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XIIth Quark Confinement & the Hadron Spectrum

Dispersive approach to QCD and hadronic contributions to electroweak observables

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Abstract. The dispersive approach to QCD is briefly overviewed and its application to the assessment of hadronic contributions to electroweak observables is discussed.

Various strong interaction processes, as well as the hadronic contributions to electroweak observables, are governed by the hadronic vacuum polarization function $\Pi(q^2)$, related Adler function $D(Q^2)$ [1], and the function $R(s)$, which is identified with the R–ratio of electron–positron annihilation into hadrons. The hadronic vacuum polarization function constitutes the scalar part of the hadronic vacuum polarization tensor

$$\Pi_{\mu\nu}(q^2) = i \int d^4x \, e^{iqx} \langle 0 \big| T\{J_\mu(x)\, J_\nu(0)\} \big| 0 \rangle = \frac{i}{12\pi^2} \left(q_\mu q_\nu - g_{\mu\nu} q^2 \right) \Pi(q^2), \tag{1}$$

whereas the definition of functions $R(s)$ and $D(Q^2)$ is given in Eqs. (3) and (4), respectively. The QCD asymptotic freedom makes it possible to study the high–energy behavior of the functions $\Pi(q^2)$ and $D(Q^2)$ directly within perturbation theory. At the same time, the description of the function $R(s)$ additionally requires the use of pertinent dispersion relations, see papers [2–6] and references therein. As for the low–energy hadron dynamics, it can only be accessed within nonperturbative approaches, for instance, analytic gauge–invariant QCD [7–10], holographic QCD [11, 12], Schwinger–Dyson equations [13–15], Bethe–Salpeter equations [16–19], lattice simulations [20–25], operator product expansion [26–30], nonlocal chiral quark model [31, 32], and others (see, e.g., Ref. [33]).

A certain nonperturbative hint about the strong interactions in the infrared domain is provided by the relevant dispersion relations. The latter are widely employed in a variety of issues of contemporary theoretical particle physics, such as, for example, the extension of applicability range of chiral perturbation theory [34, 35], the precise determination of parameters of resonances [36], the assessment of the hadronic light–by–light scattering [37], and many others (see, e.g., papers [38–54] and references therein).

Basically, the dispersion relations render the kinematic restrictions on pertinent physical processes into the mathematical form and thereby impose stringent intrinsically nonperturbative constraints on the relevant quantities. In particular, the complete set of dispersion relations (see Refs. [1–3, 55, 56])

$$\Delta\Pi(q^2, q_0^2) = (q^2 - q_0^2) \int_{m^2}^{\infty} \frac{R(\sigma)}{(\sigma - q^2)(\sigma - q_0^2)} \, d\sigma = - \int_{-q_0^2}^{-q^2} D(\zeta) \frac{d\zeta}{\zeta}, \tag{2}$$

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$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon\to 0_+} \Big[\Pi(s+i\varepsilon) - \Pi(s-i\varepsilon)\Big] = \frac{1}{2\pi i} \lim_{\varepsilon\to 0_+} \int_{s+i\varepsilon} D(-\zeta)\,\frac{d\zeta}{\zeta}, \tag{3}$$

$$D(Q^2) = -\frac{d\,\Pi(-Q^2)}{d\ln Q^2} = Q^2 \int_{m^2}^{\infty} \frac{R(\sigma)}{(\sigma+Q^2)^2}\,d\sigma \tag{4}$$

supplies substantial restraints on the infrared behavior of the functions on hand, that, in turn, plays an essential role in the study of the pertinent strong interaction processes at low energies, see papers [56–58] for the details. In Eqs. (2)–(4) $Q^2 = -q^2 > 0$ and $s = q^2 > 0$ stand for the spacelike and timelike kinematic variables, respectively, $m^2 = 4m_\pi^2$ is the hadronic production threshold, and $\Delta\Pi(q^2,\,q_0^2) = \Pi(q^2) - \Pi(q_0^2)$ denotes the subtracted hadronic vacuum polarization function.

The dispersive approach to QCD [56–58] (its preliminary formulation was discussed in Refs. [59, 60]) merges the aforementioned nonperturbative constraints with corresponding perturbative input and provides the following unified integral representations for the functions on hand:

$$\Delta\Pi(q^2,\,q_0^2) = \Delta\Pi^{(0)}(q^2,\,q_0^2) + \int_{m^2}^{\infty} \rho(\sigma)\ln\left(\frac{\sigma-q^2}{\sigma-q_0^2}\,\frac{m^2-q_0^2}{m^2-q^2}\right)\frac{d\,\sigma}{\sigma}, \tag{5}$$

$$R(s) = R^{(0)}(s) + \theta(s-m^2)\int_{s}^{\infty} \rho(\sigma)\frac{d\,\sigma}{\sigma}, \tag{6}$$

$$D(Q^2) = D^{(0)}(Q^2) + \frac{Q^2}{Q^2+m^2}\int_{m^2}^{\infty} \rho(\sigma)\frac{\sigma-m^2}{\sigma+Q^2}\,\frac{d\,\sigma}{\sigma}. \tag{7}$$

In these equations $\theta(x)$ is the unit step–function [$\theta(x) = 1$ if $x \geq 0$ and $\theta(x) = 0$ otherwise], the leading–order terms read [61, 62]

$$\Delta\Pi^{(0)}(q^2,\,q_0^2) = 2\,\frac{\varphi-\tan\varphi}{\tan^3\varphi} - 2\,\frac{\varphi_0-\tan\varphi_0}{\tan^3\varphi_0}, \qquad \sin^2\!\varphi = \frac{q^2}{m^2}, \qquad \sin^2\!\varphi_0 = \frac{q_0^2}{m^2}, \tag{8}$$

$$R^{(0)}(s) = \theta(s-m^2)\left(1-\frac{m^2}{s}\right)^{3/2}, \tag{9}$$

$$D^{(0)}(Q^2) = 1 + \frac{3}{\xi}\left[1-\sqrt{1+\xi^{-1}}\,\operatorname{arcsinh}\left(\xi^{1/2}\right)\right], \qquad \xi = \frac{Q^2}{m^2}, \tag{10}$$

and $\rho(\sigma)$ denotes the spectral density

$$\rho(\sigma) = \frac{1}{\pi}\frac{d}{d\,\ln\sigma}\,\mathrm{Im}\lim_{\varepsilon\to 0_+} p(\sigma-i\varepsilon) = -\frac{d\,r(\sigma)}{d\,\ln\sigma} = \frac{1}{\pi}\,\mathrm{Im}\lim_{\varepsilon\to 0_+} \mathrm{d}(-\sigma-\mathrm{i}\varepsilon), \tag{11}$$

with $p(q^2)$, $r(s)$, and $d(Q^2)$ being the strong corrections to the functions $\Pi(q^2)$, $R(s)$, and $D(Q^2)$, respectively, see papers [56–58] and references therein for the details. The integral representations (5)–(7) are by construction consistent with the foregoing nonperturbative constraints and constitute the "dispersively improved perturbation theory" (DPT) expressions for the functions on hand.

The rigorous calculation of the spectral density (11), which would have thoroughly accounted for both perturbative and nonperturbative aspects of hadron dynamics, is a rather challenging and hardly feasible (at least, at the present time) objective. Nonetheless, it appears that even with such incomplete input as the perturbative part of the spectral density

$$\rho_{\mathrm{pert}}(\sigma) = \frac{1}{\pi}\frac{d}{d\,\ln\sigma}\,\mathrm{Im}\lim_{\varepsilon\to 0_+} p_{\mathrm{pert}}(\sigma-i\varepsilon) = -\frac{d\,r_{\mathrm{pert}}(\sigma)}{d\,\ln\sigma} = \frac{1}{\pi}\,\mathrm{Im}\lim_{\varepsilon\to 0_+} d_{\mathrm{pert}}(-\sigma-i\varepsilon) \tag{12}$$

the integral representations (5)–(7) are capable of yielding a physically sound behavior of the functions on hand in the entire energy range. At the one–loop level Eq. (12) assumes a quite simple form,

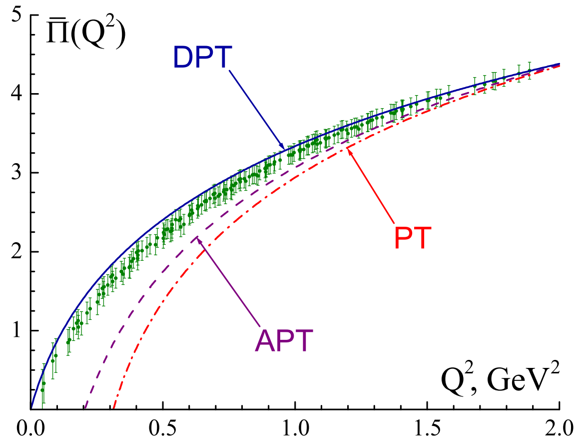

Figure 1. Hadronic vacuum polarization function within various approaches: dispersively improved perturbation theory (label DPT, Eq. (5), solid curve), its massless limit (label APT, dashed curve), perturbative approximation (label PT, dot–dashed curve), and lattice simulation data (Ref. [67], circles).

namely, $\rho^{(1)}_{\mathrm{pert}}(\sigma) = (4/\beta_0)[\ln^2(\sigma/\Lambda^2) + \pi^2]^{-1}$, where $\beta_0 = 11 - 2n_{\mathrm{f}}/3$, n_{f} denotes the number of active flavors, and Λ is the QCD scale parameter. The explicit expressions for the spectral function (12) up to the four–loop level are given in Ref. [63] (recently calculated respective four–loop perturbative coefficient can be found in Ref. [64]). The perturbative spectral function (12) will be employed hereinafter.

It has to be mentioned that in the massless limit ($m = 0$) for the case of perturbative spectral function (12) the relations for the function $R(s)$ (6) and the Adler function $D(Q^2)$ (7) become identical to those of the "analytic perturbation theory" (APT) [38] (see also Refs. [39–54]), whereas the hadronic vacuum polarization function $\Pi(q^2)$ was not addressed in the framework of the latter. However, as discussed in, e.g., papers [56–58, 65, 66], the massless limit ignores substantial nonperturbative constraints, which relevant dispersion relations impose on the functions on hand, that appears to be essential for the studies of hadron dynamics at low energies.

The dispersive approach to QCD [56–58] enables one to get rid of some inherent obstacles of the QCD perturbation theory and significantly extends its range of applicability toward the infrared domain. In particular, the Adler function (7) agrees with its experimental prediction in the entire energy range [57, 66, 68] (the studies of $D(Q^2)$ within other approaches can be found in, e.g., Refs. [69–75]). The representation (7) also complies with the results of Bethe–Salpeter calculations [19] as well as of lattice simulations [76, 77]. The dispersive approach to QCD is capable of describing OPAL (update 2012, Ref. [78]) and ALEPH (update 2014, Ref. [79]) experimental data on inclusive τ lepton hadronic decay in both vector and axial–vector channels in a self–consistent way [56, 80] (see also Refs. [65, 81]). Additionally, as one can infer from Fig. 1, the DPT expression for the hadronic vacuum polarization function (Eq. (5), solid curve) is in a good agreement with lattice simulation data [67] (circles) (the rescaling procedure described in Refs. [76, 82] was applied). The displayed in Fig. 1 result corresponds to the four–loop level, $\Lambda = 419$ MeV, and $n_{\mathrm{f}} = 2$ active flavors. Figure 1 also presents the massless limit of Eq. (5) (label APT, dashed curve) and the perturbative approximation of $\Pi(q^2)$ (label PT, dot–dashed curve). As one can infer from this figure, the perturbative approximation of $\Pi(q^2)$ contains infrared unphysical singularities, that invalidates it at low energies, whereas the APT prediction diverges in the infrared limit $Q^2 \to 0$, that makes it also inapplicable at low energies.

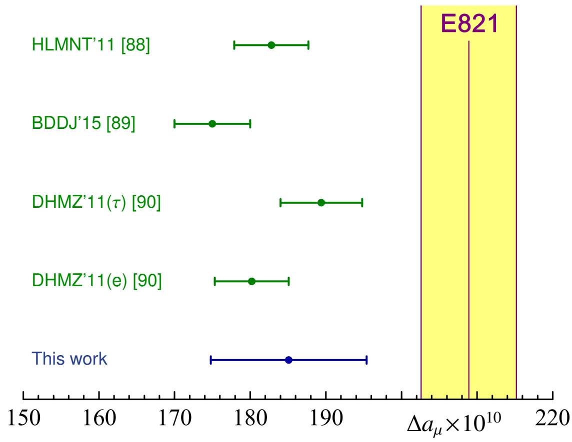

Figure 2. The subtracted muon anomalous magnetic moment ($\Delta a_\mu = a_\mu - a_0,\ a_0 = 11659 \times 10^{-7}$): theoretical evaluations (circles) and experimental measurement (Eq. (16), shaded band).

Let us address now the hadronic contributions to electroweak observables. One of the most challenging issues of the elementary particle physics, which engages the entire pattern of interactions within the Standard Model, is the muon anomalous magnetic moment $a_\mu = (g_\mu - 2)/2$. The persisting few standard deviations discrepancy between the experimental measurements [83, 84] and theoretical evaluations [85, 86] of this quantity may be an evidence for the existence of a new physics beyond the Standard Model, that brings the accuracy of an estimation of a_μ to the top of the agenda.

The uncertainty of theoretical estimation of a_μ is mainly dominated by the leading–order hadronic contribution [87]

$$a_\mu^{\text{HLO}} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_0^1 (1-x)\,\bar{\Pi}\left(m_\mu^2 \frac{x^2}{1-x}\right) dx, \qquad \bar{\Pi}(Q^2) = \Delta\Pi(0, -Q^2). \tag{13}$$

This expression involves the integration of $\Pi(q^2)$ over the low–energy range, that basically gives the major contribution to a_μ^{HLO}. However, the infrared behavior of the hadronic vacuum polarization function is inaccessible within perturbation theory. One of the ways to overcome this obstacle is to express the integrand of Eq. (13) in terms of the R–ratio of electron–positron annihilation into hadrons and substitute its low–energy behavior by the corresponding experimental measurements, see, e.g., Refs. [88–90].

At the same time, as elucidated above, the DPT expression for $\Pi(q^2)$ (5) contains no unphysical singularities, that, in turn, enables one to perform the integration in Eq. (13) in a straightforward way, namely, without invoking experimental data on R–ratio. Eventually this results in the following assessment

$$a_\mu^{\text{HLO}} = (696.1 \pm 9.5) \times 10^{-10}, \tag{14}$$

see Refs. [58, 91]. Equation (14) corresponds to the four–loop level and the quoted error accounts for the uncertainties of the parameters entering Eq. (13), their values being taken from Ref. [92]. The obtained estimation of a_μ^{HLO} (14) proves to be in a good agreement with its recent evaluations [88–90].

In addition to the leading–order hadronic contribution a_μ^{HLO} (14), the complete muon anomalous magnetic moment a_μ includes the higher–order [88] and light–by–light [93] hadronic contributions,

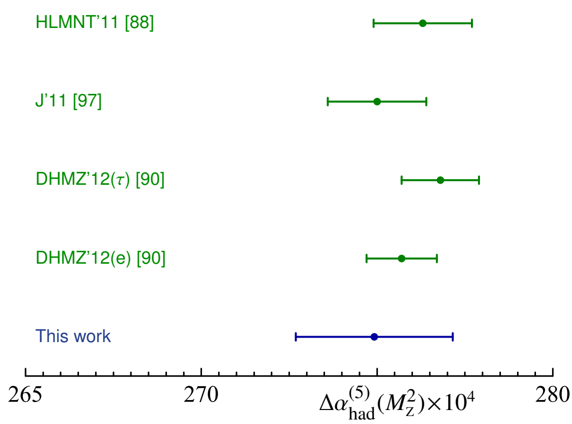

Figure 3. Theoretical evaluations of the five–flavor hadronic contribution to the shift of electromagnetic fine structure constant at the scale of Z boson mass.

the QED contribution [94], and the electroweak contribution [95], that altogether yields

$$a_\mu = (11659185.1 \pm 10.3) \times 10^{-10}, \tag{15}$$

see Refs. [58, 91]. The discrepancy between the experimental value [84, 96]

$$a_\mu^{\rm exp} = (11659208.9 \pm 6.3) \times 10^{-10} \tag{16}$$

and the estimation (15) corresponds to two standard deviations. As one can infer from Fig. 2, the obtained value of the muon anomalous magnetic moment a_μ (15) also complies with its recent assessments [88–90].

Another observable of an apparent interest is the electromagnetic running coupling

$$\alpha_{\rm em}(q^2) = \frac{\alpha}{1 - \Delta\alpha_{\rm lep}(q^2) - \Delta\alpha_{\rm had}(q^2)}, \tag{17}$$

which plays a crucial role in a variety of issues of precision particle physics. The leptonic contribution $\Delta\alpha_{\rm lep}(q^2)$ to Eq. (17) can reliably be calculated by making use of perturbation theory [98]. However, similarly to the earlier discussed case of the muon anomalous magnetic moment, the hadronic contribution to Eq. (17)

$$\Delta\alpha_{\rm had}(q^2) = -\frac{\alpha}{3\pi}\, q^2\, \mathcal{P}\!\int_{m^2}^{\infty} \frac{R(s)}{s - q^2}\, \frac{ds}{s} \tag{18}$$

involves the integration over the low–energy range and constitutes the major source of the uncertainty in the assessment of $\alpha_{\rm em}(q^2)$, see, e.g., Refs. [88, 99]. In Eq. (18) $\mathcal{P}$ stands for the "Cauchy principal value".

The five–flavor hadronic contribution to the shift of the electromagnetic fine structure constant at the scale of Z boson mass can be evaluated in the framework of DPT in the very same way as above, that leads to

$$\Delta\alpha^{(5)}_{\rm had}(M_Z^2) = (274.9 \pm 2.2) \times 10^{-4}, \tag{19}$$

see Refs. [58, 91]. The estimation (19) corresponds to the four–loop level and the quoted error accounts for the uncertainties of the parameters entering Eq. (18), their values being taken from

Ref. [92]. As one can infer from Fig. 3, the obtained assessment of $\Delta\alpha^{(5)}_{\rm had}(M_Z^2)$ (19) proves to be in a good agreement with its recent evaluations [88, 90, 97]. The corresponding value of the electromagnetic running coupling $\alpha_{\rm em}(M_Z^2)$ (18) additionally includes leptonic [98] and top quark [100] contributions, that eventually yields

$$\alpha^{-1}_{\rm em}(M_Z^2) = 128.962 \pm 0.030. \tag{20}$$

The obtained value (20) also complies with recent assessments of the quantity on hand [88, 90, 97], see papers [58, 91] and references therein for the details.

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