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CHY Representation of the Massive Six Quark Amplitude in QCD

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Abstract. The Cachazo-He-Yuan (CHY) representation for tree-level scattering amplitudes based on the so called scattering equations is proved to hold for six quarks QCD primitive amplitudes. Since the CHY representation allows to separate information into a factor which depends upon the helicities of the particles and into a factor depending on the ordering of the external particles, we prove that the external orderings of a minimal amplitude basis for three quarks pairs remain linearly independent, when viewed as the external orderings of the pure gluonic amplitudes.

1 Introduction

The CHY representation provides an elegant and compact way of writing tree amplitudes in a variety of quantum field theories, based on the solution of the so-called scattering equations. These algebraic equations introduce a space of auxiliary complex variables, whose solutions are given in terms of the kinematic invariants of the scattering process in question. For a process with n external particles, there are $(n-3)!$ inequivalent solutions to the scattering equations. Thus, one can construct an integral representation of the amplitudes that is completely localized on the solutions to the system. The representation for a variety of theories can be found in [1]-[6].
In this work we introduce the CHY representation for amplitudes in QCD. We review the details of the CHY formalism and its application to QCD and the description of a minimal amplitude basis in the language of words and shuffle algebra. A general primitive amplitude can be expanded in a minimal amplitude basis whose elements have a CHY representation with a consequent factorization of the information on the polarizations and the external ordering. Following [7] the existence of a CHY representation for an arbitrary QCD amplitude is seen to depend on the existence of a pseudoinverse of a matrix F, which is a function of the kinematic invariants of the process. We explicitly calculate the elements of the F matrix, in the case of the six quark amplitude with all different flavours, and show that F has full row rank by proving the existence of a right pseudoinverse in a general kinematic regime.

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2 The CHY Formalism

2.1 The Scattering Equations

Let $(p_i)_{1\leq i\leq n}$ denote the momenta of the external particles of an n-particle QCD scattering process. These momenta are massless for gluons, $p_g^2 = 0$, and can be massive for a quark q and its corresponding antiquark $\bar{q}$, $p_q^2 = p_{\bar{q}}^2 = m_q^2$. For each i, a solution to the scattering equations (SE) is a complex n-tuple $z = (z_1, \ldots, z_n)$ satisfying

$$f_i(z, p) = \sum_{j\neq i} \frac{2p_i \cdot p_j + 2\Delta_{ij}}{z_{ij}} = 0 \tag{1}$$

where $z_{ij} = z_i - z_j$ and $\Delta_{q_k\bar{q}_k} = \Delta_{\bar{q}_k q_k} = m_{q_k}^2$, but $\Delta_{ij} = 0$ in other cases. One particular property of the solutions to the SE is that they are invariant under Möbius transformations,

$$z \to z' = \frac{az + b}{cz + d}; \quad ad - bc = 1. \tag{2}$$

There is therefore a redundacy among the solutions of SE which can be fixed [2] such that for a given n there are $(n-3)!$ inequivalent solutions, meaning not related to each other by a Möbius transformation.

2.2 CHY Representation

Let $A_n(p;\epsilon)$ denote an n particle tree amplitude of an arbitrary field theory, written as a function of the external momenta, p, and the wavefunctions of the external particles, generically denoted by ϵ. We say that the theory has a CHY representation if any tree-level amplitude can be written in the form

$$A_n(p;\epsilon) = \int d\Omega_{CHY} \mathcal{I}(z, p, \epsilon) \tag{3}$$

where

$$d\Omega_{CHY} = \frac{d^n z}{\mathrm{vol}\,(SL(2,\mathbb{C}))} z_{ij} z_{jk} z_{ki} \prod_{a\neq i,j,k} \delta(f_a(z, p)) \tag{4}$$

is the integration measure, in which the factor vol $(SL(2,\mathbb{C}))$ cancels the gauge redundancy introduced by the Möbius invariance of the SE. The product of delta functions, independent of the choice of i, j, k, completely localizes the integral on the solutions to the SE. Formally, this is equivalent to a sum over the $(n-3)!$ solutions of the SE. For QCD, it is natural to consider primitive amplitudes, which are obtained from an arbitrary amplitude by stripping its color structure. These amplitudes then have a fixed external ordering. For an arbitrary QCD primitive amplitude, the CHY representation can be written as [8]

$$A_n(p, \epsilon; w) = i \sum_j J(p, z^j) \hat{C}_n(w, z^j) \hat{E}_n(p, \epsilon; z^j) \tag{5}$$

where z^j denotes the j-th solution to the SE, $J(z)$ is the Jacobian arising from the delta functions, and $\hat{C}_n$, $\hat{E}_n$ contain, respectively, all the information on the external cyclic ordering and the information on spin (which we compactly denote by ϵ). The argument $w = \ell_1 \ldots \ell_n$ stands for a word, that is, an ordered string of labels, which we call letters. These provide a compact way to express the dependence of the amplitude on the external ordering and to define a minimal basis of QCD amplitudes, which is fixed by the cyclic invariance, Kleiss-Kuijf (KK) relations and the Bern-Carrasco-Johansson (BCJ) relations satisfied by the amplitudes and, in the multiquark case, the vanishing of primitive amplitudes with crossed fermion lines [8]. In particular, in the gluon case,

$$\hat{C}_n(w, z^j) = C_n(w, z^j) = \frac{1}{z_{\ell_1\ell_2} z_{\ell_2\ell_3} \ldots z_{\ell_n\ell_1}} \tag{6}$$

which is known as the (standard) Parke-Taylor factor. For a fixed n, we can construct the N_b dimensional vectors A_w providing the minimal amplitudes basis for the process under consideration and the $N_s = (n-3)!$ dimensional vector of components $\hat{E}_j$ obtained by evaluating the function $\hat{E}$ at the j-th solution of the SE. Then, we can cast the CHY representation for an arbitrary QCD primitive amplitude as the matrix equation

$$A_w = i\hat{M}_{wj}\hat{E}_j \tag{7}$$

where we have defined the $N_b \times N_s$ dimensional matrix

$$\hat{M}_{wj} = J(p, z^j)\hat{C}(w, z^j). \tag{8}$$

As a consequence the CHY representation for a QCD amplitude exists if (7) can be inverted for $\hat{E}_j$ in terms of A_w. The necessary condition for this to be possible is that $N_b \leq N_s$ (Rouché-Capelli theorem), which can be seen to be satisfied for arbitrary n. Moreover, since in the gluon case $N_b = N_s$, $\hat{M}_{wj}$ is a square matrix and its inverse can be found through a property satisfied by the Parke-Taylor factors, known as Kawai-Lewellen-Tye (KLT) orthogonality [2]. In the general case, the factors $\hat{C}(w, z^j)$ are not equal to the standard Parke-Taylor factors. However, due to the Möbius invariance of the SE, a Faddeev-Popov procedure is needed to fix the redundancy in the integral representation. Therefore, one can work in a "gauge" in which $\hat{C}(w, z^j) = C(w, z^j)$ for $w \in B$, where B is the set of words which form the amplitude basis. Thus, ignoring the information about flavour, we consider a set of letters

$$\mathbb{A} = \{1, 2, \ldots, n\} \tag{9}$$

which we call an alphabet, and define the sets

$$W_0 = \{w = \ell_1 \ldots \ell_n \,|\, \ell_i \in \mathbb{A},\ \ell_i \neq \ell_j, \text{for } i \neq j\}, \tag{10}$$

$$W_2 = \{w \in W_0 |\, \ell_1 = 1,\ \ell_n = n\}, \tag{11}$$

$$B_{n_q \leq 2} = \{w \in W_0 |\, \ell_1 = 1,\ \ell_{n-1} = n-1,\ \ell_n = n\}. \tag{12}$$

The details of the basis for an arbitrary multiquark amplitude can be obtained by using the so-called Melia Basis [9]-[10]. Furthermore, since the Parke-Taylor factors contain all the information on the ordering of the external particles, they satisfy the BCJ relations

$$C(w, z^j) = F_{ww'}C(w', z^j) \tag{13}$$

where $w \in B$, $w' \in W_2$, and the factors $F_{ww'}$ form an $N_b \times N_s$ matrix, whose elements we define later. By the above construction, the function $\hat{E}$ can be found to be [8]

$$\hat{E}(z, p, \epsilon) = -i \sum_{u,v \in B_{n_q \leq 2}} \sum_{w \in B} S[u|\bar{v}]G_{uw}C(\bar{v}, z)A_n(p, w, \epsilon) \tag{14}$$

where $S[u|\bar{v}]$ is the KLT bilinear, which depends only on the kinematic invariants of the process, and the matrix G defines a right pseudoinverse to F, through

$$G = F^T(FF^T)^{-1}. \tag{15}$$

The matrix G always exists, provided that the square matrix FF^T has an inverse. Therefore, to show that there exists a CHY representation for an arbitrary QCD primitive amplitude, it is sufficient to construct the matrix FF^T and show that, for general kinematics, it has a non-vanishing determinant. The existence of the right pseudoinverse G means that the matrix F has full row rank, that is, its rows are linearly independent.

3 Matrix Elements for the Six Quark Amplitude

Now, we consider the explicit calculation of the matrix elements $F_{ww'}$ for the case of the six quark amplitude. If we denote the quarks and their corresponding antiquarks according to

$$q_1 = 1, \quad q_2 = 2, \quad q_3 = 3 \quad \bar{q}_3 = 4, \quad \bar{q}_2 = 5, \quad \bar{q}_1 = 6. \tag{16}$$

We can construct the basis

$$B = \{123456, 125346, 132546, 134256\}. \tag{17}$$

We also define the set

$$B_{n_q \le 2} = \{123456, 124356, 132456, 134256, 142356, 143256\} \tag{18}$$

which would correspond to the basis for the six gluon amplitude. The matrix $\mathbb{F}_{6q}$, whose elements are $F_{ww'}$, will be of 4×6 size and it can be regarded as the linear operator relating the multiquark amplitude with the purely gluonic one. Moreover, the elements $F_{ww'}$ are the factors appearing in the BCJ relations. Then, in order to determine the non-trivial elements (that is, those different from one or zero), one fixes the word w and using the linear expansion appearing in the BCJ relations, one can find the words w' for which the element $F_{ww'}$ is non-trivial. Since the letters $1, 5, 6$ are fixed in the elements of $B_{n_q \le 2}$ and the letters $1, 6$ are fixed in the elements of B, it is sufficient to consider the words in B which are not in $B_{n_q \le 2}$. The amplitudes with those external orderings are the only ones that will have a non-trivial linear expansion in terms of the amplitudes whose external orderings are in $B_{n_q \le 2}$. As an example of the calculation of the elements, we consider the word $w = 125346$. To determine the words $w' \in B_{n_q \le 2}$ that contribute to the linear expansion of w, we consider two subwords of w,

$$w_1 = 2, \quad w_2 = 34 \tag{19}$$

these are chosen in such a way that $w = 1w_1 5 w_2 6 \notin B_{n_q \le 2}$. Then, we calculate

$$s(w_2) = 34 + 43 \tag{20}$$

which, in general, is defined as the sum over all possible permutations of the letters in w_2. Finally, we calculate the shuffle product

$$w_1 ⧢ s(w_2) = 234 + 243 + 324 + 342 + 324 + 432 \tag{21}$$

which, for arbitrary letters $u = \ell_1 \ldots \ell_j$, $v = \ell_{j+1} \ldots \ell_r$, corresponds to the sum over all ordered permutations of the letters in u and v. The shuffle product is distributive. If we let σ denote an arbitrary term appearing on the sum in 21, we construct the word $w' = 1\sigma 56$, and we can calculate the element $F_{ww'}$ for the fixed w and each of those w'. In this particular case, we need to calculate six matrix elements. For example, we can calculate the matrix element for $w = 125346$ and $w' = 123456$. In general, every element will be a product of rational functions of the kinematic invariants. In this case,

$$F_{ww'} = \frac{\mathcal{F}(1\sigma 5|3)}{\hat{s}_{6,\ell_2,\ell_3}}\frac{\mathcal{F}(1\sigma 5|4)}{\hat{s}_{6,\ell_3}} \tag{22}$$

where $\sigma = 234$. Then, we form the string $\rho = 12345$, and we need to calculate the factors $\mathcal{F}(\rho|3)$ and $\mathcal{F}(\rho|4)$, whose structure depends on the position of the letters in w_1 and w_2, relative to the position of the same letters in σ. For a general factor $\mathcal{F}(\rho|\ell_k)$, one needs to determine the position of ℓ_k in the string $\rho = 1\sigma(n-1)$, which we denote by t_{ℓ_k}, and compare it with the positions of ℓ_{k+1} and ℓ_{k-1}; furthermore, if we let ℓ_j be the last letter appearing in w_1, which in our case is the letter 2, the position of said letter is always fixed to $t_{\ell_j} = n$, the number of particles in the scattering process. Finally, for $k = n-4$, we always have the condition $t_{\ell_{n-4}} = t_{\ell_{n-2}}$. In general, $\mathcal{F}(\rho|\ell_k) = \mathcal{F}_1 + \mathcal{F}_2$, where

$$\mathcal{F}_1 = \begin{cases} \sum_{r=1}^{t_{\ell_k}-1} \mathcal{G}(\ell_k, \rho_r) & t_{\ell_k} < t_{\ell_{k+1}} \\ -\sum_{r=t_{\ell_{k+1}}}^{n-1} \mathcal{G}(\ell_k, \rho_r) & t_{\ell_k} > t_{\ell_{k+1}} \end{cases}$$

$$\mathcal{F}_2 = \begin{cases} \hat{s}_{n,\ell_k,\ldots,\ell_{n-3}} & t_{\ell_{k-1}} < t_{\ell_k} < t_{\ell_{k+1}} \\ -\hat{s}_{n,\ell_k,\ldots,\ell_{n-3}} & t_{\ell_{k-1}} > t_{\ell_k} > t_{\ell_{k+1}} \\ 0 & Else \end{cases}$$

being

$$\hat{s}_{i_1,\ldots,i_k} = \sum_{a<b} (2p_{i_a} \cdot p_{i_b} + 2\Delta_{i_a i_b}) . \tag{23}$$

On the other hand,

$$\mathcal{G}(\ell_k, \rho_r) = \begin{cases} 2p_{\ell_k} \cdot p_{\rho_r} + 2\Delta_{\ell_k \rho_r} & \rho_r = 1, (n-1) \\ 2p_{\ell_k} \cdot p_{\rho_r} + 2\Delta_{\ell_k \rho_r} & \rho_r = \ell_t,\, t < k \\ 0 & Else. \end{cases}$$

For the factor $\mathcal{F}(\rho|3)$ in our example, since $\ell_2 = 3$ and $\ell_3 = 4$, $t_{\ell_2} < t_{\ell_3}$. Furthermore, since $\ell_1 = 2$, from the general condition $t_{\ell_j} = n$, we find $t_2 = 6$ so that $\mathcal{F}(\rho|3) = 2(p_1 + p_2) \cdot p_3$. From the general condition $t_{\ell_{n-2}} = t_{\ell_{n-4}}$, we find that $t_{\ell_3} > t_{\ell_4}$ therefore $\mathcal{F}(\rho|4) = -2p_4 \cdot p_5$.
This implies

$$F_{ww'} = -\frac{4(p_1 + p_2) \cdot p_3 (p_4 \cdot p_5)}{\hat{s}_{6,3,4}\hat{s}_{6,4}} . \tag{24}$$

The calculation of the rest of matrix elements proceeds similarly. That is, once w is fixed, one constructs the sum in (21) and calculates the element $F_{ww'}$ for each of those two words. In the case $w = w'$, $F_{ww'} = 1$ and the rest of elements are zero. With this in mind, we can arrange the matrix $\mathbb{F}_{6q}$, and find the square matrix

$$\mathbb{F}_{6q}\mathbb{F}_{6q}^T = \begin{bmatrix} 1 & 0 & 0 & D \\ 0 & 1 & A & E \\ 0 & A & A^2 + B^2 + C^2 & AE + BF + CG \\ D & E & AE + BF + CG & D^2 + E^2 + F^2 + G^2 + H^2 + I^2 \end{bmatrix} .$$

The non-zero elements are rational functions of various kinematic invariants and masses, and are given by

$$
\begin{aligned}
&A=-\frac{2[(p_1+p_3)\cdot p_4+m_3^2]}{\hat{s}_{6,4}}, \quad B=\frac{2[p_4\cdot(p_1+p_2+p_3)+m_3^2]}{\hat{s}_{4,6}}, \quad C=\frac{2p_1\cdot p_4}{\hat{s}_{4,6}}\\
&E=-\frac{4\left((p_2+p_5)\cdot p_4\right)(p_1\cdot p_3)}{\hat{s}_{6,3,4}\hat{s}_{6,4}}, \quad F=-\frac{4(p_1\cdot p_3)(p_4\cdot p_5)}{\hat{s}_{6,3,4}\hat{s}_{6,4}}\\
&G=-\frac{2p_1\cdot p_4\left(2p_3\cdot(p_2+p_5)+\hat{s}_{6,4,3}\right)}{\hat{s}_{6,4,3}\hat{s}_{6,4}}\\
&H=-\frac{2[(p_1+p_2)\cdot p_4][2p_3\cdot p_5+\hat{s}_{6,4,3}]}{\hat{s}_{6,4,3}\hat{s}_{6,4}}, \quad I=-\frac{-2p_1\cdot p_4\left(2p_3\cdot p_5+\hat{s}_{6,4,3}\right)}{\hat{s}_{6,4,3}\hat{s}_{6,4}}
\end{aligned}
\tag{25}
$$

and D is given by the expression on (24). For general kinematics, this matrix has a non-vanishing determinant. Thus, by explicitly computing the matrix elements and the determinant, we have confirmed the claim [8] that the CHY representation exists for the six quark primitive amplitude in QCD.

4 Conclusions

In most phenomenological applications one usually wants the amplitude computed from a given set of external four-momenta. In numerical applications, this is done through the Berends-Giele [11], or Britto-Cachazo-Feng-Witten recursion relations [12]. It is useful to have compact analytical formulae for the scattering amplitudes. Using spinor techniques [13] this can be done for every specific helicity configuration and every specific external ordering. This approach, however, does not give a way to identify how the result for a given amplitude changes when the external ordering or helicity configuration is changed. The CHY representation provides compact expressions for the amplitudes and allows to identify their dependence on the external ordering and the helicities by separating the information of these two set of parameters into pairs of independent factors, thus giving an alternate way to study the properties of tree-level scattering amplitudes.

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