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Pion–nucleon scattering: from chiral perturbation theory to Roy–Steiner equations

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Abstract. Ever since Weinberg's seminal predictions of the pion–nucleon scattering amplitudes at threshold, this process has been of central interest for the study of chiral dynamics involving nucleons. The scattering lengths or the pion–nucleon σ-term are fundamental quantities characterizing the explicit breaking of chiral symmetry by means of the light quark masses. On the other hand, pion–nucleon dynamics also strongly affects the long-range part of nucleon–nucleon potentials, and hence has a far-reaching impact on nuclear physics. We discuss the fruitful combination of dispersion-theoretical methods, in the form of Roy–Steiner equations, with chiral dynamics to determine pion–nucleon scattering amplitudes at low energies with high precision.⋆⋆

1 Introduction

Pion–nucleon (πN) scattering is one of the simplest processes to study chiral dynamics involving nucleons. The leading order (LO) in the chiral expansion, i.e., in the expansion in pion masses and momenta, results in the well-known low-energy theorems (LETs) for the S-wave scattering lengths, the amplitudes evaluated at threshold [3, 4]:

$$a_{0+}^{-} = \frac{M_\pi m_N}{8\pi(m_N + M_\pi)F_\pi^2} + O(M_\pi^3), \qquad a_{0+}^{+} = O(M_\pi^2). \tag{1}$$

The isospin-odd scattering length is hence predicted solely in terms of the pion (M_π) and nucleon (m_N) masses as well as the pion decay constant F_π, while the isospin-even one is suppressed. The strength of the Born term amplitudes (that do not contribute at threshold to leading order) is given in terms of the pion–nucleon coupling constant g, which is related to the axial coupling g_A by the Goldberger–Treiman relation, $g = g_A m_N / F_\pi$, up to higher orders.

Already at next-to-leading order (NLO; $O(p^2)$ in the chiral counting), the πN scattering amplitude depends on a list of low-energy constants (LECs), conventionally denoted by c_{1-4}, which are less readily determined from phenomenology. These NLO contributions tend to be large: three of the couplings (c_{2-4}) incorporate the leading low-energy effects of the $\Delta(1232)$, the lowest-lying resonant excitation of the nucleon. As the mass gap is small, $m_\Delta - m_N \sim 2M_\pi$, and as the Δ couples strongly to

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⋆⋆These proceedings borrow heavily from a previous conference contribution [1], as well as from our original article [2].

the πN system, the numerical values of the NLO LECs are somewhat larger than expected from naive dimensional analysis. To pin down these LECs accurately and consistently is an important task also in view of many nuclear physics applications: πN amplitudes constitute an important contribution to the two-pion exchange in nucleon–nucleon scattering potentials, and determine the leading long-range three-nucleon force.

A further strong incentive to study pion–nucleon scattering derives from its relation to the pion–nucleon σ-term $\sigma_{\pi N}$, defined via the scalar form factor of the nucleon

$$\sigma(t) = \frac{1}{2m_N}\langle N(p')|\hat{m}(\bar{u}u + \bar{d}d)|N(p)\rangle, \qquad t = (p' - p)^2, \qquad \hat{m} = \frac{m_u + m_d}{2}, \qquad \sigma_{\pi N} \equiv \sigma(0). \quad (2)$$

Through the Feynman–Hellmann theorem [5, 6], $\sigma_{\pi N}$ determines the light-quark content of the nucleon mass,

$$\sigma_{\pi N} = \hat{m}\frac{\partial m_N}{\partial \hat{m}} = -4c_1 M_\pi^2 + O(M_\pi^3), \quad (3)$$

where we have already indicated the leading term in the chiral expansion of this quantity: it is given in terms of the LEC c_1, which in turn should be fixed from πN scattering. This is the leading approximation to the Cheng–Dashen LET [7, 8], which we will discuss in more detail below. The σ-term has garnered strong interest beyond the hadron physics community due to its relation to the scalar couplings of the nucleon, which are prerequisite for a consistent interpretation of direct-detection dark matter searches [9–11].

2 Roy–Steiner equations for πN scattering

The predictive power of chiral symmetry can be vastly increased by combining chiral perturbation theory (ChPT) with dispersive techniques. In particular, for $\pi\pi$ scattering, the use of Roy equations [12] has led to a determination of the low-energy $\pi\pi$ scattering amplitude with unprecedented accuracy [13, 14]. In the case of πN scattering, a full system of partial-wave dispersion relations (PWDRs) has to include dispersion relations for two distinct physical processes, $\pi N \to \pi N$ (s-channel) and $\pi\pi \to \bar{N}N$ (t-channel), and the use of $s \leftrightarrow t$ crossing symmetry will intertwine s- and t-channel equations. Roy–Steiner (RS) equations are a set of coupled PWDRs that, in contrast to $\pi\pi$ Roy equations, are derived from hyperbolic dispersion relations [15], which automatically relate the different channels in the πN system. Solving the RS equations for πN, in particular once combined with the pionic-atom constraints on the scattering lengths [16, 17], can provide a remarkably precise representation of the πN amplitude at low energies. The construction of a complete system of RS equations for πN scattering has been presented in detail in [18]. For the s-channel partial waves, they read [15]

$$\begin{aligned} f_{l+}^I(W) &= N_{l+}^I(W) + \frac{1}{\pi}\int\limits_{4M_\pi^2}^{\infty} dt' \sum_J \Big\{G_{lJ}(W,t')\,\mathrm{Im}\, f_+^J(t') + H_{lJ}(W,t')\,\mathrm{Im}\, f_-^J(t')\Big\} \\ &+ \frac{1}{\pi}\int\limits_{W_+}^{\infty} dW' \sum_{l'=0}^{\infty}\Big\{K_{ll'}^I(W,W')\,\mathrm{Im}\, f_{l'+}^I(W') + K_{ll'}^I(W,-W')\,\mathrm{Im}\, f_{(l'+1)-}^I(W')\Big\}, \end{aligned} \quad (4)$$

where due to G-parity only even/odd angular momenta J contribute for isospin $I = +/-$, respectively. The kernels $K_{ll'}^I(W,W)$, $G_{lJ}(W,t)$, and $H_{lJ}(W,t)$ are known analytically, and $N_{l+}^I(W)$ denotes the partial-wave projections of the pole terms.

The strategy for the solution of the RS equations is as follows. In the s-channel, the six S- and P-waves $f_{l\pm}^I$, with $I = \pm$ for the isospin index and orbital angular momentum l, are considered dynamically below a matching point $s_{\rm m}$, whereas the imaginary parts of higher partial waves for all s, the imaginary parts of the S- and P-waves above $s_{\rm m}$, and, potentially, inelasticities below $s_{\rm m}$ are required as input. We choose the matching point at its optimal value $s_{\rm m} = (1.38\,\text{GeV})^2$ [18]. In contrast, there are only three S- and P-waves in the t-channel, $f_\pm^J$, with total angular momentum J and the subscript referring to parallel/antiparallel antinucleon–nucleon helicities.

Given that data in the t-channel reaction $\pi\pi \to \bar{N}N$ become available only above the two-nucleon threshold, the solution of the t-channel equations is subject to the additional complication of the large pseudophysical region $4M_\pi^2 \leq t \leq 4m_N^2$ in this reaction. The amplitudes required for the t-channel integrals need to be reconstructed from unitarity. While for every partial wave $\pi\pi$ intermediate states generate by far the dominant contribution, a coupled-channel $\pi\pi/\bar{K}K$ treatment is required for the S-wave [19], which accounts for the occurrence of the $f_0(980)$ resonance.

Once the t-channel problem is solved, the resulting t-channel partial waves are used as input for the s-channel problem, which then reduces to the form of conventional $\pi\pi$ Roy equations. Eventually, a full solution of the system can be obtained by iterating this procedure until all partial waves and parameters are determined self-consistently. The s-channel phase shifts below the matching point are represented in suitable parametrizations whose free parameters, together with the subtraction constants, are determined by minimizing the difference between the left-hand side (LHS) and right-hand side (RHS) of (4). We found that the solution is stabilized substantially when the S-wave scattering lengths, known very precisely from pionic atoms [2, 16, 17], are imposed as constraints on the system,

$$a_{0+}^{1/2} = (169.8 \pm 2.0) \times 10^{-3} M_\pi^{-1}, \qquad a_{0+}^{3/2} = (-86.3 \pm 1.8) \times 10^{-3} M_\pi^{-1}. \tag{5}$$

The minimization then provides us with a new set of subthreshold parameters and S- and P-wave phase shifts.

We have performed a full error analysis, where the uncertainty estimates include a number of effects [2]: first, we vary the input for the matching condition as well as for the energy region above the matching point and higher partial waves, both regarding different partial-wave analyses [20–23] and truncations of the partial-wave expansion. Furthermore, we vary the πN coupling constant within $g^2/(4\pi) = 13.7(2)$ [16, 17] and investigate the sensitivity to the parametrization of the low-energy phase shifts used in the solution. Second, we observe that there are important correlations in the phase space of parameters, leading to flat fit minima, but with significantly different values for the subthreshold parameters. To account for this effect, we generated a statistical ensemble of solutions exploring these shallow minima while imposing sum rules for the higher subthreshold parameters, and took the observed distribution as an additional source of uncertainty. Third, we propagate the errors in the scattering lengths, which crucially enter as constraints in the minimization, to the results for the subthreshold parameters.

The corresponding results for the s-channel partial-wave phase shifts are plotted in Fig. 1; the solutions for the t-channel are shown in [2]. The complete list of resulting subthreshold parameters is also given and compared to the Karlsruhe–Helsinki (KH80) values [20, 21] in [2]; here we just quote the two most important ones in connection to the σ-term,

$$\begin{aligned} d_{00}^+ &= (-1.361 \pm 0.032) M_\pi^{-1} && (\text{KH80:} \quad (-1.46 \pm 0.10) M_\pi^{-1}),\\ d_{01}^+ &= (1.155 \pm 0.016) M_\pi^{-3} && (\text{KH80:} \quad (1.14 \pm 0.02) M_\pi^{-3}). \end{aligned} \tag{6}$$

We also keep track of the correlations between subthreshold parameters, obtaining a 13×13 covariance matrix that encodes uncertainties and correlations of the 13 subthreshold parameters, which is relevant for the matching to ChPT.

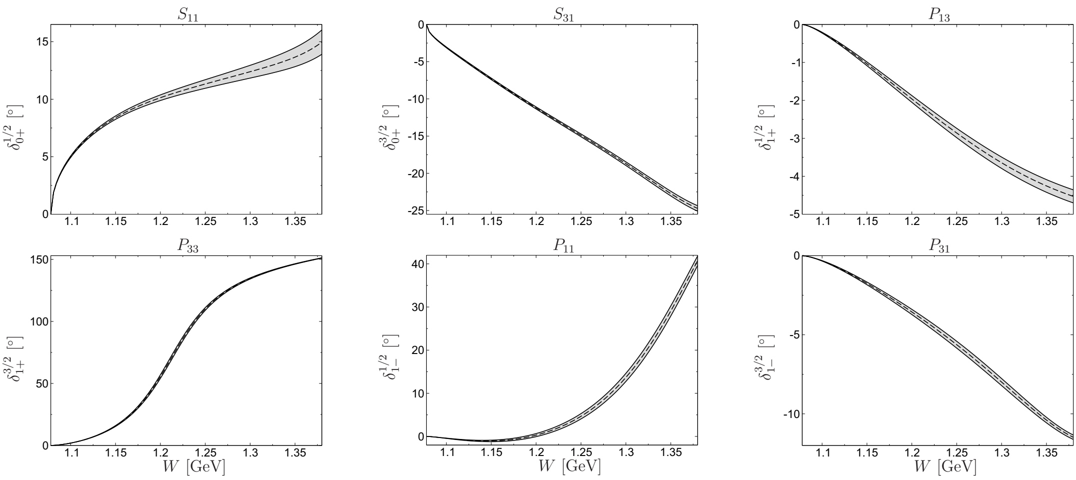

Figure 1. Error bands for the πN phase shifts; dashed lines indicate the central curves. Figures taken from [2].

3 Consequences for the πN σ-term

The Cheng–Dashen LET [7, 8] relates the Born-term-subtracted isoscalar amplitude evaluated at the Cheng–Dashen point ($\nu = 0, t = 2M_\pi^2$) to the scalar form factor of the nucleon, evaluated at momentum transfer $t = 2M_\pi^2$,

$$\bar{D}^+(0, 2M_\pi^2) = \sigma(2M_\pi^2) + \Delta_R, \tag{7}$$

where Δ_R represents higher-order corrections in the chiral expansion. These corrections are expected to be very small: we use the estimate $|\Delta_R| \lesssim 2\,\text{MeV}$ [24], derived from resonance saturation for the $O(p^4)$ LECs. In practice, the relation (7) is often rewritten as

$$\sigma_{\pi N} = \sigma(0) = \Sigma_d + \Delta_D - \Delta_\sigma - \Delta_R, \tag{8}$$

with correction terms $\Delta_\sigma = \sigma(2M_\pi^2) - \sigma_{\pi N}$, $\Delta_D = \bar{D}^+(0, 2M_\pi^2) - \Sigma_d$, $\Sigma_d = F_\pi^2(d_{00}^+ + 2M_\pi^2 d_{01}^+)$. Δ_σ measures the curvature in the scalar form factor, while Δ_D parametrizes contributions to the πN amplitude beyond the first two terms in the subthreshold expansion. As shown in [25], although these corrections are large individually due to strong rescattering in the isospin-0 $\pi\pi$ S-wave, they cancel to a large extent in the difference. For the numerical analysis we will use $\Delta_D - \Delta_\sigma = (-1.8 \pm 0.2)\,\text{MeV}$ [19]. Based on (7) the RS results for the subthreshold parameters translate immediately to a corresponding value of $\sigma_{\pi N}$. To illustrate the dependence of the σ-term on the scattering lengths used as input to the solution, we expand Σ_d linearly around the central values and find

$$\Sigma_d = (57.9 \pm 0.9)\,\text{MeV} + \sum_{I_s} c_{I_s} \Delta a_{0+}^{I_s}, \qquad c_{1/2} = 0.24\,\text{MeV}, \qquad c_{3/2} = 0.89\,\text{MeV}, \tag{9}$$

where $\Delta a_{0+}^{I_s}$ gives the deviation from the scattering lengths extracted from hadronic atoms in units of $10^{-3} M_\pi^{-1}$. (9) leads to $\Sigma_d = (46 \pm 4)\,\text{MeV}$ if the KH80 scattering lengths are used, in excellent agreement with the original KH80 value $\Sigma_d = (50 \pm 7)\,\text{MeV}$. In contrast, our central solution corresponds to $\Sigma_d = (57.9 \pm 1.9)\,\text{MeV}$, and thus to a significant increase compared to the early estimates. Including also isospin-breaking corrections [26–28], the final result [29]

$$\sigma_{\pi N} = (59.1 \pm 3.5)\,\text{MeV}, \tag{10}$$

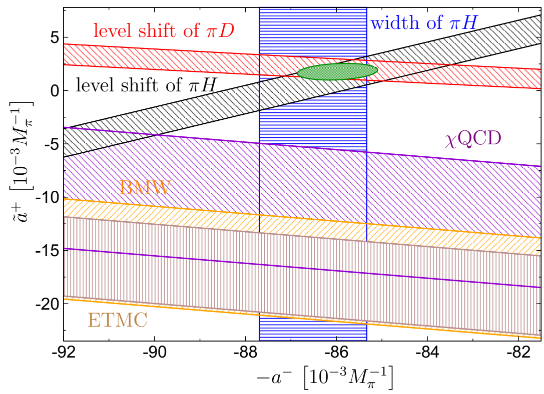

Figure 2. Constraints on the πN scattering lengths from pionic atoms and lattice σ-terms. Figure taken from [38].

does amount to a significant increase compared to the "canonical value" of $\sigma_{\pi N} \sim 45$ MeV. Already 4.2 MeV are due to new corrections to the LET (and thereof 3.0 MeV from isospin breaking); the remaining increase of nearly 10 MeV is dictated by experiment: the new scattering lengths from pionic atoms determine the position of the σ-term on the curve approximately described by (9).

Our result for the σ-term (10) seems to be somewhat at odds with a series of recent lattice calculations of the same quantity, performed near or at physical pion masses, which yield values

$$\begin{aligned} &\sigma_{\pi N} = 38(3)(3)\,\text{MeV} && \text{(BMW [30])}, && \sigma_{\pi N} = 44.4(3.2)(4.5)\,\text{MeV} && (\chi\text{QCD [31]}),\\ &\sigma_{\pi N} = 37.2(2.6)(^{+1.0}_{-0.6})\,\text{MeV} && \text{(ETMC [32])}, && \sigma_{\pi N} = 35.0(6.1)\,\text{MeV} && \text{(RQCD [33])}. \end{aligned} \tag{11}$$

Such smaller values are more consistent with analyses of flavor $SU(3)$ breaking in the baryon spectrum and the OZI rule for scalar strangeness matrix elements of the nucleon (see the discussion in [34] and references therein). However, these lattice results are in significant tension with the pionic-atoms spectroscopy measurements [35–37], interpreted in [16, 17]: the relation (9) can be inverted in a way that every value of $\sigma_{\pi N}$ corresponds to a linear constraint in the plane of πN S-wave scattering lengths [38]. The constraints corresponding to the lattice results of [30–32] are shown in Fig. 2, compared to the bands extracted from pionic atoms, illustrating the disagreement quite clearly. A lattice calculation of the πN scattering lengths may be a good way to illuminate the cause of this discrepancy.

4 Matching to chiral perturbation theory

The matching to ChPT is one of the most fundamental applications of the RS solution, since it offers a unique opportunity for a systematic determination of πN LECs [39]. One expects the chiral expansion to work best in the subthreshold region, far away from singularities, where the amplitude can be described solely by a polynomial in the Mandelstam variables. The matching is thus most conveniently performed by equating the chiral expansion for the subthreshold parameters to the RS results.

The πN amplitude at N^{3}LO, $\mathcal{O}(p^4)$, involves four NLO LECs, c_i, four (combinations of) N^{2}LO, and five N^{3}LO LECs [40], corresponding to the 13 subthreshold parameters that receive contributions from LECs in a fourth-order calculation. Inverting the expressions for the subthreshold parameters, we obtain all 13 LECs including the corresponding correlation coefficients [39]; Table 1 only shows the NLO LECs explicitly. Comparing the different extractions up to N^{3}LO, the convergence pattern

Table 1. Results for the NLO πN LECs at different orders in the chiral expansion [2, 39].

	NLO	N^2LO	N^3LO
c_1 [GeV^{-1}]	-0.74 ± 0.02	-1.07 ± 0.02	-1.11 ± 0.03
c_2 [GeV^{-1}]	1.81 ± 0.03	3.20 ± 0.03	3.13 ± 0.03
c_3 [GeV^{-1}]	-3.61 ± 0.05	-5.32 ± 0.05	-5.61 ± 0.06
c_4 [GeV^{-1}]	2.17 ± 0.03	3.56 ± 0.03	4.26 ± 0.04

for the c_i looks reasonably stable. In contrast, the convergence of some of the higher-order LECs proves to be more problematic, which can partly be explained by loop contributions enhanced by large $\Delta(1232)$ effects. The errors for the LECs at a given chiral order are negligible compared to the uncertainties to be attached to the chiral expansion itself.

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