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Fermions in 5D brane world models

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Abstract. In the present manuscript the fermion fields in the background of 5D brane world models with compact extra dimension are examined. It is shown that the only case that allows one to perform the Kaluza-Klein decomposition in a mathematically consistent way without unnatural fine-tunings and possible pathologies, is the one which does not admit localization of the zero mode. The report is based on the results presented in [1].

1 Setup and equations of motion

Let us take a five-dimensional space-time with the compact extra dimension forming the orbifold S^1/Z_2 with the coordinate $-L \leq z \leq L$ and the points $-z$ and z identified. Let us consider the following form of the background metric, which is standard in five-dimensional brane world models:

$$ds^2 = e^{2\sigma(z)}\eta_{\mu\nu}dx^\mu dx^\nu - dz^2, \tag{1}$$

where $\sigma(-z) = \sigma(z)$. We do not specify the explicit form of the solution for $\sigma(z)$.

As a model-independent example, we consider a theory with the action of the general form

$$\begin{aligned} S = \int d^4xdz\sqrt{g}\Big(&E^M_N i\bar{\Psi}_1\Gamma^N\nabla_M\Psi_1 + E^M_N i\bar{\Psi}_2\Gamma^N\nabla_M\Psi_2 \\ &-F(z)\left(\bar{\Psi}_1\Psi_1 - \bar{\Psi}_2\Psi_2\right) - G(z)\left(\bar{\Psi}_2\Psi_1 + \bar{\Psi}_1\Psi_2\right)\Big), \end{aligned} \tag{2}$$

where $M, N = 0, 1, 2, 3, 5$, $\Gamma^\mu = \gamma^\mu$, $\Gamma^5 = i\gamma^5$, ∇_M is the covariant derivative containing the spin connection, E^M_N is the vielbein. The fields are also supposed to satisfy the orbifold symmetry conditions

$$\Psi_1(x, -z) = \gamma^5\Psi_1(x, z), \qquad \Psi_2(x, -z) = -\gamma^5\Psi_2(x, z), \tag{3}$$

the functions $F(z)$ and $G(z)$ are such that $F(-z) = -F(z)$ and $G(-z) = G(z)$. Due to the absence of chirality in five-dimensional space-time, we have to take two five-dimensional spinor fields Ψ_1 and Ψ_2 [2–4]. Here the function $F(z)$ is responsible for the localization of the zero Kaluza-Klein mode, whereas the function $G(z) \not\equiv 0$ provides its nonzero mass — for $G(z) \equiv 0$ there always exist the modes with zero four-dimensional mass [5].

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For the case of metric (1), action (2) can be rewritten in the form (see, for example, [4, 6] for the explicit form of the vielbein and spin connections)

$$S = \int d^4xdze^{4\sigma}\left(e^{-\sigma}i\bar{\Psi}_1\gamma^\mu\partial_\mu\Psi_1 - \bar{\Psi}_1\gamma^5\left(\partial_5+2\sigma'\right)\Psi_1 + e^{-\sigma}i\bar{\Psi}_2\gamma^\mu\partial_\mu\Psi_2\right.$$
$$\left.-\bar{\Psi}_2\gamma^5\left(\partial_5+2\sigma'\right)\Psi_2 - F(z)\left(\bar{\Psi}_1\Psi_1-\bar{\Psi}_2\Psi_2\right) - G(z)\left(\bar{\Psi}_2\Psi_1+\bar{\Psi}_1\Psi_2\right)\right).$$

The equations of motion for the fields Ψ_1 and Ψ_2 take the form

$$e^{-\sigma}i\gamma^\mu\partial_\mu\Psi_1 - \gamma^5\left(\partial_5+2\sigma'\right)\Psi_1 - F\Psi_1 - G\Psi_2 = 0, \tag{4}$$
$$e^{-\sigma}i\gamma^\mu\partial_\mu\Psi_2 - \gamma^5\left(\partial_5+2\sigma'\right)\Psi_2 + F\Psi_2 - G\Psi_1 = 0. \tag{5}$$

From (4) and (5) it is not difficult to obtain the second-order differential equations for the components of the fields Ψ_1 and Ψ_2:

$$-e^{-2\sigma}\Box\Psi_1 + \Psi_1'' + 5\sigma'\Psi_1' + (6\sigma'^2+2\sigma'')\Psi_1 - (G^2+F^2-e^{-\sigma}(e^\sigma F)'\gamma^5)\Psi_1$$
$$+e^{-\sigma}(e^\sigma G)'\gamma^5\Psi_2 = 0, \tag{6}$$
$$-e^{-2\sigma}\Box\Psi_2 + \Psi_2'' + 5\sigma'\Psi_2' + (6\sigma'^2+2\sigma'')\Psi_2 - (G^2+F^2+e^{-\sigma}(e^\sigma F)'\gamma^5)\Psi_2$$
$$+e^{-\sigma}(e^\sigma G)'\gamma^5\Psi_1 = 0. \tag{7}$$

We see that equations (6), (7) remain coupled in the general case, whereas we expect that in a consistent theory the corresponding equations should be five-dimensional second-order differential equations, which contain the derivatives in the four-dimensional coordinates only in the form $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ and provide a correct mass spectrum for the Kaluza-Klein modes. Equations for each component of the fields Ψ_1 and Ψ_2 separately can be obtained only for the special choice

$$G(z) \equiv Me^{-\sigma(z)}, \tag{8}$$

where M is a constant. In this case equations (6), (7), (4) and (5) give for the lowest mode

$$\Psi_1 = C_f\exp\left[-\int\limits_0^z F(y)dy - 2\sigma(z)\right]\psi_L(x), \quad i\gamma^\mu\partial_\mu\psi_L - M\psi_R = 0, \quad \gamma^5\psi_L = \psi_L, \tag{9}$$

$$\Psi_2 = C_f\exp\left[-\int\limits_0^z F(y)dy - 2\sigma(z)\right]\psi_R(x), \quad i\gamma^\mu\partial_\mu\psi_R - M\psi_L = 0, \quad \gamma^5\psi_R = -\psi_R, \tag{10}$$

where C_f is a normalization constant. The fields ψ_L and ψ_R are localized in the vicinity of the same point in the extra dimension because they have the same wave function; taken together they make up a four-dimensional Dirac fermion with mass M. Depending on the form of the function $F(z)$, the zero mode can be localized at any point of the orbifold (see, for example, [3, 5, 7]).

In order to decouple equations (6) and (7) for $(e^\sigma G)' \not\equiv 0$, first let us consider the left-handed parts of the spinor fields $\Psi_1^L = \gamma^5\Psi_1^L$, $\Psi_2^L = \gamma^5\Psi_2^L$. Equations (6) and (7) for Ψ_1^L and Ψ_2^L can be rewritten in the operator form as

$$\begin{pmatrix}\hat{L} & 0\\ 0 & \hat{L}\end{pmatrix}\begin{pmatrix}\Psi_1^L\\ \Psi_2^L\end{pmatrix} + \hat{\Lambda}_L\begin{pmatrix}\Psi_1^L\\ \Psi_2^L\end{pmatrix} = 0, \tag{11}$$

where the operator $\hat{L}$ and the matrix $\hat{\Lambda}_L$ are defined as

$$\hat{L} = -e^{-2\sigma}\Box + \partial_5^2 + 5\sigma'\partial_5 + 6\sigma'^2 + 2\sigma'' - G^2 - F^2, \qquad \hat{\Lambda}_L = \begin{pmatrix}e^{-\sigma}(e^\sigma F)' & e^{-\sigma}(e^\sigma G)'\\ e^{-\sigma}(e^\sigma G)' & -e^{-\sigma}(e^\sigma F)'\end{pmatrix}. \tag{12}$$

The form of equation (11) implies that the decoupling of the equations of motion for the components of the fermion fields is equivalent to the diagonalization of the matrix $\hat{\Lambda}_L$ (and the corresponding matrix $\hat{\Lambda}_R$). This can be done for [1]

$$F(z) \equiv 0, \tag{13}$$

$$G(z) \equiv \gamma \,\mathrm{sign}(z) F(z) + M e^{-\sigma}, \quad \text{any} \quad F(z); \tag{14}$$

$$F(z) \equiv \gamma \,\mathrm{sign}(z) e^{-\sigma}, \quad \text{any} \quad G(z); \tag{15}$$

$$G(z) \equiv K_1 \,\delta(z) + K_2 \,\delta(z - L) + M e^{-\sigma}, \quad \text{any} \quad F(z); \tag{16}$$

where γ, M, K_1 and K_2 are constants.

To examine in more detail the general case, in which the functions $F(z)$ and $G(z)$ do not satisfy the conditions (8) or (13)–(16), let us again, for simplicity, consider the left parts Ψ_1^L, Ψ_2^L of the spinor fields and represent equations (6) and (7) as

$$\hat{L}\Psi_1^L + e^{-\sigma}(e^{\sigma}F)'\Psi_1^L + e^{-\sigma}(e^{\sigma}G)'\Psi_2^L = 0, \tag{17}$$

$$\hat{L}\Psi_2^L - e^{-\sigma}(e^{\sigma}F)'\Psi_2^L + e^{-\sigma}(e^{\sigma}G)'\Psi_1^L = 0, \tag{18}$$

where the operator $\hat{L}$ is defined in (12). If the appropriate diagonalization of the matrix $\hat{\Lambda}_L$ in (12) is impossible, then the only obvious way is to obtain separate equations for the fields Ψ_1^L, Ψ_2^L. For example, equation for Ψ_1^L takes the form

$$\left(\hat{L} - e^{-\sigma}(e^{\sigma}F)'\right) \frac{\hat{L} + e^{-\sigma}(e^{\sigma}F)'}{e^{-\sigma}(e^{\sigma}G)'}\Psi_1^L - e^{-\sigma}(e^{\sigma}G)'\Psi_1^L = 0. \tag{19}$$

Analogous equations can be obtained for Ψ_2^L, Ψ_1^R, Ψ_2^R. Thus, we get fourth-order differential equations of motion (recall that the operator $\hat{L}$ contains second derivatives). In the general case such equations describe more degrees of freedom than the second-order differential equations and, in principle, may contain serious pathologies like tachyons or even ghosts [8] (see the detailed discussion of this issue in [1]). The appearance of the fourth-order equations of motion for the general form of $F(z)$ and $G(z)$ is a nonperturbative effect, so it is hard to believe that the perturbation analysis, which is often used to examine fermion sector in brane world models, can adequately describe the theory in the general case. Indeed, the perturbation theory can provide solutions for some of the physical degrees of freedom of the theory, which can be obtained perturbatively. However, the rest of the possible physical degrees of freedom may appear to be lost when one uses perturbation theory in such nontrivial cases. Thus, the general case with $(e^{\sigma}G)' \not\equiv 0$ and $F(z) \not\equiv 0$, naively leading to equations of form (19), should be carefully and thoroughly examined before considering its phenomenological consequences.

Equation (19) is still a fourth-order differential equation even for $F(z) \equiv 0$, whereas it was noted above that the system of equations (6) and (7) can be decoupled for $F(z) \equiv 0$. This would-be contradiction can be easily resolved — in fact, the corresponding fourth-order differential equations can be reduced to second-order differential equations [1].

2 Kaluza-Klein decomposition

Below the Kaluza-Klein decompositions will be performed for the following three cases:

1. $G(z) \equiv M e^{-\sigma(z)}$; any $F(z)$.
2. $F(z) \equiv 0$; any $G(z)$.
3. $G(z) \equiv K \,\delta(z - L)$; any $F(z)$.

The fine-tuned cases (14) and (15) appear to be more complicated and deserve an additional study.

2.1 $G(z) \equiv Me^{-\sigma(z)}$, any $F(z)$

It is convenient to represent the five-dimensional fields as [1]

$$\Psi_1 = \sum_{n=0}^{\infty}\left(f_n(z)\psi_L^n(x) + \sqrt{\frac{m_n+M}{m_n-M}}\tilde{f}_n(z)\hat{\psi}_R^n(x)\right), \quad \Psi_2 = \sum_{n=0}^{\infty}\left(f_n(z)\psi_R^n(x) - \sqrt{\frac{m_n+M}{m_n-M}}\tilde{f}_n(z)\hat{\psi}_L^n(x)\right), \tag{20}$$

where m_n is the Kaluza-Klein mass which will be defined later. According to the orbifold symmetry conditions (3) for the fields Ψ_1 and Ψ_2, the functions $f_n(z)$ and $\tilde{f}_n(z)$ satisfy the symmetry conditions $f_n(-z) = f_n(z), \tilde{f}_n(-z) = -\tilde{f}_n(z)$. Substituting the decomposition into equations (6), (7), we get the following equations for the wave functions $f_n(z)$, $\tilde{f}_n(z)$:

$$e^{-2\sigma}(m_n^2 - M^2)f_n + f_n'' + 5\sigma' f_n' + (6\sigma'^2 + 2\sigma'')f_n - (F^2 - \sigma' F - F')f_n = 0, \tag{21}$$
$$e^{-2\sigma}(m_n^2 - M^2)\tilde{f}_n + \tilde{f}_n'' + 5\sigma' \tilde{f}_n' + (6\sigma'^2 + 2\sigma'')\tilde{f}_n - (F^2 + \sigma' F + F')\tilde{f}_n = 0. \tag{22}$$

Since all the coefficients in (21) and (22) are all even in z and real, we can always find real solutions to these equations satisfying the symmetry conditions presented above. It is easy to show that whenever equations (21), (22) hold, the following system of equations also holds (see [1]):

$$f_n' + (2\sigma' + F)f_n = (m_n + M)e^{-\sigma}\tilde{f}_n, \qquad \tilde{f}_n' + (2\sigma' - F)\tilde{f}_n = -(m_n - M)e^{-\sigma}f_n. \tag{23}$$

With the help of (21) and (22) it is possible to show that the orthonormality conditions

$$\int_{-L}^{L} e^{3\sigma} f_n f_k dz = \delta_{nk}, \qquad \int_{-L}^{L} e^{3\sigma}\tilde{f}_n\tilde{f}_k dz = \frac{m_n - M}{m_n + M}\delta_{nk} \tag{24}$$

can be imposed, whereas $\int_{-L}^{L} e^{3\sigma} f_n \tilde{f}_k dz = 0$ for all n and k because of the symmetry conditions.

Substituting decomposition (20) into action (4), taking into account (23) and integrating over the coordinate of the extra dimension, we get the effective four-dimensional action

$$S_{eff} = \int d^4x\Bigg(i\bar{\psi}\gamma^\mu\partial_\mu\psi - M\bar{\psi}\psi + \sum_{n=1}^{\infty}\Big(i\bar{\psi}_n\gamma^\mu\partial_\mu\psi_n + i\bar{\hat{\psi}}_n\gamma^\mu\partial_\mu\hat{\psi}_n$$
$$-M(\bar{\psi}_n\psi_n - \bar{\hat{\psi}}_n\hat{\psi}_n) - \sqrt{m_n^2 - M^2}\left(\bar{\psi}_n\hat{\psi}_n + \bar{\hat{\psi}}_n\psi_n\right)\Big)\Bigg), \tag{25}$$

where $\psi = \psi_L^0 + \psi_R^0$ (see also equations (9) and (10)) and $\psi_n = \psi_L^n + \psi_R^n$, $\hat{\psi}_n = \hat{\psi}_L^n + \hat{\psi}_R^n$ for $n \geq 1$. To diagonalize the mass matrix, we will use the rotation

$$\psi_n(x) = \psi_{1,n}(x)\cos(\theta_n) + \psi_{2,n}(x)\sin(\theta_n), \qquad \hat{\psi}_n(x) = \psi_{1,n}(x)\sin(\theta_n) - \psi_{2,n}(x)\cos(\theta_n) \tag{26}$$

for $n \geq 1$, where $\tan(2\theta_n) = \frac{\sqrt{m_n^2 - M^2}}{M}$; supplemented by the subsequent redefinition $\psi_{2,n} \to \gamma^5\psi_{2,n}$. Finally, we get

$$S_{eff} = \int d^4x\left(i\bar{\psi}\gamma^\mu\partial_\mu\psi - M\bar{\psi}\psi + \sum_{n=1}^{\infty}\sum_{i=1}^{2}\left(i\bar{\psi}_{i,n}\gamma^\mu\partial_\mu\psi_{i,n} - m_n\bar{\psi}_{i,n}\psi_{i,n}\right)\right). \tag{27}$$

2.2 $F(z) \equiv 0$, any $G(z)$

From the very beginning it is convenient to use the combinations $\Psi_A = \frac{1}{\sqrt{2}}(\Psi_1 + \Psi_2)$, $\Psi_B = \frac{1}{\sqrt{2}}(\Psi_1 - \Psi_2)$. With these notations equations of motion, following from (6), (7) with $F(z) \equiv 0$, can be rewritten as

$$-e^{-2\sigma}\Box\Psi_{A,B} + \Psi''_{A,B} + 5\sigma'\Psi'_{A,B} + (6\sigma'^2 + 2\sigma'')\Psi_{A,B} - G^2\Psi_{A,B} \pm (G' + \sigma' G)\gamma^5\Psi_{A,B} = 0. \tag{28}$$

Solutions to these equations, corresponding to a four-dimensional mass m, can be represented as

$$\Psi_A = f(z)\psi^A_L(x) + f(-z)\psi^A_R(x), \qquad \Psi_B = f(z)\psi^B_R(x) + f(-z)\psi^B_L(x), \tag{29}$$

where the equation for the function $f(z)$, corresponding to the four-dimensional mass m, takes the form

$$e^{-2\sigma}m^2 f(z) + f''(z) + 5\sigma' f'(z) + (6\sigma'^2 + 2\sigma'')f(z) - (G^2 - \sigma' G - G')f(z) = 0. \tag{30}$$

According to the general theory [9], the solutions to equation (30) make up an orthonormal set of eigenfunctions $f_n(z)$, the lowest eigenvalue m_0 being simple. One can also show that $m_n^2 \geq 0$.

The Kaluza-Klein decomposition for the fields Ψ_1 and Ψ_2 take the form [1]

$$\Psi_1 = \frac{1}{\sqrt{2}}\left(f_{+,0}(z)\psi_L(x) - f_{-,0}(z)\psi_R(x) + \sum_{n=1}^{\infty}\sum_{k=1}^{2}\left(f_{+,n,k}(z)\psi^{n,k}_L(x) - f_{-,n,k}(z)\psi^{n,k}_R(x)\right)\right), \tag{31}$$

$$\Psi_2 = \frac{1}{\sqrt{2}}\left(f_{+,0}(z)\psi_R(x) + f_{-,0}(z)\psi_L(x) + \sum_{n=1}^{\infty}\sum_{k=1}^{2}\left(f_{+,n,k}(z)\psi^{n,k}_R(x) + f_{-,n,k}(z)\psi^{n,k}_L(x)\right)\right), \tag{32}$$

where $f_{+,0}(z) = f_0(z) + f_0(-z)$, $f_{-,0}(z) = f_0(z) - f_0(-z)$, $f_{+,n,k}(z) = f_{n,k}(z) + f_{n,k}(-z)$, $f_{-,n,k}(z) = f_{n,k}(z) - f_{n,k}(-z)$; whereas the corresponding first-order equations, which will be necessary for obtaining the effective four-dimensional action, take the form [1]

$$f'_0(z) + 2\sigma'(z)f_0(z) + G(z)f_0(z) = m_0 e^{-\sigma(z)}f_0(-z), \tag{33}$$

$$f'_{n,1}(z) + 2\sigma'(z)f_{n,1}(z) + G(z)f_{n,1}(z) = m_{n,1}e^{-\sigma(z)}f_{n,1}(-z), \tag{34}$$

$$f'_{n,2}(z) + 2\sigma'(z)f_{n,2}(z) + G(z)f_{n,2}(z) = -m_{n,2}e^{-\sigma(z)}f_{n,2}(-z). \tag{35}$$

Here $m_0 \geq 0$, $m_{n,k} > 0$. Since solutions to equation (33), (34) and (35) satisfy equation (30), the following orthonormality conditions can be imposed:

$$\int_{-L}^{L} e^{3\sigma} f_{n,k}(z)f_{j,l}(z)dz = \int_{-L}^{L} e^{3\sigma} f_{n,k}(-z)f_{j,l}(-z)dz = \frac{1}{2}\delta_{n,j}\delta_{k,l}, \qquad \int_{-L}^{L} e^{3\sigma} f_0^2(z)dz = \frac{1}{2} \tag{36}$$

$$\int_{-L}^{L} e^{3\sigma} f_0(z)f_{n,k}(z)dz = \int_{-L}^{L} e^{3\sigma} f_0(-z)f_{n,k}(-z)dz = 0, \qquad n \geq 1. \tag{37}$$

Substituting the Kaluza-Klein decomposition (31) and (32) into (4), using equations (33), (34), (35), the orthonormality conditions (36)–(37) when integrating over the coordinate of the extra dimension z, defining $\psi = \psi_L + \psi_R$, $\psi_{n,1} = \psi^{n,1}_L + \psi^{n,1}_R$, $\psi_{n,2} = \psi^{n,2}_L + \psi^{n,2}_R$ and then using the redefinition $\psi_{n,2} \to \gamma^5\psi_{n,2}$, we arrive at

$$S_{eff} = \int d^4x\left(i\bar{\psi}\gamma^\mu\partial_\mu\psi - m_0\bar{\psi}\psi + \sum_{n=1}^{\infty}\sum_{k=1}^{2}\left(i\bar{\psi}_{n,k}\gamma^\mu\partial_\mu\psi_{n,k} - m_{n,k}\bar{\psi}_{n,k}\psi_{n,k}\right)\right). \tag{38}$$

One can expect that in the most cases all the eigenvalues $m_{n,k}$, $n \geq 1$ are simple — it was noted in [10] that in general the double eigenvalues are not common, whereas the Fourier case is not typical. However, the double eigenvalues are still possible. For example, the case $G(z) \equiv Me^{-\sigma}$, $F(z) \equiv 0$, for which the Kaluza-Klein decomposition can be performed in two different ways presented in the previous subsection and in this subsection, leads to double eigenvalues. Of course, both ways lead to the same four-dimensional effective theory and can be connected explicitly [1].

2.3 $G(z) \equiv K\,\delta(z-L)$, any $F(z)$

Now we consider the third case with $G(z) \equiv K\,\delta(z-L)$, where $K > 0$ is a constant. It describes the Higgs field located exactly on the brane. Without loss of generality we take this brane to be the one at $z = L$.

It is clear that everywhere except the point $z = L$ the following five-dimensional equations hold:

$$-e^{-2\sigma}\Box\Psi_1 + \Psi_1'' + 5\sigma'\Psi_1' + (6\sigma'^2 + 2\sigma'')\Psi_1 - (F^2 - e^{-\sigma}(e^{\sigma}F)'\gamma^5)\Psi_1 = 0, \tag{39}$$

$$-e^{-2\sigma}\Box\Psi_2 + \Psi_2'' + 5\sigma'\Psi_2' + (6\sigma'^2 + 2\sigma'')\Psi_2 - (F^2 + e^{-\sigma}(e^{\sigma}F)'\gamma^5)\Psi_2 = 0. \tag{40}$$

These equations are not coupled and suggest the decomposition (for simplicity, we keep only a single mode)

$$\Psi_1 = f(z)\psi_L(x) + \tilde{f}(z)\psi_R(x), \qquad \Psi_2 = f(z)\hat{\psi}_R(x) - \tilde{f}(z)\hat{\psi}_L(x). \tag{41}$$

According to the symmetry conditions and equations (39), (40), the functions $f(-z) = f(z)$ and $\tilde{f}(-z) = -\tilde{f}(z)$ are supposed to satisfy the equations

$$e^{-2\sigma}m^2 f + f'' + 5\sigma' f' + (6\sigma'^2 + 2\sigma'')f - (F^2 - \sigma' F - F')f = 0, \tag{42}$$

$$e^{-2\sigma}m^2 \tilde{f} + \tilde{f}'' + 5\sigma' \tilde{f}' + (6\sigma'^2 + 2\sigma'')\tilde{f} - (F^2 + \sigma' F + F')\tilde{f} = 0, \tag{43}$$

where $\Box\psi_{L,R} + m^2\psi_{L,R} = 0$ and $\Box\hat{\psi}_{L,R} + m^2\hat{\psi}_{L,R} = 0$. Note that here m is not an eigenvalue, but just a parameter, and the functions $f(z)$ and $\tilde{f}(z)$ are not eigenfunctions — equations (42) and (43) do not hold at $z = L$. So $f(z)$ and $\tilde{f}(z)$ are just solutions to equations (42) and (43) with some parameter m which is not defined yet.

Analogously to the previous cases, it is possible to show that whenever equations (42), (43) hold, the following system of equations also holds:

$$f'(z) + (2\sigma' + F)f(z) = me^{-\sigma}\tilde{f}(z), \qquad \tilde{f}'(z) + (2\sigma' - F)\tilde{f}(z) = -me^{-\sigma}f(z), \tag{44}$$

again everywhere except the point $z = L$. Substituting (41) into equations (4), (5) with $G(z) \equiv K\delta(z-L)$ and using (44), we get everywhere except $z = L$

$$i\gamma^\mu\partial_\mu\psi_L - m\psi_R = 0, \qquad i\gamma^\mu\partial_\mu\psi_R - m\psi_L = 0, \qquad i\gamma^\mu\partial_\mu\hat{\psi}_L - m\hat{\psi}_R = 0, \qquad i\gamma^\mu\partial_\mu\hat{\psi}_R - m\hat{\psi}_L = 0, \tag{45}$$

which are the standard four-dimensional Dirac equations. At the point $z = L$ equations (4) and (5) give

$$\hat{\psi}_R(x) = -\lim_{\epsilon\to+0}\frac{2\tilde{f}(L-\epsilon)}{Kf(L)}\psi_R(x), \qquad \hat{\psi}_L(x) = -\lim_{\epsilon\to+0}\frac{Kf(L)}{2\tilde{f}(L-\epsilon)}\psi_L(x) \tag{46}$$

for any x. In derivation of (46), we have used the regularization $\tilde{f}(z)\delta(z-L) \sim \mathrm{sign}(z-L)\delta(z-L) = 0$ (in order to support the existence of the delta-function in the five-dimensional action, the function $\tilde{f}(z)$ should be discontinuous at $z = L$). Finally, using (46) we can find that equations (45) are consistent if the conditions

$$\lim_{\epsilon\to+0}\frac{Kf(L)}{2\tilde{f}(L-\epsilon)} = -\beta_i, \qquad i = 1,2, \qquad \beta_1 = 1, \qquad \beta_2 = -1 \tag{47}$$

hold. This condition defines the allowed values of m, i.e., the mass spectrum of the theory. It can be shown explicitly that the values $m^2 < 0$ are impossible in the model at hand, see [1] for details.

Let us label the solutions to equation (47) as $m_{n,i}$ and suppose that all the masses are different. The complete decomposition of the five-dimensional fields has the form [1]

$$\Psi_1 = f_0(z)\psi_L(x) + \tilde{f}_0(z)\psi_R(x) + \sum_{n=1}^{\infty}\sum_{i=1}^{2}\left(f_{n,i}(z)\psi_L^{n,i}(x) + \tilde{f}_{n,i}(z)\psi_R^{n,i}(x)\right), \tag{48}$$

$$\Psi_2 = f_0(z)\psi_R(x) - \tilde{f}_0(z)\psi_L(x) + \sum_{n=1}^{\infty}\sum_{i=1}^{2}\beta_i\left(f_{n,i}(z)\psi_R^{n,i}(x) - \tilde{f}_{n,i}(z)\psi_L^{n,i}(x)\right), \tag{49}$$

where we have used (46) and (47). Here the modes with $i = 1$ are defined by (47) with $\beta_1 = 1$, whereas the modes with $i = 2$ are defined by (47) with $\beta_2 = -1$.

With (47), it is possible to choose the following nonstandard orthonormality conditions (see [1] for the proof):

$$\int_{-L}^{L} e^{3\sigma}\left(f_{n,i}(z)f_{k,j}(z) + \beta_i\beta_j\tilde{f}_{n,i}(z)\tilde{f}_{k,j}(z)\right)dz = \int_{-L}^{L} e^{3\sigma}\left(\beta_i\beta_j f_{n,i}(z)f_{k,j}(z) + \tilde{f}_{n,i}(z)\tilde{f}_{k,j}(z)\right)dz = \delta_{nk}\delta_{ij}.$$

The last useful step is to find the equations instead of equations (44), which are valid in the whole extra dimension. Indeed, equation for the mass spectrum (47) relates the values of $f_n(L)$ and $\tilde{f}_n(L)$, so we can use it to supply (44) by extra terms in order to obtain the systems of equations [1]

$$f_0' + (2\sigma' + F)f_0 = m_0 e^{-\sigma}\tilde{f}_0, \qquad \tilde{f}_0' + (2\sigma' - F)\tilde{f}_0 - \frac{K}{\beta_0}f_0\,\delta(z-L) = -m_0 e^{-\sigma}f_0 \tag{50}$$

$$f_{n,i}' + (2\sigma' + F)f_{n,i} = m_{n,i}e^{-\sigma}\tilde{f}_{n,i}, \qquad \tilde{f}_{n,i}' + (2\sigma' - F)\tilde{f}_{n,i} - \frac{K}{\beta_i}f_{n,i}\,\delta(z-L) = -m_{n,i}e^{-\sigma}f_{n,i}, \tag{51}$$

which are valid for any z. Substituting (48), (49) into five-dimensional action (4), using the orthonormality conditions and equations (50)–(51) and defining $\psi = \psi_L^0 + \psi_R^0$, $\psi_{n,i} = \psi_L^{n,i} + \psi_R^{n,i}$, we get

$$S_{eff} = \int d^4x\left(i\bar{\psi}\gamma^\mu\partial_\mu\psi - m_0\bar{\psi}\psi + \sum_{n=1}^{\infty}\sum_{i=1}^{2}\left(i\bar{\psi}_{n,i}\gamma^\mu\partial_\mu\psi_{n,i} - m_{n,i}\bar{\psi}_{n,i}\psi_{n,i}\right)\right). \tag{52}$$

3 Discussion

Let us briefly discuss the results presented above.

1. As was shown in Section 1, the general case may contain rather pathological behavior in the form of fourth-order differential equations of motion for the components of the five-dimensional spinor fields. At the moment it is unclear how to perform the Kaluza-Klein decomposition in this case — in all the cases discussed in Section 2 the independent second-order equations of motion for the components of the five-dimensional fields (or of their linear combinations) provided the complete set of possible physical degrees of freedom of the four-dimensional effective theory, whereas the corresponding first-order differential equations allow one to get rid of the terms with the functions $F(z)$, $G(z)$ and with the derivative in the coordinate of the extra dimension in (4) and to obtain the four-dimensional effective action in a consistent way. It is not clear how to perform an analogous analysis starting from the fourth-order equations like equation (19).

2. The case $F(z) \not\equiv 0$, $G(z) \equiv Me^{-\sigma}$ does not contain any pathological behavior and the consistent Kaluza-Klein decomposition can be performed in this case. However, this case may contain some serious drawbacks, at least for $\sigma(z) \not\equiv 0$. Indeed, the existence of the functions $\sigma(z) \not\equiv 0$, $F(z) \not\equiv 0$ and $G(z) \equiv Me^{-\sigma}$, whatever the origin of these functions is, imply that the extra dimension is not uniform in z. Thus, one can expect that the backreaction of the bulk fields on the background metric (or possible quantum corrections) are also nonuniform in z, which may violate the fine-tuned relation between the vacuum profile of the Higgs field and the form of the background metric. Even if the violation is very small, the effect is nonperturbative, so it will lead to the problems discussed in the previous item.

   Analogous reasonings can be applied to the fine-tuned cases (14) and (15).

3. The case $G(z) \equiv K\,\delta(z-L)$, $F(z) \not\equiv 0$ also admits a consistent Kaluza-Klein decomposition. However, it leads to the discontinuous wave functions of the modes. This discontinuity imply that such a wave function is just an approximation for a continuous, but very rapidly varying in the vicinity of $z = L$ wave function, whereas the delta-function-like $G(z)$ is an approximation of a very narrow and peaked, but continuous profile of the Higgs field. However, any modification of the delta-function (such as, for example, $\delta(z-L) \to$ Gaussian function) in this case leads to impossibility to diagonalize the matrix $\hat{\Lambda}_L$ in (12) and, consequently, to fourth-order equations of motion for the components of the five-dimensional fermion fields.

4. Contrary to the previous cases, the choice $F(z) \equiv 0$ looks the most natural and safe from this point of view — indeed, the Kaluza-Klein decomposition can be performed in a mathematically consistent way for any $G(z)$ without the necessity for any fine-tuning. However, this case is not interesting from phenomenological point of view.

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