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On the relativistic $^3D_1$ partial-wave contribution to the bound three-nucleon system

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Abstract. The bound state of three nucleons is investigated using the Faddeev equations within the Bethe-Salpeter approach. The relativistic and nonrelativistic nucleon-nucleon interaction is chosen in a multirank separable form. The extension for partial-states with $L > 0$ is done. Three partial-wave states - $^1S_0$,$^3S_1$ and $^3D_1$ - are taken into account. The Gauss quadrature method is used to calculate the integrals and find the triton binding energy by iterations.

1 Introduction

Three-body calculations in nuclear physics are of great interest used for the description of three-nucleon bound states (^3He, T), processes of elastic, inelastic and deep inelastic scattering the leptons off light nuclei and also the hadron-deuteron reactions (for example, $pd \to pd$, $pd \to ppn$). The investigation of the nuclei ^3He and T is also interesting because it allows us to investigate further (in addition to the case of the deuteron) evolution of the bound nucleon thereby contributing to the explanation of so-called EMC-effect. In quantum mechanics the Faddeev equations are commonly used to describe the three-particle systems. The main feature of Faddeev equations is that all particles interact through a pair potential.

However at the high momentum transfer relativistic effects should be taken into account. The Bethe-Salpeter (BS) [1] equation is one of the most consistent approaches to describe the NN interaction. In this formalism, one has to deal with a system of nontrivial integral equations for both the NN scattered states and the bound state – the deuteron. To solve a system of integral equations, it is convenient to use a separable *Ansatz* [2] for the interaction kernel in the BS equation. In this case, one can transform integral equations into a system of algebraic linear ones which is easy to solve. Parameters of the interaction kernel are found from an analysis of the phase shifts and inelasticity, low-energy parameters and deuteron properties (binding energy, moments, etc.).

The relativistic three-particle systems are described by the Faddeev equations within the BS approach - so called Bethe-Salpeter-Faddeev equations. All nucleons have equal masses and the scalar propagators instead of spinor ones are used for simplicity. The spin-isospin structure of the nucleons is taken into account by using the so-called recoupling-coefficient matrix. The work mainly follows the ideas of the paper [3]. In the paper [4] only S partial-states were considered for the one-rank

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kernel. In this paper the relativistic $^3D_1$ partial-wave is added into formalism using the three-rank Graz-II kernel [5].

The paper is organized as follows: in Sec. 2 the two-particle problem is considered, in Sec. 3 three-particle equations and partial-wave decomposition are described. In Sec. 4 the calculations and results are given. The summary is in the Sec. 5.

2 Two-particle case

Since the formalism of the Faddeev equations is based on the properties of the pair nucleon-nucleon interaction only some conclusions of two-body problem are given here.

The Bethe-Salpeter equation for the relativistic two-particle system is taken in the following form:

$$T(p,p';s) = V(p,p') + \frac{i}{4\pi^3}\int d^4k\, V(p,k)\, G(k;s)\, T(k,p';s) \tag{1}$$

where $T(p,p';s)$ is the two-particle T matrix and $V(p,p')$ - kernel (potential) of the nucleon-nucleon interaction. The free two-particle Green function $G(k;s)$ is expressed, for simplicity, through the scalar propagator of the nucleons

$$G^{-1}(k;s) = [(P/2+k)^2 - m_N^2 + i\epsilon][(P/2-k)^2 - m_N^2 + i\epsilon]. \tag{2}$$

To solve equation (1) the separable *Ansatz* for the nucleon-nucleon potential $V(p,p')$ is used

$$V_{LL'}(p_0,p,p_0',p') = \sum_{n_1 n_2} \lambda_{n_1 n_2} g_{n_1}^{(L)}(p_0,p) g_{n_2}^{(L')}(p_0',p'). \tag{3}$$

In this case the two-particle T matrix has the following simple form:

$$T_{LL'}(p_0,p,p_0',p',s) = \sum_{n_1 n_2} \tau_{n_1 n_2}(s) g_{n_1}^{(L)}(p_0,p) g_{n_2}^{(L')}(p_0',p'). \tag{4}$$

where

$$[\tau^{-1}(s)]_{n_1 n_2} = [\lambda^{-1}]_{n_1 n_2} - \frac{i}{4\pi^3}\sum_{L=0,2}\int_{-\infty}^{\infty} dk_0 \int_0^{\infty} k^2\, dk\, g_{n_1}^{(L)}(k_0,k) g_{n_2}^{(L')}(k_0,k)\, G(k_0,k;s). \tag{5}$$

3 Three-particle case

Neglecting the three-particles interaction we write the equations for three-particle amplitude $T = \sum_{i=1}^{3} T^{(i)}$ in the following form:

$$\begin{bmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} - \begin{bmatrix} 0 & T_1G_1 & T_1G_1 \\ T_2G_2 & 0 & T_2G_2 \\ T_3G_3 & T_3G_3 & 0 \end{bmatrix} \begin{bmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{bmatrix} \tag{6}$$

where G_i is the free two-particle (j and n) Green function (ijn is cyclic permutation of (1,2,3)):

$$G_i(k_j,k_n) = S^{(j)}(k_j) S^{(n)}(k_n). \tag{7}$$

The two-particle amplitude T_i satisfies the Bethe-Salpeter equation (see previous section).

For the system of equal-mass particles the Jacobi momenta can be written in the following form:

$$p_i = \frac{1}{2}(k_j - k_n),\ q_i = \frac{1}{3}K - k_i,\ K = k_1 + k_2 + k_3. \tag{8}$$

Using expressions (8) the amplitude T (6) can be rewritten as

$$T^{(i)}(p_i, q_i; p'_i, q'_i; K) = (2\pi)^4\delta^{(4)}(q_i - q'_i)T_i(p_i; p'_i; P) \tag{9}$$
$$-i\int \frac{dp''_i}{(2\pi)^4} T_i(p_i; p''_i; P)G_i(k''_j, k''_n)[T^{(j)}(p''_j, q''_i; p'_i, q'_i; K) + T^{(n)}(p''_i, q''_i; p'_i, q'_i; K)],$$

where P is the total two-body momentum.

For the bound state the total momentum squared $s = K^2$ is fixed at the mass of the bound state (triton) $M_B = \sqrt{s} = 3m_N - E_B$. The equation (9) becomes homogeneous for the amplitude $\Psi^{(i)}(p_i, q_i; s)$

$$\Psi^{(i)}(p_i, q_i; s) = \langle p_i, q_i | T^{(i)} | M_B \rangle \equiv \Psi_{LM}(p, q; s), \tag{10}$$

For the equal-mass case all $\Psi^{(i)}$ functions are equal to each other and we can write

$$\Psi(p, q; K) = -i\int \frac{dq'}{(2\pi)^4}\Big[T(p, -\frac{q}{2} - q'; P)S(\frac{K}{3} + q + q')S(\frac{K}{3} - q')\Psi(q + \frac{q'}{2}, q'; K)$$
$$+T(p, \frac{q}{2} + q'; P)S(\frac{K}{3} + q + q')S(\frac{K}{3} - q')\Psi(-q - \frac{q'}{2}, q'; K)\Big]. \tag{11}$$

To perform the partial-wave decomposition several steps should be made:

1. construct two-body state in its rest frame;

2. boost two-body state into the three-particle rest frame;

3. construct three-body state.

The details of the partial-wave decomposition ideology for the quasi-potential equation can be found in the paper [6].

All steps should be made if the particles are considered as spin-one-half with corresponding spinor propagators $S^{(i)}$. However, as it was said in previous section we simplify the problem and consider the nucleons with scalar propagators (2). In this case the spin-isospin structure of the system can be presented as a simple multiplicative coefficient.

The total orbital angular momentum of the triton then can be written as $\boldsymbol{L} = \boldsymbol{l} + \boldsymbol{\lambda}$, where $\boldsymbol{l}$ is the angular momentum corresponding to a nucleon pair with relative momentum $\boldsymbol{p}$ and $\boldsymbol{\lambda}$ is the angular momentum corresponding to relative momentum $\boldsymbol{q}$.

To separate the angular dependence the amplitude can be written in the three-particle rest frame in the following form:

$$\Psi_{LM}(p, q; s) = \sum_{a\lambda} \Psi^{(a)}_{\lambda L}(p_0, |\mathbf{p}|, q_0, |\mathbf{q}|; s)\mathcal{Y}^{(a)}_{\lambda LM}(\hat{\mathbf{p}}, \hat{\mathbf{q}}), \tag{12}$$

with the angular part

$$\mathcal{Y}^{(a)}_{\lambda LM}(\hat{\mathbf{p}}, \hat{\mathbf{q}}) = \sum_{m\mu} C^{LM}_{lm\lambda\mu} Y_{lm}(\hat{\mathbf{p}})Y_{\lambda\mu}(\hat{\mathbf{q}}), \tag{13}$$

where the two-nucleon state with spin s, angular momentum l and total momentum j ($a \equiv {}^{2s+1}l_j$) is introduced, C are the Clebsch-Gordan coefficients and Y are the spherical harmonics.

If one considers two-nucleon separable interaction (with n_i being the separable index) the amplitude $\Psi^{(a)}_{\lambda L}$ can be written as

$$\Psi^{(a)}_{\lambda L}(p_0, p, q_0, q; s) = \sum_{n_1 n_2} g^{(a)}_{n_1}(p_0, p)\tau^{(a)}_{n_1 n_2}\Phi^{(a)}_{\lambda L n_2}(q_0, q; s), \tag{14}$$

where functions $\Phi^{(a)}_{\lambda Li}$ satisfy the following system of integral equations:

$$\Phi^{(a)}_{\lambda L n_2}(q_0, q; s) = \frac{i}{4\pi^3}\sum_{a'\lambda'}\sum_{n_3 n_4}\int_{-\infty}^{\infty} dq'_0 \int_0^{\infty} q'^2 dq' \times$$
$$Z^{(aa')}_{\lambda\lambda' n_2 n_3}(q_0, q; q'_0, q'; s)\frac{\tau^{(a')}_{n_3 n_4}[(\frac{2}{3}\sqrt{s} + q'_0)^2 - q'^2]}{(\frac{1}{3}\sqrt{s} - q'_0)^2 - q'^2 - m_N^2 + i\epsilon}\Phi^{(a')}_{\lambda' n4}(q'_0, q'; s), \tag{15}$$

with

$$Z^{(aa')}_{\lambda\lambda' n_2 n_3}(q_0, q; q'_0, q'; s) = C_{(aa')}\int d\cos\vartheta_{qq'} K^{(aa')}_{\lambda\lambda' L}(q, q', \cos\vartheta_{qq'})$$
$$\frac{g^{(a)}_{n_2}(-\frac{1}{2}q_0 - q'_0, |\mathbf{q}/2 + \mathbf{q}'|)g^{(a')}_{n_3}(q_0 + \frac{1}{2}q'_0, |\mathbf{q} + \mathbf{q}'/2|)}{(\frac{1}{3}\sqrt{s} + q_0 + q'_0)^2 - (\mathbf{q} + \mathbf{q}')^2 - m_N^2 + i\epsilon} \tag{16}$$

and

$$K^{(aa')}_{\lambda\lambda' L}(q, q', \cos\vartheta_{qq'}) = (4\pi)^{3/2}\frac{\sqrt{2\lambda + 1}}{2L + 1}$$
$$\sum_{mm'} C^{Lm}_{lm\lambda 0} C^{Lm}_{l'm'\lambda' m-m'} Y^*_{lm}(\vartheta, 0) Y_{l'm'}(\vartheta', 0) Y_{\lambda' m-m'}(\vartheta_{qq'}, 0) \tag{17}$$

where

$$\cos\vartheta = (\frac{q}{2} + q'\cos\vartheta_{qq'})/|\frac{\mathbf{q}}{2} + \mathbf{q}'|, \qquad \cos\vartheta' = (q + \frac{q'}{2}\cos\vartheta_{qq'})/|\mathbf{q} + \frac{\mathbf{q}'}{2}|$$

and

$$C_{(aa')} = \begin{bmatrix} \frac{1}{4} & -\frac{3}{4} & -\frac{3}{4} \\ -\frac{3}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \tag{18}$$

is the spin-isospin recoupling-coefficient matrix with $(a) = {}^1S_0, {}^3S_1, {}^3D_1$.

In the case of $L = 0$ we have $l = \lambda = 0$ or $l = \lambda = 2$ and expressions for non-zero K-functions are:

$$\begin{aligned} K_{00} &= 1 \\ K_{02} &= 4\pi A \\ K_{20} &= \sqrt{4\pi}Y^*_{20}(\vartheta, 0) \\ K_{22} &= \sqrt{(4\pi)^3}Y^*_{20}(\vartheta, 0)A \\ A &= \sum_{m'} C^{00}_{2m'2-m'} Y_{2m'}(\vartheta', 0) Y_{2-m'}(\vartheta_{qq'}, 0), \end{aligned} \tag{19}$$

and the K-matrix is:

$$K_{(aa')} = \begin{bmatrix} K_{00} & K_{00} & K_{02} \\ K_{00} & K_{00} & K_{02} \\ K_{20} & K_{20} & K_{22} \end{bmatrix} \tag{20}$$

In above formulae the Lorenz transformation from the rest system of two particles to the rest system of three particles is omitted. Since the g-functions depend on invariant p^2 momentum squared and for S partial-wave states the angular part is constant the only variable which should be transformed is the argument of spherical harmonics for $^3D_1$ partial-wave state. We believe that such transformation gives small contribution to the binding energy of three-nucleon system. However it should be done for nucleons with spinor propagators.

The system of integral equations (15-17) has several singularities in q'_0 complex plain. However in the case of the bound three-particle system ($\sqrt{s} < 3m_N$) all singularities do not cross the path of integration on q'_0 and thus do not affect to the Wick-rotation procedure $q'_0 \to iq'_4$.

The system (15-17) after the Wick-rotation procedure is well analytically defined and can be solved using various numerical methods. One of them is discussed in the next section.

4 Solution and results

In the paper the iterations method is used to solve the system of integral equations. The mappings for the variable of integration on q $[0, \infty)$ and q_4 $(-\infty, \infty)$ to $[-1, 1]$ interval are introduced.

As it is shown in [7] the binding energy of the three-nucleon system can be found using the following condition for iterated solutions (see details in [7]):

$$\lim_{n\to\infty} \frac{\Phi_n(s)}{\Phi_{n-1}(s)}\Big|_{s=M_B^2} = 1 \qquad (21)$$

where n is a number of iterations.

The Graz-II kernels of interaction [5] with three probabilities ($p_D = 4, 5, 6\%$) of the $^3D_1$-state are used for calculations. The obtained results are shown in table 1 both for nonrelativistic and relativistic cases and for S-states and $(S + D)$ partial-wave states. The experimental value is 8.48 MeV.

Table 1. Three-nucleon binding energy calculated with Graz-II separable NN kernel.

Potential	Nonrelativistic		Relativistic	
	$^1S_0, ^3S_1$	$^1S_0, ^3S_1, ^3D_1$	$^1S_0, ^3S_1$	$^1S_0, ^3S_1, ^3D_1$
GRAZ-II, $p_D = 4$ %	8.372	8.334	8.628	8.617
GRAZ-II, $p_D = 5$ %	7.964	7.934	8.223	8.217
GRAZ-II, $p_D = 6$ %	7.569	7.548	7.832	7.831

As it is seen from the table 1 kernels with different p_D can give results for binding energy both bigger and smaller than the experimental value. The relativistic results are systematically higher then nonrelativistic ones to about 3%. Also the $^3D_1$ partial-state gives rather small contribution to the three-nucleon binding energy - about -0.5%.

Summary

In the paper three-body system is investigated using Bethe-Salpeter-Faddeev equations. In the calculations the three-rank Graz-II potential is used. The BSF integral equations are solved using the iterations method. The binding energy of the triton and amplitudes of the $^1S_0$, $^3S_1$ and $^3D_1$ partial-wave states of the triton are obtained.

It was shown that kernels with different p_D can give results for binding energy both bigger and smaller than the experimental value. The relativistic results are systematically higher then nonrelativistic ones to about 3% and the $^3D_1$ partial-state gives rather small contribution to the three-nucleon binding energy - about -0.5%.

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References

[1] E.E. Salpeter, H.A. Bethe, Phys. Rev. **84**, 1232 (1951)
[2] S.G. Bondarenko, V.V. Burov, A.V. Molochkov, G.I. Smirnov, H. Toki, Prog. Part. Nucl. Phys. **48**, 449 (2002)
[3] G. Rupp and J. A. Tjon, Phys. Rev. C **37**, 1729 (1988)
[4] S. Bondarenko, V. Burov, S. Yurev, EPJ Web of Conferences **108**, 02015 (2016)
[5] L. Mathelitsch, W. Plessas, W. Schweiger, Phys. Rev. C **26**, 65 (1982). G. Rupp, J.A. Tjon, Phys. Rev. C. **45**, 2133 (1992)
[6] A. Stadler, F. Gross, M. Frank, Phys. Rev C **56**, 2365 (1997)
[7] R. A. Malfliet and J. A. Tjon, Nucl. Phys. A **127**,161-168 (1969)