

$SU(N) \rightarrow Z(N)$ dual superconductor models: the magnetic loop ensemble point of view

Luis E. Oxman^{1,a}

¹*Instituto de Física, Universidade Federal Fluminense,
Campus da Praia Vermelha, Niterói, 24210-340, RJ, Brazil.*

Abstract.

In this work, we initially discuss some physical properties of effective $SU(N) \rightarrow Z(N)$ YMH models, emphasizing the important role of valence gluons. Next, we review how adjoint fields are naturally generated as an effective description of “adjoint” loops in $4D$. Finally, we discuss the consequences that can be learnt from this point of view, and briefly comment on some improvements.

1 Introduction

Dual superconductivity is a promising scenario to understand confinement in pure Yang-Mills (YM) theories [1]-[3]. On the one hand, many different dual superconductor models where the confining string is represented by a *classical* vortex solution have been explored (see [4]-[22] and references therein). On the other, results obtained from Monte Carlo simulations have been understood in terms of the proliferation of magnetic degrees of freedom, detected in the lattice (see [23] [32] and references therein). The latter correspond to ensembles of *quantum* objects such as center vortices, monopoles and chains, which capture the path-integral measure of pure YM. These approaches can be thought of as (dual non-Abelian) versions of the understanding of superconductors in terms of the Guinzburg-Landau wave functional and the condensation of Cooper pairs, respectively. Analyzing to what extent the existing lattice phenomenology is accommodated and the relation with well-established *quantum* ensemble descriptions are two important tasks to be pursued.

On the side of effective models, those based in the bosonic sector of supersymmetric theories have been extensively studied. In particular, non-Abelian strings in $U(N)_{\text{gauge}} \times SU(N)_{\text{flavor}}$ YMH models with fundamental Higgs fields were introduced in refs. [10], [14], [16]. In the SSB phase, there is a remnant global flavor-locking $SU(N)_{C+F}$ symmetry which equips confining strings with non Abelian degrees of freedom. In ref. [16], a non-supersymmetric model with $SU(N) \times U(1)$ gauge symmetry and a similar color-flavor locking vacuum was introduced, with decay rates of quasi-stable k -strings [17] similar to those present in large N pure YM. Although it is constructed in terms of N flavors of fundamental Higgs fields, the presence of the $U(1)$ factor leads to good N -ality properties. As is well-known, this important property (see ref. [33]) can also be implemented in models with at least N flavors of adjoint Higgs fields [34]-[36]. In ref. [20], they were implemented in the bosonic part

^ae-mail: oxman@if.uff.br

of $\mathcal{N} = 2$ super QCD with a soft breaking mass term. Relying on phenomenological arguments, they were proposed to describe the confining string in pure YM (see refs. [4]-[9] and references therein). In ref. [22], we pointed out that ensembles of monopoles carrying adjoint charges could be naturally represented by models based on a set of adjoint Higgs fields. This type of magnetic degree has been detected in lattice calculations of pure YM, interpolating different center vortices to form closed chains [32]. In ref. [37], assuming phenomenological information to characterize an ensemble of magnetic “adjoint” loops, such as tension, stiffness and (magnetic) color degrees of freedom (d.o.f.), we showed how adjoint fields can be generated.

In this talk, we shall briefly review some of these ideas and discuss the consequences that can be learnt from this point of view. We shall also comment on some improvements in the description of non Abelian ensembles of loops; a detailed discussion will be reported elsewhere.

2 $SU(N) \rightarrow Z(N)$ models with adjoint scalars

Let us consider a general Yang-Mills-Higgs (YMH) model with a set of adjoint fields $\psi_I \in \mathfrak{su}(N)$,

$$\frac{1}{2} \langle D_\mu \psi_I, D^\mu \psi_I \rangle + \frac{1}{4g^2} \langle F_{\mu\nu} - J_{\mu\nu}, F^{\mu\nu} - J^{\mu\nu} \rangle - V_{\text{Higgs}}(\psi_I), \quad (1)$$

$$D_\mu = \partial_\mu + \Lambda_\mu \wedge, \quad F_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu + \Lambda_\mu \wedge \Lambda_\nu,$$

where Λ_μ is a non Abelian (dual) gauge field and I is a flavor index; the color index can be made explicit by expanding the fields in a Lie algebra basis T_A , $\psi_I = \psi_I^A T_A$. We are using the notation $X \wedge Y = -i[X, Y]$, and $\langle X, Y \rangle$ for the Killing form. The source, $J_{\mu\nu} = \vec{J}_{\mu\nu}|_q T_q$, $\vec{J}_{\mu\nu} = 2\pi 2N \vec{w}_e s_{\mu\nu}$ depends on a weight \vec{w}_e of the quark representation.¹ Each static source contributes to $s_{\mu\nu}$ with,

$$s_{0i} = 0, \quad s_{ij} = -\epsilon_{ijk} \int ds \frac{dx_k}{ds} \delta^{(3)}(x - x(s)),$$

where $x(\sigma)$ is a Dirac line ending (starting) at the quark location. Higgs potentials that drive $SU(N) \rightarrow Z(N)$ are characterized by configurations of absolute minima with the property $U \psi_I^0 U^{-1} = \psi_I^0$ just if $U \in Z(N)$, that is, $\mathcal{M} = SU(N)/Z(N)$. As is well known, this phase is characterized by:

- ***N*-ality**: This is a consequence of $\Pi_1(\mathcal{M}) = Z(N)$. The asymptotic behavior of a center string is *locally* a pure gauge (but not *globally*) that can be written in terms of the phase,

$$S = e^{i\varphi \vec{\beta} \cdot \vec{T}} \quad , \quad \vec{\beta} = 2N \vec{w} \quad , \quad \vec{\beta} \cdot \vec{T} = \vec{\beta}|_q T_q. \quad (2)$$

For the fundamental representation, there are N weights \vec{w}_i (fundamental colors). Infinite adjoint strings are trivial, as the asymptotic behavior $S \sim e^{i\varphi 2N \vec{\alpha} \cdot \vec{T}}$ ($\vec{\alpha}$ a root) is a closed loop in $SU(N)$. Then, it can be continuously deformed to avoid any defect at the origin ($\Pi_1(SU(N)) = 0$).²

- **Normal mesons**: A pair of quark sources, with fundamental weights $\vec{w}, -\vec{w}$, are introduced by the Dirac lines in fig. 1a. The absolute minimum is characterized by S in eq. (2). Around the Dirac lines, the field profiles should be close to the true vacuum. In this way, the field configuration will be a (singular) pure gauge in that region, and the string-like singularity in F_{ij} cancels the Dirac lines (see fig. 1b). On the other hand, between the quarks, the phase defect implies a false vacuum region. At the end, a finite center string with smooth energy density is induced (see fig. 1c).

¹A weight \vec{w} is defined by the eigenvalues of the Cartan generators T_q corresponding to one common eigenvector.

$$[T_q, T_p] = 0 \quad , \quad T_q \text{ eigenvector} = \vec{w}|_q \text{ eigenvector} \quad , \quad q, p = 1, \dots, N-1.$$

²The roots are the weights of the adjoint representation, which acts via commutators $[T_q, E_\alpha] = \vec{\alpha}|_q E_\alpha$.

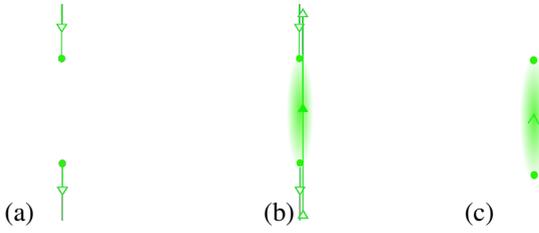


Figure 1. Here, we show: (a) the external Dirac lines, (b) their cancellation against similar terms in F_{ij} and (c) the final induced finite string.

2.1 Additional physical properties

• **Hybrid mesons:** As shown in refs. [11], [12], non-Abelian string models have solutions where different strings can be interpolated by monopoles (forming complexes). The latter were interpreted as confined string-attached gluons [16], [18] (for a similar interpretation in the context of supersymmetric theories, see ref. [21]). These configurations were discussed for the first time in ref. [38], in the SU(2) case. In ref. [22], motivated by previous work on the description of chains in the continuum, using defects of a local Lie basis [39], an analysis of center strings interpolated by monopoles was presented, pointing to some physical consequences. In this regard, lattice calculations predict a rich spectrum of exotic mesons. Some of them correspond to $qq\bar{q}'$ hybrids, where a nonsinglet color pair and a valence gluon form a colorless state. This state is currently searched by a collaboration based at the Jefferson Lab (GlueX) [40].

The (infinite) hybrid glue is induced by fundamental sources $\vec{w}, -\vec{w}'$, with $\vec{w} \neq \vec{w}'$. This state is associated with a non Abelian phase (θ, ϕ are spherical angles),

$$S = e^{i\varphi \vec{\beta} \cdot \vec{T}} W(\theta) \quad , \quad W(\theta) = e^{i\theta \sqrt{N} T_\alpha} \quad , \quad \vec{\alpha} = \vec{w} - \vec{w}' .$$

Around the north pole, $S \sim e^{i\varphi \vec{\beta} \cdot \vec{T}}$, while around the south pole, using that $W(\pi)$ is a Weyl reflection, we get, $S \sim W(\pi) e^{i\varphi \vec{\beta} \cdot \vec{T}}$. As gauge transformations act on the left, this is locally equivalent to the behavior $\sim e^{i\varphi \vec{\beta} \cdot \vec{T}}$. Note that S gives a well-defined mapping for the local Cartan directions on S^2 , $n_q = S T_q S^{-1}$, with a point-like defect at the origin. Of course, in the complete ansatz, some profile functions must tend to zero at this singularity, and others must tend to zero on those line-like singularities where the guiding centers of the strings are located. The gauge invariant monopole charge is obtained from the (dual) field strength projection along the *local* Cartan directions n_q , $\vec{Q}_m = 2\pi 2N (\vec{w} - \vec{w}') = 2\pi 2N \vec{\alpha}$. This is a root, or weight of the adjoint representation, thus leading the dual monopole to be identified with a valence gluon with adjoint color $\vec{\alpha} = \vec{w} - \vec{w}'$.

• **Valence gluons are confined:** Another important property of the *dynamical* dual monopole is that it cannot exist as an isolated object. This comes about as $\mathcal{M} = \text{Ad}(\text{SU}(N))$ is a compact group, so that $\Pi_2(\mathcal{M}) = 0$.

• **Adjoint quarks are not confined:** Consider a pair of adjoint sources at a finite distance, with weights $\vec{\alpha}, -\vec{\alpha}$, respectively. We could proceed as for normal mesons, considering $S_{\text{Abe}} = e^{i\varphi 2N \vec{\alpha} \cdot \vec{T}}$, $\vec{\alpha} = \vec{w} - \vec{w}'$, thus obtaining a linearly growing (finite) energy. However, at some point, another type of configuration will be preferred. We can also consider the non Abelian phase [41], $S_{\text{non-Ab}} = e^{i\varphi \vec{\beta} \cdot \vec{T}} W(\gamma) e^{-i\varphi \vec{\beta} \cdot \vec{T}}$, where γ, φ are bipolar angles. Around $\gamma \sim 0$ (see fig. 2a), $S_{\text{non-Ab}} \sim e^{i\varphi 2N \vec{\alpha} \cdot \vec{T}}$. Then, to cancel the Dirac lines that run between the sources and infinity, the field profiles must tend to a true vacuum in that region. In addition, for $\gamma \sim \pi$ (see fig. 2b), $S_{\text{non-Ab}} \sim W(\pi)$. That is, between the

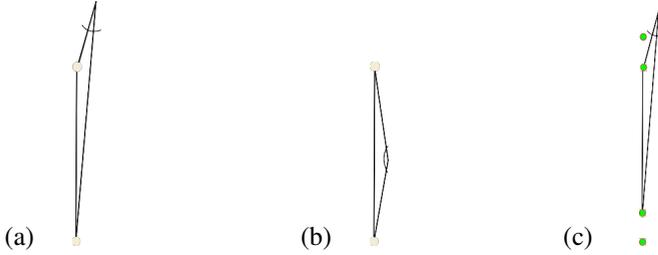


Figure 2. Bipolar coordinates with foci at a pair of adjoint quarks, (a) $\gamma \sim 0$, (b) $\gamma \sim \pi$, and at inner quarks in double pairs of fundamental quarks (c).

quarks there is no phase defect (nor Dirac lines) and the fields can also assume values close to the true vacuum. Then, the energy density will only be significant around the quarks, where $S_{\text{non-Abel}}$ looks like the non Abelian phase that characterizes a valence gluon (in spherical coordinates around each one of the sources). In other words, the energy minimization will induce a dynamical dual monopole around each adjoint quark, screening them by the formation of a pair of adjoint-quark/valence-gluon bound states. This is the correct picture for the breaking of the adjoint string [42].

• **Difference-in-areas for doubled pairs of fundamental quarks:** In ref. [43], it was shown that double-winding Wilson loops in SU(2) gauge theory, computed by Monte Carlo simulations, obey a difference-in-areas law, in agreement with the center vortex model of confinement. On the other hand, Abelian pictures lead to a sum of areas (see [43] and references therein). Here, we will show that SU(2) \rightarrow Z(2) effective YMH models also agree with Monte Carlo simulations. For this aim, let us consider four fundamental SU(2) quarks as shown in fig. 2c. The two upper (lower) quarks have negative (positive) weight³, doubling the simple scheme in fig. 1a. When the upper and lower sets are sufficiently apart, the absolute minima will be given by $S_{\text{non-Abel}} = e^{i\varphi\beta T_1} W(\gamma) e^{i\varphi\beta T_1}$. When $\gamma \sim 0$, the behavior $S_{\text{non-Abel}} \sim e^{i\varphi 2\beta T_1}$ will introduce a doubled string-like singularity in F_{ij} . This will cancel the doubled Dirac lines between the outer quarks and infinity (region I), and will partially cancel the single Dirac line between the upper (lower) quarks (region II). When $\gamma \sim \pi$ we have the regular behavior $S_{\text{non-Abel}} \sim W(\pi)$, and no Dirac lines (region III). Then, the smooth energy density requirement leads to true vacuum in regions I and III. In addition, there will be a state similar to a fundamental string between the upper (lower) quarks with lengths L_u (L_l). In other words, the generated action for this static configurations goes like $TL_{u+} + TL_l = TL_o - TL_i$, where L_o (L_i) is the distance between the outer (inner) quarks, and T is the (infinite) configuration lifetime.

The possibility of a fundamental string between, say, the upper quarks (each carrying positive charge w) is due to the induction of a valence gluon (with adjoint charge $-2w$) around the inner quark. This partial screening also occurs at the inner lower quark location.

3 Ensembles and fields

Lattice calculations in pure YM show that ensembles of magnetic objects capture the main contribution to the path-integral in the infrared regime (see [26, 31, 32] and references therein). In particular, the center vortices that arise in lattice center gauges are good at describing N -ality and have a physical density scaling. A large percentage of them end at monopoles forming closed chains (see [32] and references therein). Similarly to the dual description of valence gluons, the monopole component in

³For SU(2), weights are one-component. The fundamental ones are $\pm w$, $w = \frac{1}{2\sqrt{2}}$.

chains can be described by Weyl transformations, so they carry adjoint charges labelled by the roots of $\mathfrak{su}(N)$. Here, we review the description of ensembles of loops that carry an adjoint charge, equipped with non Abelian (magnetic) color d.o.f. [37]. We also clarify some points regarding the projection over a reduced sector of well-defined color states.

As is well-known, ensembles of one-dimensional objects lead in general to effective field models [44]-[45]. The sum over loops can be seen as a sum over different numbers of particle worldlines, that is, a second quantized field theory represented by a path-integral. The loops can be characterized by phenomenological properties: tension τ , stiffness ξ^{-1} (which is important for the continuum limit), interactions among them, etc. We also considered the coupling,

$$\int ds \left[\frac{1}{2} (\bar{z}_c \dot{z}_c - \dot{\bar{z}}_c z_c) - i u_\mu I_A \Lambda_\mu^A(x(s)) \right] , \quad I_A = M_A|_{cd} \bar{z}_c z_d \quad (3)$$

which was introduced in ref. [46] to describe a classical relativistic particle interacting with a non Abelian gauge field. In our work, it was included to make contact with typical terms in the dual effective YMH models. The z_a 's, $a = 1, \dots, D$, are complex variables; D is the dimension of the group representation under consideration. In particular, for adjoint loops we have $D = N^2 - 1$ and the matrix elements of $M_A = R(T_A)$, $A = 1, \dots, N^2 - 1$, are proportional to the $\mathfrak{su}(N)$ structure constants. The partition function for the ensemble of adjoint loops was then defined by,

$$Z = \int [D\phi][D\phi_A] e^{-W[\phi, \phi_A]} \sum_n Z_n , \quad (4)$$

$$Z_n = \int [Dm]_n \exp \left[- \sum_{k=1}^n \int_0^{L_k} ds_k (\cdot)_k \right] , \quad u_\mu = \frac{dx_\mu}{ds} \in S^3 ,$$

$$(\cdot) = \tau + \frac{1}{2} (\bar{z}_c \dot{z}_c - \dot{\bar{z}}_c z_c) + \frac{1}{2\xi} \dot{u}_\mu \dot{u}_\mu - i u_\mu I_A \Lambda_\mu^A(x) + \phi(x) + I_A \phi^A(x) ,$$

$$[Dm]_n \equiv \frac{1}{n!} \int_0^\infty \frac{dL_1}{L_1} \frac{dL_2}{L_2} \dots \frac{dL_n}{L_n} \int dv_1 dv_2 \dots dv_n \int [Dv(s)_1]_{v_1, v_1}^{L_1} \dots [Dv(s)_n]_{v_n, v_n}^{L_n}$$

$$v : x, u, z \quad , \quad dv = d^4x d^3u dz d\bar{z} ,$$

where n sums over the number of loops and W encodes some correlations among them; in particular, excluded volume effects (density-density interactions) are implemented with a ϕ^2 -term. Similarly, (magnetic) color-dependent density interactions are introduced by means of a ϕ_A^2 -term in W . The measure $[Dv(s)]_{v, v}^L$ refers to a single (smooth) loop of size L that starts and ends at a given set of variables $v = x, u, z$; it integrates over every possible shape. The associated weight $q(v, v, L)$ can be obtained from the end-to-end probability for an open line $q(v, v_0, L)$, after identifying the initial and final points. In this manner, the sum over loops becomes,

$$\sum_n Z_n = e^{\int_0^\infty \frac{dL}{L} \int dv q(v, v, L)} , \quad q(v, v_0, L) = \int [Dv(s)]_{v, v_0}^L e^{-\int_0^L ds (\cdot)} . \quad (5)$$

The effective field model was obtained as follows. The probability $q(v, v_0, L)$ to start at x_0 , with tangent u_0 and z_0 , and end at x with u, z can be studied by relying on the equilibrium theory of inhomogeneous polymers [47]. For this aim, we considered a Chapman-Kolmogorov recurrence relation

for diffusion in v -space (polymer growth),

$$q_j(x, x_0, u, u_0, \bar{z}, z_0) = \int d^4 x' d^3 u' dz' d\bar{z}' e^{-\tau\Delta L} e^{(\bar{z}-\bar{z}')\cdot z'} \times \\ \psi(u-u') e^{-\omega(x,u,\bar{z},z')\Delta L} \delta(x-x'-u\Delta L) q_{j-1}(x', x_0, u', u_0, \bar{z}', z_0), \\ \psi(u-u') = \mathcal{N} e^{-\frac{1}{2\xi}\Delta L\left(\frac{u-u'}{\Delta L}\right)^2}, \quad \omega(x, u, \bar{z}, z') = \phi(x) - i u_\mu \Lambda_\mu^A(x) T_{cd}^A \bar{z}^c z'^d + \phi^A(x) T_{cd}^A \bar{z}^c z'^d.$$

Taking the initial condition, $q_0(x, x_0, u, u_0, \bar{z}, z_0) = \delta(x-x_0) \delta(u-u_0) e^{\bar{z}\cdot z_0}$, a discretized version of q for a size $L = M \Delta L$ polymer was obtained upon M iterations. Then, using the recurrence, a Fokker-Plank equation was derived by relating the probability to get x, u, z with M monomers and the probability to get $x' = x + u' \Delta L, u', z'$ with $M + 1$ (see fig. 3),

$$\partial_L q = \left[-\tau - \phi(x) + \frac{\xi}{\pi} \hat{L}_u^2 - u_\mu \partial_\mu + (i u_\mu \Lambda_\mu^A - \phi^A) M_{cd}^A \bar{z}^c \frac{\partial}{\partial \bar{z}^d} \right] q. \quad (6)$$

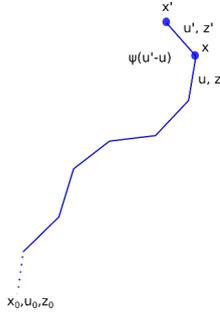


Figure 3. Polymer growth, from M to $M + 1$ monomers.

The path-integral over the z -sector (cf. eq. (5)) can also be written as a transition between coherent color states,

$$q(v, v_0, L) = \int [Dx(s)]_{x,x_0}^L [Du(s)]_{u,u_0}^L \langle z | P \left\{ e^{-\int_0^L ds \hat{H}(s)} \right\} | z_0 \rangle, \quad (7)$$

$$\hat{H}(s) = \left(\tau + \frac{1}{2\xi} \dot{u}_\mu \dot{u}_\mu + \phi(x(s)) \right) \hat{I} - i u_\mu \Lambda_\mu^A(x(s)) M_{cd}^A \hat{a}_c^\dagger \hat{a}_d + \phi^A(x(s)) M_{cd}^A \hat{a}_c^\dagger \hat{a}_d, \quad (8)$$

where $|z\rangle$ stands for the overcomplete basis of coherent states $|z_1, \dots, z_D\rangle$, $\hat{a}_c |z\rangle = z_c |z\rangle$, in a linear space of general (magnetic) color states [37]. Therefore,

$$\sum_n Z_n = \exp \int_0^\infty \frac{dL}{L} \int d^4 x d^3 u \int [Dx(s)]_{x,x}^L [Du(s)]_{u,u}^L \text{Tr} P \left\{ e^{-\int_0^L ds \hat{H}(s)} \right\}. \quad (9)$$

This trace is originated from the $dz d\bar{z}$ integration. Switching to the occupation number basis $|N\rangle$,⁴

$$\sum_n Z_n = \exp \int_0^\infty \frac{dL}{L} \int d^4 x d^3 u \sum_N \mathcal{Q}_{NN}, \quad (10)$$

$$\mathcal{Q}_{NM} = \int [Dx(s)]_{x,x}^L [Du(s)]_{u,u}^L \langle N | P \left\{ e^{-\int_0^L ds \hat{H}(s)} \right\} | M \rangle \\ = \int dz d\bar{z} dz_0 d\bar{z}_0 e^{-\frac{\xi z}{2}} e^{-\frac{\bar{z}_0 \cdot z_0}{2}} \bar{\psi}_N(z) \psi_M(\bar{z}_0) q(v, v_0, L). \quad (11)$$

⁴ N denotes the tuple of occupation numbers (N_1, \dots, N_D) .

A partial contribution to $\sum_N Q_{NN}$ comes from states $|N\rangle$ of the form $|0, \dots, 1, \dots, 0\rangle$, with one non-trivial entry $N_a = 1$, $a = 1, \dots, D$. They have well-defined color a , and are represented by the wavefunction $\bar{\psi}(z) = z^a$. Organizing the associated matrix elements in a reduced $D \times D$ matrix $Q|^{cd} = Q^{cd}$, eq. (6) was projected into the reduced Fokker-Plank equation,

$$\left[(\partial_L - (\xi/\pi) \hat{L}_u^2 + (\tau + \phi) 1 + \phi^A M_A + u \cdot D \right] Q(x, x_0, u, u_0, L) = 0. \quad (12)$$

$$Q(x, x_0, u, u_0, 0) = \delta(x - x_0) \delta(u - u_0) 1, \quad D_\mu = 1 \partial_\mu - i \Lambda_\mu^A M^A.$$

In this manner, taking the semiflexible limit (small stiffness), we arrived at,

$$\int d^4x d^3u Q(x, x, u, u, L) \approx \int d^4x \langle x | e^{-LO} | x \rangle, \quad O = -\frac{\pi}{12\xi} D_\mu D_\mu + (\phi + \tau) 1 + \phi^A M_A,$$

$$\sum_n Z_n = \exp \int_0^\infty \frac{dL}{L} \int d^4x d^3u \sum_{a=1}^D Q_{aa}(x, x, u, u, L) + \dots = \exp[-\text{Tr} \ln O] \dots \quad (13)$$

The dots represent the contribution originated from other sectors, which produce effective fields carrying product representations of the original D -dimensional representation. Finally, performing the path-integral in eq. (4), and using the adjoint representation, the sum over the ensemble was obtained in terms of an effective complex adjoint Higgs field $\zeta = \psi_1 + i\psi_2$, $Z = \int [D\psi] e^{-\int d^4x \mathcal{L}_{\text{eff}}}$,

$$\mathcal{L}_{\text{eff}} = \frac{1}{2} \langle D_\mu \psi_I, D^\mu \psi_I \rangle + \frac{\mu^2}{2} \langle \psi_I, \psi_I \rangle + \frac{\lambda}{4} \langle \psi_I, \psi_I \rangle \langle \psi_J, \psi_J \rangle + \frac{\eta}{4} \langle \psi_I \wedge \psi_J, \psi_I \wedge \psi_J \rangle, \quad (14)$$

where $\mu^2 \propto \tau \xi$, and ψ_I , $I = 1, 2$ is a pair of Hermitian adjoint Higgs fields.

3.1 Non Abelian coupling and group coherent states

In the non-Abelian coupling (3), instead of using linear variables z_c we can consider nonlinear (Gilmore-Perelomov) group coherent states for the adjoint representation,

$$|z\rangle = \tilde{R} |u_\alpha\rangle, \quad \tilde{R} \in \text{Ad}(\text{SU}(N)),$$

where $|u_\alpha\rangle$ is the weight vector for the highest weight $\vec{\alpha}$ of the adjoint representation. This is an overcomplete basis over a D -dimensional vector space ($D = N^2 - 1$), $\int d(G/H) |z\rangle \langle z| = I$. In this case,

$$\frac{dx_\mu}{ds} I_A \Lambda_\mu^A - i z_c \dot{z}_c = \frac{dx_\mu}{ds} \text{Tr} (\tilde{R}^{-1} \Lambda_\mu \tilde{R} + i \tilde{R}^{-1} \partial_\mu \tilde{R}) (\vec{\alpha} \cdot \vec{M}), \quad (15)$$

and instead of eqs. (7) and (8), we have [49],

$$q(v, v_0, L) = \int [Dx(s)]_{x, x_0}^L [Du(s)]_{u, u_0}^L \langle z | P \left\{ e^{-\int_0^L ds H(s)} \right\} | z_0 \rangle, \quad (16)$$

$$H(s) = \left(\tau + \frac{1}{2\xi} \dot{u}_\mu \dot{u}_\mu + \phi(x(s)) \right) I - i u_\mu \Lambda_\mu^A(x(s)) M_A + \phi^A(x(s)) M_A.$$

In this respect, note that the path-integral over the z -sector in eq. (5) (when $\phi_A = 0$) is proportional to the Petrov-Diakonov representation of transition elements of a (dual) Wilson line, computed between group coherent states. Then, eq. (13) gets replaced by,

$$\sum_n Z_n = \exp \int_0^\infty \frac{dL}{L} \int d^4x d^3u \sum_{a=1}^D \Gamma_{aa}(x, x, u, u, L), \quad (17)$$

$$\Gamma_{ba}(x, x_0, u, u_0, L) = \int [Dx(s)]_{x, x_0}^L [Du(s)]_{u, u_0}^L P \left\{ e^{-\int_0^L ds H(s)} \right\} \Big|_{ba}. \quad (18)$$

The discretized version of this path-integral, in matrix form, can again be obtained from a Chapman-Kolmogorov recurrence relation. Moreover, using again the recurrence to relate M and $M + 1$ steps, we can show that Γ satisfies eq. (12), with Γ in the place of Q , see ref. [49]. Then, the consideration of Gilmore-Perelemov group coherent states leads directly to the former reduced Fokker-Planck equation, and a representation only based on effective adjoint fields is obtained, without product representations.

3.2 Higgs potentials and flavor symmetry

Now, let us discuss some specific features of $SU(N) \rightarrow Z(N)$ YMH models motivated from the ensemble point of view. In the $SU(2)$ case, the simplest model is based on a pair of real adjoint scalars [34]-[36],

$$V_{\text{Higgs}}(\psi_1, \psi_2) = \frac{\mu_1^2}{2} \langle \psi_1, \psi_1 \rangle + \frac{\mu_2^2}{2} \langle \psi_2, \psi_2 \rangle + \frac{\lambda_1}{4} \langle \psi_1, \psi_1 \rangle^2 + \frac{\lambda_2}{4} \langle \psi_2, \psi_2 \rangle^2 + \frac{\gamma}{2} \langle \psi_1, \psi_1 \rangle \langle \psi_2, \psi_2 \rangle + \frac{\beta}{2} \langle \psi_1, \psi_2 \rangle^2 \quad (19)$$

($\lambda_1, \lambda_2 > 0$). To drive $SU(2) \rightarrow Z(2)$ SSB, the following conditions must be satisfied: $\mu_1^2, \mu_2^2 < 0$, $\lambda_2 \frac{\mu_1^2}{\mu_2^2} > \gamma + \beta$, $\lambda_1 \frac{\mu_2^2}{\mu_1^2} > \gamma + \beta$. Here, the last term is essential for $SU(2) \rightarrow Z(2)$ SSB, as for nontrivial $\langle \psi_1, \psi_1 \rangle$, $\langle \psi_2, \psi_2 \rangle$ it favors a minimization with linearly independent fields, ψ_1 and ψ_2 . Then, in this case, the SSB phase cannot display an additional global $U(1)$ flavor symmetry,

$$\psi'_1 = \cos \omega \psi_1 + \sin \omega \psi_2 \quad , \quad \psi'_2 = -\sin \omega \psi_1 + \cos \omega \psi_2 . \quad (20)$$

This symmetry would be present for $\mu_1^2 = \mu_2^2$, $\lambda_1 = \lambda_2 = \gamma$ and, necessarily, vanishing β . On the other hand, any effective field model derived from a loop ensemble is expected to possess this symmetry, which is related to the equivalence between loops with different orientations. In particular, the potential in eq. (14) is flavor symmetric,

$$V_{\text{Higgs}}(\psi_1, \psi_2) = \frac{\mu^2}{2} [\langle \psi_1, \psi_1 \rangle + \langle \psi_2, \psi_2 \rangle] + \frac{\lambda}{4} [\langle \psi_1, \psi_1 \rangle + \langle \psi_2, \psi_2 \rangle]^2 + \frac{\eta}{2} \langle \psi_1 \wedge \psi_2, \psi_1 \wedge \psi_2 \rangle . \quad (21)$$

If $\mu^2 < 0$, $\lambda, \eta > 0$, there is no $SU(2) \rightarrow Z(2)$ SSB, as the last term tends to align ψ_1 and ψ_2 . However, this phase does appear for $\eta < 0$, $2\lambda + \eta > 0$. Another possibility is to embed $U(1)$ in $SO(3) = \text{Ad}(SU(2))$ by including a third adjoint field ψ_3 . In general, $\text{Ad}(SU(N))$ flavor symmetry was implemented in ref. [22]. For this objective, we considered $N^2 - 1$ flavors, $I \rightarrow A = 1, \dots, N^2 - 1$ and the Higgs potential,

$$V_{\text{Higgs}} = c + \frac{\mu^2}{2} \langle \psi_A, \psi_A \rangle + \frac{\kappa}{3} f_{ABC} \langle \psi_A \wedge \psi_B, \psi_C \rangle + \frac{\lambda}{4} \langle \psi_A \wedge \psi_B, \psi_A \wedge \psi_B \rangle . \quad (22)$$

When $\mu^2 < \frac{2}{9} \frac{\kappa}{\lambda}$, the absolute minima are given by Lie bases associated with structure constants f_{ABC} , thus driving $SU(N) \rightarrow Z(N)$ SSB. These vacua have a color-flavor locking symmetry $\text{Ad}(SU(N))_{C+F}$ similar to that proposed in refs. [10], [14], [16]. In addition, at $\mu^2 = 0$ center string field equations get Abelianized [41], with $N - 1$ fields pointing along the Cartan directions, and frozen at vacuum values. The remaining $N(N - 1)$ real adjoint fields are along off-diagonal directions. They can be combined as $\phi_\alpha E_\alpha$, where E_α are root vectors (with positive weight $\vec{\alpha}$) and $\phi_{\alpha_1}, \phi_{\alpha_2}, \dots$ are $N(N - 1)/2$ complex scalar fields. This Abelianized content makes contact with effective models for monopole ensembles in pure YM, based on Abelian projection and Abelian dominance [48].

4 Conclusions

In this work, we initially discussed some physical properties of effective $SU(N) \rightarrow Z(N)$ YMH models emphasizing the important role of valence gluons, which are represented by confined (dual) monopoles. Among them, we list: i) The possibility of hybrid mesons, currently searched by the GlueX collaboration, ii) The correct picture for the adjoint string breaking between adjoint quarks, iii) Difference-in-areas for doubled pairs of fundamental $SU(2)$ quarks, understood as the minimum energy for doubled quark-antiquark pairs.

Next, we reviewed how adjoint fields are naturally generated as an effective description of “adjoint” loops in $4D$, with tension, stiffness, and (magnetic) color degrees of freedom. On the other hand, monopole loops with adjoint charges have been detected in center vortex ensembles that capture the path-integral measure in infrared YM theories. This gives further support to look for dual superconductor models with adjoint Higgs fields. In this respect, we discussed the simplest ensemble of loops in $SU(2)$ showing that the symmetry between different loop orientations is manifested as a $U(1)$ flavor symmetry, which can guide the construction of the effective model. This could also be accommodated in a larger $Ad(SU(2))$ flavor symmetry realized in recently studied models, and extended to the case of $SU(N)$.

Answering what is the appropriate field content and symmetries will require further steps to characterize ensembles in pure YM, elucidating the origin of magnetic non Abelian d.o.f. and the role of center vortices in the model construction. The latter is a particularly difficult task as center vortices are two-dimensional objects (for a treatment in the continuum, see refs. [26]-[28]). On the other hand, this is the component that can pierce Wilson loops to produce (N -ality dependent) center elements. Here, we anticipated some simpler steps. Namely, the projection of effective fields into the adjoint representation, without product representations, by coupling magnetic color degrees via Gilmore-Perelomov group coherent states. Indeed, in ref. [49], we will show that these degrees are naturally originated by combining center gauges that detect magnetic defects in the continuum [50] together with a non Abelian Hodge decomposition.

Acknowledgements

The Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) is acknowledged for the financial support.

References

- [1] Nambu, Phys. Rev. **D10**, (1974) 4262.
- [2] Mandelstam, Phys. Rep. **23C** (1976) 245.
- [3] G. 't Hooft, Nucl. Phys. B138 (1978) 1.
- [4] S. Maedan, Y. Matsubara and T. Suzuki, Prog. of Theor. Phys. **84** (1990) 130.
- [5] Y. Koma, E. M. Ilgenfritz, H. Toki, and T. Suzuki, Phys. Rev. **D64** (2001) 011501(R).
- [6] Y. Koma, M. Koma, E. M. Ilgenfritz and T. Suzuki, Phys. Rev. **D68** (2003) 114504.
- [7] M. Baker, J. S. Ball and F. Zachariasen, Phys. Rev. **D41** (1990) 2612.
- [8] M. Baker, J. S. Ball and F. Zachariasen, Phys. Rev. **D44** (1991) 3328.
- [9] M. Baker, N. Brambilla, H. G. Dosch, and A. Vairo, Phys. Rev. **D58** (1998) 034010.
- [10] A. Hanany and D. Tong, JHEP **0307** (2003) 037
- [11] A. Hanany and D. Tong, JHEP 0404 (2004) 066.
- [12] D. Tong, Phys. Rev. **D69** (2004) 065003.

- [13] D. Tong, *Ann. of Phys.* **324** (2009) 30.
- [14] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, *Nucl. Phys. B* **673** (2003) 187.
- [15] K. Konishi, *Lect. Notes Phys.* **737** (2008) 471.
- [16] A. Gorsky, M. Shifman and A. Yung, *Phys. Rev.* **D71** (2005) 045010.
- [17] A. Armoni and M. Shifman, *Nucl. Phys.* **B671** (2003) 67.
- [18] M. Shifman and A. Yung, *Phys. Rev. D* **70**, 045004 (2004).
- [19] K. Konishi and L. Spanu, *Int. J. Mod. Phys.* **A18** (2003) 249.
- [20] Marco A. C. Kneipp and Patrick Brockill, *Phys. Rev.* **D64** (2001) 125012.
- [21] Marco A. C. Kneipp, *Phys. Rev.* **D76** (2007) 125010.
- [22] L. E. Oxman, *J. High Energy Phys.* **03** (2013) 038.
- [23] G. Mack and V. B. Petkova, *Ann. Phys.* **123** (1979) 442; *ibid.* **125** (1980) 117.
- [24] L. Del Debbio, M. Faber, J. Greensite and S. Olejnik, *Phys. Rev.* **D55** (1997) 2298.
- [25] K. Langfeld, H. Reinhardt and O. Tennert, *Phys. Lett.* **B419** (1998) 317; *ibid.* **B431** (1998) 141.
- [26] M. Engelhardt and H. Reinhardt, *Nucl. Phys.* **B567** (2000) 249.
- [27] H. Reinhardt, *Nucl. Phys.* **B628** (2002) 133.
- [28] M. Engelhardt, M. Quandt, and H. Reinhardt, *Nucl. Phys.* **B685** (2004) 227.
- [29] Ph. de Forcrand and M. Pepe, *Nucl. Phys.* **B598** (2001) 557.
- [30] F. V. Gubarev, A. V. Kovalenko, M. I. Polikarpov, S. N. Syritsyna and V. I. Zakharov, *Phys. Lett.* **B574** (2003) 136.
- [31] J. Greensite, *Prog. Part. Nucl. Phys.* **51** (2003) 1.
- [32] J. Greensite, *An introduction to the confinement problem* (Springer, Berlin-Heidelberg, 2011).
- [33] B. Lucini and M. Teper, *Phys. Lett.* **B501** (2001) 128.
- [34] H. J. de Vega, *Phys. Rev.* **D18** (1978) 2932.
- [35] H. J. de Vega and F. A. Schaposnik, *Phys. Rev.* **D34** (1986) 3206.
- [36] J. Heo and T. Vachaspati, *Phys. Rev.* **D58** (1998) 065011.
- [37] L. E. Oxman, G. C. Santos Rosa and B. F. I. Teixeira, *J. Phys. A: Math. Theor.* **47** (2014) 305401.
- [38] M. Hindmarsh and T. W. B. Kibble, *Phys. Rev. Lett.* **55** (1985) 2398.
- [39] L. E. Oxman, *JHEP* **0812** 2008 089.
- [40] GlueX website: <http://www.glueX.org/GlueX/Home.html>
- [41] L. E. Oxman and D. Vercauteren, arXiv:1603.01105.
- [42] S. Kratochvila and P. de Forcrand, *Nucl. Phys.* **B671** (2003) 103.
- [43] Jeff Greensite and Roman Höllwieser, *Phys. Rev.* **D91** (2015) 054509.
- [44] K. Bardakci and S. Samuel, *Phys. Rev.* **D18** (1978) 2849.
- [45] M. B. Halpern and W. Siegel, *Phys. Rev.* **D16** (1977) 2486.
- [46] A. P. Balachandran, P. Salomonson, B. Skagerstam and J. Winnberg, *Phys. Rev.* **D15** (1977) 2308.
- [47] G. H. Fredrickson, *The Equilibrium Theory of Inhomogeneous Polymers* (2nd edition, Clarendon Press, Oxford, 2006), p. 452.
- [48] D. Antonov, *Surveys High Energ. Phys.* **14** (2000) 265.
- [49] L. E. Oxman, in preparation.
- [50] L. E. Oxman and G. C. Santos-Rosa, *Phys. Rev.* **D92** (2015) 125025.