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Fuzzy Geometry of Commutative Spaces and Quantum Dynamics

S.N. Mayburov[1,a]

[1] *Lebedev Institute of Physics, Moscow, Leninski pr. 53*

Abstract. Fuzzy topology and geometry considered as the possible mathematical framework for novel quantum-mechanical formalism. In such formalism the states of massive particle m correspond to the elements of fuzzy manifold called fuzzy points. Due to the manifold weak topology, m space coordinate x acquires principal uncertainty σ_x and described by the positive, normalized density $w(\vec{r}, t)$ in 3-dimensional case. It's shown that the evolution of m state on such 3-dimensional manifold corresponds to Shroedinger dynamics of massive quantum particle.

1 Introduction

It's well known that quantum mechanics (QM) can be consistently described by several alternative formalisms such as, Shroedinger or standard one, algebraic QM, functional integral, etc. [1]. Each of them possesses its own advantages and thus the development of new QM formalisms can bring, in principle, the additional opportunities. In this note we consider the temptative QM formalism based on the specific fuzzy calculus. During the last 50 years the fuzzy set theory and other branches of fuzzy mathematics were applied in a wide range of scientific areas, such as biology, economics and computer science [2, 3]. Meanwhile, from the early days of its formulation and development the significant similarity of that theory and QM formalism was noticed [4–6]. In particular, it was argued that the generic parameter uncertainties which is the cornerstone of fuzzy mathematics are, in fact, similar to QM observable uncertainties [4, 5]. Furthermore, the analysis of quantum particle coordinate measurement evidences that such coordinate can be described as the fuzzy observable [7]. It was shown also that fuzzy phase space is the appropriate tool for the description of noncommutative QM observable measurements [8]. Successful applications of fuzzy methods to particular QM problems demonstrate the close connection between fuzzy mathematics and QM formalism.

Concerning with the general reformulation of QM in fuzzy terms, until now such studies were performed only in the context of fuzzy (multivalued) logics ([9] and refs. therein). In distinction, the approach considered here operates with standard logics and exploits the formalism based on the results of fuzzy topology and geometry [5, 6, 10]. Its application permits to consider the object phase space equipped with fuzzy topology and derive from that the structure of state space which turns out to be equivalent to QM Hilbert space. As the result, it permits to reduce the number of axioms necessary for QM formalism and simplify the description of quantum object evolution. Some intermediate results

[a] e-mail: mayburov@sci.lebedev.ru

on this formalism development were published previously in [11, 12]. In its essence the considered formalism is geometric; the different approaches to QM geometrization studied extensively in the last years, first of all, for the application to quantum gravity theory [13, 14].

2 Geometric Fuzzy Structures

Here we'll consider some examples illustrating the properties of fuzzy structures important for QM formalism construction, the detailed reviews on fuzzy topology and geometry can be found in [6, 10]. Let's start from the case of discrete set, remind that if S is the totally ordered set, then for its elements $\{a_i\}$ the ordering relation between all its element pairs $a_k \leq a_l$ (or vice versa) is fulfilled. However, for partially ordered set (Poset) S^p, some its element pairs can obey to the incomparability relations (IR) between them: $a_j \sim a_k$. If this is the case, then both $a_j \leq a_k$ and $a_k \leq a_j$ propositions are false [17]. Such set can possess some nontrivial properties, to illustrate them, consider poset $S^p = A^p \cup B$, which includes the subset of 'incomparable' elements $A^p = \{a_j\}$, and ordered subset $B = \{b_i\}$. For the simplicity suppose that in B the element's indexes grow correspondingly to their ordering, so that $\forall\, i,\ b_i \leq b_{i+1}$. The relations between an arbitrary pairs a_j, a_i or a_i, b_l can be ordered, as well as incomparable. Let's consider B interval $\{b_l, b_n\}$ and suppose that some A^p element a_j is confined in $\{b_l, b_n\}$, i.e. $b_l \leq a_j$; $a_j \leq b_n$, and simultaneously a_j is incomparable with all internal $\{b_l, b_n\}$ elements: $b_i \sim a_j$; $\forall\, i$; $l + 1 \leq i \leq n - 1$. In this case a_j is 'smeared' over such interval, which is the rough analogue of a_j coordinate uncertainty relative to B 'coordinate', if to consider the ordered sequence of B elements $\{b_i\}$ as the analogue of coordinate axe. The generalization of poset structure is tolerance space for which the ordering relations can be nontransitive [5].

The next step to fuzzy topology is to detalize the described IR properties introducing the fuzzy relations between S^p elements. For that purpose one can put in correspondence to each a_j, b_i pair of S^p set the nonnegative weight $w^j_i \geq 0$ with norm $\sum_i w^j_i = 1$, so that w^j_i characterizes the rate of closeness (membership) between a_j, b_i [2, 3]. For the example considered above, one can ascribe $w^j_i \neq 0$ to all b_i inside $\{b_l, b_n\}$ interval, $w^j_i = 0$ for other b_i. To illustrate how such structure can appear for the continuum consider the continuous set S^f such that $S^f = A^p \cup X$ where A^p is the same discrete subset, X is the continuous ordered subset. If the flat metrics $\mathcal{M}(x, x')$ is defined on X, then it's equivalent to R^1 real number axe. Then, in this framework the fuzzy relations between elements a_j, x are described by real, nonnegative functions $w^j(x) \geq 0$ with the norm $\int w^j dx = 1$ [3]. In this case $\{a_j\}$ called the fuzzy numbers $\tilde{x}_j$ or in geometric framework the fuzzy points (FPs) [2, 3, 15]. In such approach the ordered point $x_a \in X$ is characterized by $w^a(x) = \delta(x - x_a)$, hence the ordered points and FPs can be regarded formally on the same ground. As the result, the obtained formalism can be considered also as the generalization of 1-dimensional Euclidian geometry for which the positions of some elements $\{a_j\}$ are described by the positive normalized function $w^j(x)$ on R^1; w^j dispersion σ_x characterizes a_j coordinate uncertainty.

In 3-dimensional case as the fundamental set one can consider the set: $C^f = A^p \cup X'$ where A^p defined above, X' is the continuous set. Suppose that on X' 3−dimensional linear space R^3 with flat metrics $\mathcal{M}_{ij}$ is defined. Then FP a_j is described by the nonnegative function $w^j(\vec{r})$ with norm $\int w^j d^3r = 1$. Such C^f is called fuzzy manifold and will be denoted also as $\tilde{R}^3$. Note that in fuzzy mathematics the different FP definitions are exploited, we use here the one given in [3, 15]. Previously, such formalism was called fuzzy (or digital) geometry [6, 15], however, in modern physics such term denotes some noncommutative theories [16], so to stress the difference, we shall exploit for our formalism the term the commutative fuzzy geometry (CFG).

In such formalism $w^j(x)$ *a priori* doesn't have any probabilistic meaning but only the algebraic one, characterizing the properties of fuzzy value $\tilde{x}_j$. To describe the distinction between the fuzzy structure and probabilistic one, the correlation $\kappa_0(x, x')$ defined over w_j support can be introduced; thus if $w_j(x_{1,2}) \neq 0$, then $\forall x_1, x_2$; $\kappa_0(x_1, x_2) = 1$ for FP a_j and $\kappa_0(x_1, x_2) = 0$ for probabilistic a_j distribution. Thus a_j state G on X is described by two functions $G = \{w_j(x), \kappa_0(x, x')\}$. Such description illustrates the difference of such formalism from stochastic geometry, which exploits practically infinite set of parameters. As will be shown below, the similar bilocal correlations can describe some properties of quantum objects.

3 Structure of Fuzzy States

In the considered framework the massive particle of 1-dimensional classical mechanics corresponds to the ordered point $x_a(t) \in X$. By the analogy, in our theory called fuzzy mechanics (FM), in 1-dimensional case the particle m corresponds to FP $a(t)$ and characterized by normalized positive density $w(x, t)$. Beside $w(x)$, m physical state $|g\}$, which called the fuzzy state, can depend on other m degrees of freedom (DFs) which will define its evolution. As the example, consider m average velocity:

$$\bar{v}(t) = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} x w(x, t) dx = \int_{-\infty}^{\infty} x \frac{\partial w}{\partial t} dx \tag{1}$$

It's reasonable to assume that in general $\bar{v}(t)$ can be independent of $w(x, t)$, below we shall look for additional m state DFs in form of real functions $q_{1,\dots,n}(x, t)$. Let's suppose that in FM m evolution is local, in particular, that:

$$\frac{\partial w}{\partial t}(x, t) = -\Phi(w, q_1, \dots, q_n) \tag{2}$$

where Φ is an arbitrary function which depends only on $w(x, t)$, $q_{1,\dots,n}(x, t)$. From w norm conservation it follows that:

$$\int_{-\infty}^{\infty} \Phi(x, t) dx = -\int_{-\infty}^{\infty} \frac{\partial w}{\partial t}(x, t) dx = -\frac{\partial}{\partial t}\int_{-\infty}^{\infty} w(x, t) dx = 0 \tag{3}$$

If to substitute: $\Phi = \frac{\partial J}{\partial x}$ where $J(x)$ is some differentiable function, then eq. (3) demands:

$$J(\infty, t) - J(-\infty, t) = 0 \tag{4}$$

For normalized density w at $x \to \pm\infty$, $w(x, t) \to 0$, and such assumption is, in fact, trivial. If it's fulfilled, then $J(x)$ can be considered as w flow, and eq. (2) is equivalent to 1-dimensional flow continuity equation [21]:

$$\frac{\partial w}{\partial t} = -\frac{\partial J}{\partial x} \tag{5}$$

$J(x)$ can be decomposed formally as: $J = w(x)v(x)$ where $v(x)$ corresponds to 1-dimensional w flow velocity [21]. In these terms eq.(5) can be written as:

$$\frac{\partial w}{\partial t} = -v\frac{\partial w}{\partial x} - \frac{\partial v}{\partial x} w \tag{6}$$

By the analogy with fluid dynamics, we shall assume that $v(x)$ can be considered as the novel m DF, and there is no other independendent DFs on which m state $|g\}$ can depend. In this case 1-dimensional matrix $g^o = \{w, v\}$ describes the observational $|g\}$ representation, such g^o supposedly contains the information about the expectation values of all m observables. To make the theory most simple it's sensible to look for dynamical $|g\}$ representation η for which the evolution equations will be linear.

This isn't fulfilled for g^o, because the eq. (6) which describes the evolution of its component w is nonlinear. We shall suppose that such η can be also represented as the array of real functions $\{\eta_i(x)\}$, $i = 1, n_a$ which can constitute the complex or quoternionic number or some other algebraic object. Then, the most general η ansatz is $\eta_j(x) = \Upsilon^j_x(w, v)$ where Υ^j_x are some w, v functionals and x is their parameter. For $|g\}$ characterized by two DFs w, v, it's instructive to start the search of minimal $|g\}$ ansatz from complex $\eta(x)$, so $n_a = 2$, not assuming yet that η is normalized as $w(x) = |\eta|^2$.

In FM the evolution of massive particle supposedly can be characterized also by m velocity 'as the whole' $u(t)$ with expectation value $\bar{u}(t)$. Such $u(t)$ is described by the corresponding normalized distribution $w_u(u, t)$ which can be defined in u measurements. Thus, in FM framework, analogously to m coordinate x, such u also can be considered as fuzzy value $\tilde{u}(t)$. Obviously, $\bar{u}(t)$ coincides with $\bar{v}(t)$ of (1) and so it related to m flow velocity $v(x)$:

$$\bar{u} = \int_{-\infty}^{\infty} v(x)w(x)dx \tag{7}$$

In place of u, below it will be convenient to use the variable $p = \mu u$ where μ is the theory constant; for the corresponding distribution $\omega(p)$ it gives $w_u(u) = \mu\omega(\mu u)$. As proposed above, $|g\}$ contains the information on the expectation values of any m observable Q_i, hence any such value should be described by some η functional. In particular, it should be: $\omega(p) = F_p(\eta)$ where F_p is parameter dependent functional. For complex η, it can be shown that F_p related to $\eta(x)$ fourier transform. To prove it and calculate η and ω, let's introduce the auxiliary form: $\varphi(p) = \omega^{\frac{1}{2}} e^{i\beta(p)}$, here $\beta(p)$ is the auxilary real function on which the final ω ansatz wouldn't depend. From the above arguments, $\eta(x)$ is also equal to:

$$\eta(x) = f_x(w, v)e^{i\lambda_x(w,v)} \tag{8}$$

where $f_x(w, v)$, $\lambda_x(w, v)$ are some real functionals. We shall look for ω, β functions such that φ fourier decomposition on X is equal to:

$$\varphi(p) = \int_{-\infty}^{\infty} \eta(x)e^{-ipx}dx = \int_{-\infty}^{\infty} f_x e^{i\lambda_x - ipx}dx \tag{9}$$

w_p is normalized distribution, so the application of Plancherele formulae to that norm gives [19]:

$$\int_{-\infty}^{\infty} \omega(p)dp = \int_{-\infty}^{\infty} \varphi(p)\varphi^*(p)dp = \int_{-\infty}^{\infty} f_x^2(w)dx = 1 \tag{10}$$

To define f_x, one can consider w variation for the equality:

$$\int_{-\infty}^{\infty} [f_x^2(w, v) - w]dx = 0 \tag{11}$$

with additional δw constraint: $\int \delta w dx = 0$. Application of Du Bois-Reymond lemma [18] to eq. (11) together with this constraint gives $f_x^2 = w(x)$, so f_x doesn't depend on $v(x)$. Then $\bar{p}$ can be calculated anew from 2-nd Plancherele formulae :

$$\bar{p} = \int_{-\infty}^{\infty} p\varphi(p)\varphi^*(p)dp = \int_{-\infty}^{\infty} \frac{\partial \lambda_x}{\partial x} f_x^2 dx = \int_{-\infty}^{\infty} \frac{\partial \lambda_x}{\partial x} w(x)dx \tag{12}$$

From the comparison with eq. (7) and $\bar{p} = \mu\bar{u}$ it follows: $\lambda_x = \gamma(x) + \chi(w)$ where γ is the functional:

$$\gamma(x) = \mu \int_{-\infty}^{x} v(\xi)d\xi + c_\gamma \tag{13}$$

here c_γ is an arbitrary real number. $\chi(w)$ is an arbitrary real function which obeys to the condition:

$$\int_{-\infty}^{\infty} \chi(w)\frac{\partial w}{\partial x}dx = 0 \tag{14}$$

and is the analogue of η gauge. The resulting m state in x-representation is equal to:

$$\eta(x) = w^{\frac{1}{2}}(x)e^{i\gamma+i\chi} \tag{15}$$

is the vector (ray) of complex Hilbert space $\mathcal{H}$. $\omega(p)$ and $\beta(p)$ can be calculated from eq. (9) as functions of χ. In particular:

$$\omega(p) = |\int_{-\infty}^{\infty} w^{\frac{1}{2}} e^{i\gamma+i\chi-ipx}dx|^2 \tag{16}$$

is independent of $\beta(p)$, so ω is just w, v functional. $\beta(p)$ is, in fact, the analogue of $\gamma(x)$ for $|g\}$ in p-representation described by $\varphi(p)$.

4 Linear Model of Fuzzy Dynamics

Obtained m state: $\eta = e^{i\chi}g$, where $g(x,t)$ is standard QM wave function, so that $\eta(x,t)$ is its trivial map. Thus one can study first $g(x,t)$ evolution, and then basing on obtained results $\eta(x,t)$ properties will be defined . Evolution equation for g is supposed to be of the first order in time, i.e.:

$$i\frac{\partial g}{\partial t} = \hat{H}g. \tag{17}$$

In general $\hat{H}$ can be the nonlinear operator, for the simplicity we shall consider first the linear case and turn to nonlinear one in the next section. Free m evolution is invariant relative to x space shift to arbitrary x_0 distance performed by the operator $\hat{W}(x_0) = \exp(x_0\frac{\partial}{\partial x})$. Because of it, the corresponding operator $\hat{H}_0$ should commute with $\hat{W}(x_0)$ for the arbitrary x_0, i.e. $[\hat{H}_0, \frac{\partial}{\partial x}] = 0$. It holds only if $\hat{H}_0$ is differential polinom of the form:

$$\hat{H}_0 = -\sum_{l=1}^{n} b_{2l}\frac{\partial^{2l}}{\partial x^{2l}} \tag{18}$$

where b_{2l} are arbitrary real constants. Suppose that the action of external field on m can be accounted in $\hat{H}$ additively: $\hat{H} = \hat{H}_0 + V(x,t)$ where V is real, nonsingular function. Let's rewrite eq. (17) separating w, γ derivatives:

$$i\frac{\partial g}{\partial t} = (i\frac{\partial w^{\frac{1}{2}}}{\partial t} - w^{\frac{1}{2}}\frac{\partial \gamma}{\partial t})e^{i\gamma} = e^{i\gamma}\hat{Z}g \tag{19}$$

where $\hat{Z} = e^{-i\gamma}\hat{H}$. Hence:

$$\frac{\partial w^{\frac{1}{2}}}{\partial t} = im(\hat{Z}g) \tag{20}$$

Yet if to substitute $v(x)$ by $\gamma(x)$ in eq. (6) and transform it to $w^{\frac{1}{2}}$ time derivative, then:

$$\frac{\partial w^{\frac{1}{2}}}{\partial t} = -\frac{1}{\mu}\frac{\partial w^{\frac{1}{2}}}{\partial x}\frac{\partial \gamma}{\partial x} - \frac{1}{2\mu}w^{\frac{1}{2}}\frac{\partial^2 \gamma}{\partial x^2} \tag{21}$$

Plainly, to make the formalism consistent, the right parts of equations (20) and (21) should coincide for arbitrary w, γ, otherwise $\hat{H}$ ansatz would be incompatible with w flow continuity described by eqs. (6, 21). Hence $\hat{H}$ ansatz can be obtained from their comparison term by term. In particular, the imaginary part of $\hat{Z}g$ includes the highest γ derivative as the term: $b_{2l}w^{\frac{1}{2}}\frac{\partial^{2l}\gamma}{\partial x^{2l}}$, yet for eq. (21) the highest γ derivative is proportional to $w^{\frac{1}{2}}\frac{\partial^2\gamma}{\partial x^2}$. As the result, it gives: $b_2 = \frac{1}{2\mu}$, for all $l > 1$ it follows that $b_{2l} = 0$, only in this case both expressions for $\frac{\partial w^{\frac{1}{2}}}{\partial t}$ would coincide. Thus g free evolution is described by the only $\hat{H}_0$ term $b_2 = \frac{1}{2\mu}$, so $\hat{H}$ is Schroedinger Hamiltonian for particle with mass μ.

In this framework, the observable p corresponds to the operator $\hat{p} = -i\frac{\partial}{\partial x}$ acting on $g(x)$. Thus, x and p observables are described by the linear self-adjoint operators, which obey to the commutation relation $[\hat{x}, \hat{p}] = i$. By the analogy, we suppose that all m PV observables $\{Q\}$ are the linear, self-adjoint operators on $\mathcal{H}$. The flow velocity $v(x)$ isn't observable, but can be formally defined as the ratio of $J(x), w(x)$ observable expectation values, where w observable is described by the projection operator $\hat{\Pi}(x)$; the operator $\hat{J}(x)$ considered in [20]. As was noticed earlier, the particle evolution in QM in some aspects is similar to the motion of continuous media [21]. This analogy is exploited in hydrodynamical QM model (QFD) [22, 23], its connection with FM will be discussed elsewhere.

Plainly, $\gamma(x)$ corresponds to $|g\}$ quantum phase, so that:

$$k(x, x') = \gamma(x) - \gamma(x')$$

describes the phase correlation between the state components in x, x'. Thus pure FM state can be characterized by the density $w(x)$ and the array of bilocal geometric correlations $\{\kappa_l(x, x')\}$, the first of them: $\kappa_0(x, x')$ was introduced in sect. 2. Until now we've considered only the pure fuzzy states, i. e. the states which aren't the probabilistic mixture of several pure states. Analogously to QM, the mixed states in FM can be defined via the density matrixes, i.e. the positive, trace one operators ρ on $\mathcal{H}$ [1]. In particular, for pure m states:

$$\rho(x, x') = g(x)g^*(x') = [w(x)w(x')]^{\frac{1}{2}}e^{ik(x,x')}$$

is equivalent to $g(x)$, yet such $|g\}$ representation demonstrates in the open the correlation structure of m pure states. Thus, the most consistent FM state ansatz is given by the density matrix ρ. However, the evolution equations for pure states in form of ρ are more complicated then for Dirac vector $g(x)$, and because of it, we shall exploit $g(x)$ throughout our paper.

5 General Fuzzy Dynamics

In the previous section 1-dimensional FM formalism was derived from CFG premises assuming that $|g\}$ evolution is linear and its phase component $\chi(w)$ can be neglected. Now these assumptions will be dropped one by one and general formalism derived. Concerning with nonlinear evolution, the conditions of QM linearity were reconsidered recently by Jordan, and turn out to be essentially weaker than Wigner theorem asserts [24]. In particular, it was proved that if the evolution maps the set of all pure states one to one onto itself, and for arbitrary mixture of orthogonal states $\rho(t) = \sum P_i(t)\rho_i(t)$ all P_i are independent of time, then such evolution is linear. Here $\rho_i(x, x', t) = \psi_i(x, t)\psi_i^*(x', t)$ are the density matrixes of orthogonal pure states ψ_i. Yet for the considered FM formalism first condition is, in fact, generic: no mixed (i.e. probabilistic) state can appear in the evolution of pure state in such geometrodynamical approach. The second condition involves the probabilistic mixture of such orthogonal pure states and also seems to be rather weak assumption.

Now let's return to $\eta(x)$ ansatz of (15), note first that if $\chi(w) \neq 0$ then its appearance demands to introduce at least two additional theory parameters. It can be shown also that $\hat{H}$ linearity demands that $\chi(w) = 0$. Really, it's easy to check that η evolution equation:

$$i\frac{\partial \eta}{\partial t} = \hat{H}\eta \tag{22}$$

for linear $\hat{H}$ ansatz is incompatible with w flow continuity expressed by eq. (6,21) due to the appearance of additional $\hat{H}$ terms which contain χ derivatives. Hence resulting $\eta(x,t)$ coincides with QM wave function, i.e. is Dirac vector in Hilbert space $\mathcal{H}$. Obtained results don't exclude the existence of alternative solutions for $\eta(x)$, it seems, however, that the obtained ansatz is the minimal one.

As follows from the results of sect. 2, 3-dimensional FM, in fact, doesn't demand any principal modification of described formalism. In that case FP a_j is described by the normalized, nonnegative function $w^j(\vec{r})$. The particle m corresponds to FP $a(t)$ characterized by $w(\vec{r},t)$, then analogously to sect. 3, given w evolution depends on local parameters only, it can be expressed as:

$$\frac{\partial w}{\partial t}(\vec{r},t) = -\Phi(\vec{r},t) \tag{23}$$

where Φ is an arbitrary local function. Then from w norm conservation 3-dimensional flow continuity equation follows:

$$\frac{\partial w}{\partial t} = -div\vec{J} \tag{24}$$

One can decompose $\vec{J} = w\vec{v}$ and consider w flow velocity $\vec{v}(\vec{r})$ as independent $|g\}$ parameter. m state supposedly depends on 4 DFs, $w, \vec{v}$, however, analogously to 1-dimensional case, we'll look for minimal m state ansatz $|g\}$ in form of the the complex $w, \vec{v}$ functional $\eta(\vec{r}) = \Upsilon_{\vec{r}}(w, \vec{v})$. For m as the whole, its velocity is supposedly characterized by fuzzy vector $\tilde{\vec{u}}$ which corresponds to distribution $w_u(\vec{u})$, so that:

$$\langle \vec{u} \rangle = \int \vec{u} w_u(\vec{u}) d^3u = \int \vec{v}(\vec{r}) w(\vec{r}) d^3r \tag{25}$$

m kinematical fuzzy momentum defined as: $\tilde{\vec{p}} = \mu\tilde{\vec{u}}$. From that, analogously to eqs. (7 - 15), standard QM ansatz for m state obtained: $g(\vec{r}) = w^{\frac{1}{2}} e^{i\gamma}$, for which g phase $\gamma(\vec{r})$ obeys to the equality $\mu\vec{v} = grad(\gamma)$. m state $g(\vec{r})$ should be defined unambiguously, due to it, this equality imposes constraints on $\vec{v}(\vec{r})$ field [1].

Considering g linear evolution, for free m evolution its operator $\hat{H}_0$ should be the even polinom of the form:

$$\hat{H}_0 = -\sum_{l=1}^{n} b_{2l} \frac{\partial^{2l}}{\partial \vec{r}^{2l}} \tag{26}$$

If the external field action can be described by the addition of real function V to it:

$$\hat{H} = \hat{H}_0 + V(\vec{r},t) \tag{27}$$

then from $\frac{\partial g}{\partial t}$ the term $\frac{\partial w^{\frac{1}{2}}}{\partial t}$ can be extracted and expressed via corresponding w, γ
$\vec{r}$-derivatives. From their comparison with corresponding $\hat{H}g$ derivatives the schroedinger equation is

obtained for m evolution. The applicability of Jordan theorem to 3-dimensional $\hat{H}$ is obvious, because the derivation of $\hat{H}$ linearity doesn't depend on the dimensiality of coordinate space. The same is true for the proof of uniqueness of $\eta(\vec{r}, t)$ ansatz, i.e. that $\chi(w) = 0$.

Planck constant $\hbar = 1$ in our FM ansatz, but the same value ascribed to it in relativistic unit system in which the velocity of light $c = 1$; in FM framework $\hbar$ only connects x, p scales and doesn't have any other meaning. Note that superposition principle doesn't need to be postulated separately in such formalism. Rather, it follows that the sum of two physical m states $g_{1,2}$ with proper complex coefficients $a_{1,2}$ is also the physical m state. In the described derivation of evolution equation we didn't assume Galilean invariance of FM, rather in our approach it follows from the obtained ansatz, if the reference frame (RF) is regarded as the physical object with mass $\mu \to \infty$ [11, 25].

In our approach the state space is defined by the underlying geometry and corresponding dynamics i.e. is derivable concept. For states of massive particle m it was found to be equivalent to $\mathcal{H}$, but, in principle, it can be different for other systems. The similar features possess the formalism of algebraic QM where the state space is defined by the observable algebra and system dynamics, as the result, the state space can differ from QM Hilbert space [1].

6 Conclusion

As was noticed already, QM can be described by several alternative formalisms, of them the most notorious are algebraic QM and Schroedinger or standard formalism [1]. To discuss the possible advantages of FM formalism it's instructive to compare it with the latter one. From the formal side, standard QM exploits two fundamental structures of different nature: the space-time manifold $R^3 * T$ and function space $\mathcal{H}$ defined on R^3. In distinction, FM formalism involves only one basic structure - the fuzzy manifold $\tilde{R}^3 * T$. FM physical state densities $w(\vec{r})$ are $\tilde{R}^3$ fuzzy points, the equivalence of such states to $\mathcal{H}$ Dirac vectors was proved here. In standard QM the evolution equation or postulated *ad hoc* or derived assuming Galilean invariance of object states [1]. In FM the Schroedinger equation is derived assuming only space-time shift and rotational invariance which are essentially weaker assumptions. The quantum-classical transition in such theory is essentially more simple, it's just the transition from $\tilde{R}^3$ manifold to R^3 one, for which the classical particles correspond to ordered points.

As we know, general relativity is essentially geometric theory, but the attempts to quantize gravity suffer the serious difficulty even at axiomatics level. Hence such geometric QM formalism can help to construct quantum theory of gravity. It can be useful also for development of gauge field theory, which is mainly geometric [26]; in addition, its implications can be important for the analysis of QM foundations. Currently, the main impact of QM geometrization studies is done on the exploit of Hilbert manifolds ([13, 14] and refs. therein), however, the results obtained up to now have quite abstract form, and their applicability to particular problems is questionable. The considered FM formalism possesses simple and logical axiomatics which origin is basically geometrical, so it can become the appropriate part of QM geometrization program.

In conclusion, we have shown that the quantization of elementary systems can be derived directly from axiomatic of set theory and topology together with the natural assumptions about system evolution. It allows to suppose that the quantization phenomenon has its roots in foundations of mathematics [1]. Our approach permits to construct QM formalism starting from geometric concepts and structures only, so in these aspects it's analogous to general relativity construction. In the same time the considered fuzzy manifold describes the possible variant of fundamental pregeometry which is basic component of some quantum gravity theories [16].

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