like the standard non-renormalization theorems because the quantum calculation of $\mathcal{N} = (1,0)$ supergraphs should give invariants in the *off-shell* $\mathcal{N} = (1,0)$ superspace, not in the *on-shell* $\mathcal{N} = (1,1)$ superspace. So the above property seems not enough to explain the absence of the double-trace divergences and some additional piece of reasoning is needed.

7 Summary and outlook

Following refs. [14], [17], [18] and [21], the off-shell $\mathcal{N} = (1,0)$ and on-shell $\mathcal{N} = (1,1)$ harmonic superfield approaches were reviewed and argued to be efficient for constructing higher-dimensional invariants in the $\mathcal{N} = (1,0)$ and $\mathcal{N} = (1,1)$ SYM theories. The $\mathcal{N} = (1,1)$ SYM constraints were solved in terms of harmonic $\mathcal{N} = (1,0)$ superfields. This allowed to explicitly construct the full set of the dimensions $\underline{d = 8}$ and $\underline{d = 10}$ superfield invariants possessing $\mathcal{N} = (1,1)$ on-shell supersymmetry.

All possible $\underline{d = 6}$ $\mathcal{N} = (1,1)$ invariants were demonstrated to be on-shell vanishing, thereby proving the UV finiteness of $\mathcal{N} = (1,1)$ SYM at one loop.

The off-shell $\underline{d = 8}$ invariants which would be non-vanishing on shell, are absent. Assuming that the $\mathcal{N} = (1,0)$ supergraphs yield integrals over the full $\mathcal{N} = (1,0)$ harmonic superspace, this means the absence of two-loop counterterms as well.

Two $\underline{d = 10}$ invariants were constructed as integrals over the whole $\mathcal{N} = (1,0)$ harmonic superspace. The single-trace invariant can be rewritten as an integral over the $\mathcal{N} = (1,1)$ superspace, while the double-trace one cannot. This property combined with an additional reasoning (e.g., based on the algebraic renormalization approach [25]) could explain why the double-trace invariant is UV protected.

Some further lines of development:

(a). To construct the $\underline{d \geq 12}$ invariants in the $\mathcal{N} = (1,1)$ SYM theory using the on-shell $\mathcal{N} = (1,1)$ harmonic superspace techniques.
(b). To apply the same method for constructing the Born-Infeld action with the manifest off-shell $\mathcal{N} = (1,0)$ and hidden on-shell $\mathcal{N} = (0,1)$ supersymmetries.
(c). To develop an analogous on-shell harmonic $\mathcal{N} = 4, 4D$ superspace approach to the $\mathcal{N} = 4, 4D$ SYM theory in the $\mathcal{N} = 2$ superfield formulation (by solving the $\mathcal{N} = 4$ SYM constraints in terms of $\mathcal{N} = 2$ superfields) and apply it to the problem of constructing the $\mathcal{N} = 4$ SYM effective action.
(d). Applications in supergravity? The absence of the double-trace divergent structures in the 3-loop amplitude in $\mathcal{N} = (1,1)$ SYM theory is similar to the absence of 3-loop and 4-loop divergences for the four-graviton amplitudes in $\mathcal{N} = 4, 4D$ and $\mathcal{N} = 5, 4D$ supergravities [26], [27], [28], [29]. All these UV divergence cancelations could find a common explanation within the harmonic superspace approach [1].

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