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Critical behaviour of (2+1)-dimensional QED: $1/N$-corrections

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Abstract. We present recently obtained results for dynamical chiral symmetry breaking studied within $(2+1)$-dimensional QED with N four-component fermions. The leading and next-to-leading orders of the $1/N$ expansion are computed exactly in an arbitrary non-local gauge.

1 Introduction

In these Proceedings we present the results of our recent papers [1, 2], where the critical behavior of Quantum Electrodynamics in $2+1$ dimensions (QED_3) has been studied. QED_3 is described by the Lagrangian:

$$L = \overline{\Psi}(i\hat{\partial} - e\hat{A})\Psi - \frac{1}{4}F_{\mu\nu}^2\,, \tag{1}$$

where Ψ is taken to be a four component complex spinor. In the presence of N fermion flavours, the model has a $U(2N)$ symmetry. A (parity-invariant) fermion mass term, $m\overline{\Psi}\Psi$, breaks this symmetry to $U(N)\times U(N)$. In the massless case, loop expansions are plagued by infrared divergences. The latter soften upon analyzing the model in a $1/N$ expansion [3]. Since the theory is super-renormalizable, the mass scale is then given by the dimensionless coupling constant: $a = Ne^2/8$, which is kept fixed as $N \to \infty$. Early studies of this model [4, 5] suggested that the physics is rapidly damped at momentum scales $p \gg a$ and that a fermion mass term breaking the flavour symmetry is dynamically generated at scales which are orders of magnitude smaller than the intrinsic scale a. Since then, dynamical chiral symmetry breaking (DχSB) in QED_3 and the dependence of the dynamical fermion mass on N have been the subject of extensive studies, see, *e.g.*, [1, 2, 4–13].

One of the central issue is related to the value of the critical fermion number, N_c, which is such that DχSB takes place only for $N < N_c$. An accurate determination of N_c is of crucial importance to understand the phase structure of QED_3 with far reaching implications from particle physics to planar condensed matter physics systems having relativistic-like low-energy excitations [14]. It turns out that the values that can be found in the literature vary from $N_c \to \infty$ [4, 6] corresponding to DχSB for all values of N, all the way to $N_c \to 0$ in the case where no sign of DχSB is found [7].

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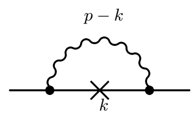

Figure 1. LO diagram to the dynamically generated mass $\Sigma(p)$. The crossed line denotes mass insertion.

Of importance to us in the following, is the approach of Appelquist et al. [5] who found that $N_c = 32/\pi^2 \approx 3.24$ by solving the Schwinger-Dyson (SD) gap equation using a leading order (LO) $1/N$-expansion. Lattice simulations in agreement with a finite non-zero value of N_c can be found in [8]. Soon after the analysis of [5], Nash approximately included next-to-leading order (NLO) corrections and performed a partial resummation of the wave-function renormalization constant at the level of the gap equation; he found [9]: $N_c \approx 3.28$.

Recently, in [1], the NLO corrections could be computed exactly in the Landau gauge upon refining the analysis of [10]. This led to $N_c \approx 3.29$, a value which is surprisingly close to the one of Nash in [9]. More recently, in [2], the results of [1] were generalized to an arbitrary non-local gauge [15]. Moreover, it was shown in [2] that a resummation of the wave-function renormalization yields a strong suppression of the gauge dependence of the critical fermion flavour number, $N_c(\xi)$ where ξ is the gauge fixing parameter, which is such that DχSB takes place for $N < N_c(\xi)$. Neglecting the gauge-dependent terms yields $N_c = 2.8469$, that coincides with results in [11]. In the general case, it is found that: $N_c(1) = 3.0084$ in the Feynman gauge, $N_c(0) = 3.0844$ in the Landau gauge and $N_c(2/3) = 3.0377$ in the $\xi = 2/3$ gauge where the leading order fermion wave function is finite. These results suggest that DχSB should take place for integer values $N \leq 3$. Using a very different method, Herbut obtained [12] a close value: $N_c \approx 2.89$.

It is the purpose of this work to review some of the basic steps of papers [1, 2] which represent an essential improvement with respect to Nash's approximate NLO results derived some 30 years ago.

2 Schwinger-Dyson equations

With the conventions of [1], the inverse fermion propagator is defined as: $S^{-1}(p) = [1 + A(p)]\,(i\hat{p} + \Sigma(p))$ where $A(p)$ is the fermion wave function and $\Sigma(p)$ is the dynamically generated parity-conserving mass which is taken to be the same for all the fermions. The SD equation for the fermion propagator may be decomposed into scalar and vector components as follows:

$$\tilde{\Sigma}(p) = \frac{2a}{N}\,\mathrm{Tr}\int \frac{d^3k}{(2\pi)^3}\,\frac{\gamma^\mu D_{\mu\nu}(p-k)\Sigma(k)\Gamma^\nu(p,k)}{[1+A(k)]\,(k^2+\Sigma^2(k))}\,, \quad A(p)p^2 = -\frac{2a}{N}\,\mathrm{Tr}\int \frac{d^3k}{(2\pi)^3}\,\frac{D_{\mu\nu}(p-k)\hat{p}\gamma^\mu\hat{k}\Gamma^\nu(p,k)}{[1+A(k)]\,(k^2+\Sigma^2(k))}\,, \tag{2}$$

where $\tilde{\Sigma}(p) = \Sigma(p)[1 + A(p)]$, $D_{\mu\nu}(p)$ is the photon propagator in the non-local ξ-gauge:

$$D_{\mu\nu}(p) = \frac{P^{\xi}_{\mu\nu}(p)}{p^2\,[1+\Pi(p)]}, \quad P^{\xi}_{\mu\nu}(p) = g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\,, \tag{3}$$

$\Pi(p)$ is the polarization operator and $\Gamma^\nu(p,k)$ is the vertex function. In the following, (2) will be studied for an arbitrary value of the gauge-fixing parameter ξ. All calculations will be performed with the help of the standard rules of perturbation theory for massless Feynman diagrams as in [16], see also the recent short review [17]. For the most complicated diagrams, the Gegenbauer polynomial technique will be used following [18].

3 Gap equation at leading order

The LO approximations in the $1/N$ expansion are given by: $A(p) = 0$, $\Pi(p) = a/|p|$ and $\Gamma^\nu(p,k) = \gamma^\nu$, where the fermion mass has been neglected in the calculation of $\Pi(p)$. A single diagram contributes to the gap equation (2) at LO, see figure 1, and the latter reads:

$$\Sigma(p) = \frac{8(2+\xi)a}{N} \int \frac{d^3k}{(2\pi)^3} \frac{\Sigma(k)}{(k^2 + \Sigma^2(k))\,[(p-k)^2 + a\,|p-k|]} \,. \tag{4}$$

Following [5], we consider the limit of large a and linearize (4) which yields:

$$\Sigma(p) = \frac{8(2+\xi)}{N} \int \frac{d^3k}{(2\pi)^3} \frac{\Sigma(k)}{k^2\,|p-k|} \,. \tag{5}$$

The mass function may then be parameterized as [5]: $\Sigma(k) = B\,(k^2)^{-\alpha}$, where B is arbitrary and the index α has to be self-consistently determined. Using this Ansatz, (5) leads to the LO gap equation: ($\beta^{-1} = \alpha(1/2 - \alpha)$ and $L \equiv \pi^2 N$)

$$1 = \frac{(2+\xi)\beta}{L} \quad \text{and} \quad \alpha_\pm = \frac{1}{4}\left(1 \pm \sqrt{1 - \frac{16(2+\xi)}{L}}\right), \tag{6}$$

which reproduces the solution given by Appelquist et al. [5]. The gauge-dependent critical number of fermions: $N_c \equiv N_c(\xi) = 16(2+\xi)/\pi^2$, is such that $\Sigma(p) = 0$ for $N > N_c$ and $\Sigma(0) \simeq \exp[-2\pi/(N_c/N - 1)^{1/2}]$, for $N < N_c$. Thus, DχSB occurs when α becomes complex, that is for $N < N_c$.

The gauge-dependent fermion wave function may be computed in a similar way. At LO, (2) simplifies as:

$$A(p)p^2 = -\frac{2a}{N} \mathrm{Tr} \int \frac{d^Dk}{(2\pi)^D} \frac{P^\xi_{\mu\nu}(p-k)\hat{p}\gamma^\mu\hat{k}\gamma^\nu}{k^2|p-k|} \,, \tag{7}$$

where the integral has been dimensionally regularized with $D = 3-2\varepsilon$. Taking the trace and computing the integral on the r.h.s. yields:

$$A(p) = \frac{\overline{\mu}^{2\varepsilon}}{p^{2\varepsilon}}\, C_1(\xi) \ + \mathrm{O}(\varepsilon)\,, \quad C_1(\xi) = +\frac{2}{3\pi^2 N}\left((2-3\xi)\left[\frac{1}{\varepsilon} - 2\ln 2\right] + \frac{14}{3} - 6\xi\right), \tag{8}$$

where the $\overline{MS}$ parameter $\overline{\mu}$ has the standard form $\overline{\mu}^2 = 4\pi e^{-\gamma_E}\mu^2$ with the Euler constant γ_E. We note that in the $\xi = 2/3$-gauge, the value of $A(p)$ is finite and $C_1(\xi = 2/3) = +4/(9\pi^2 N)$. From (8), the LO wave-function renormalization constant may be extracted: $\lambda_A = \mu(d/d\mu)A(p) = 4(2-3\xi)/(3\pi^2 N)$ a result which coincides with the one of [19].

4 Next-to-leading order

We now consider the NLO contributions and parametrize them as:

$$\Sigma^{(\mathrm{NLO})}(p) = \left(\frac{8}{N}\right)^2 B\,\frac{(p^2)^{-\alpha}}{(4\pi)^3}\ (\Sigma_A + \Sigma_1 + 2\,\Sigma_2 + \Sigma_3)\,, \tag{9}$$

where each contribution to the linearized gap equation is represented graphically in figure 2. The gap equation has the following general form:

$$1 = \frac{(2+\xi)\beta}{L} + \frac{\overline{\Sigma}_A(\xi) + \overline{\Sigma}_1(\xi) + 2\,\overline{\Sigma}_2(\xi) + \overline{\Sigma}_3(\xi)}{L^2}\,, \quad \overline{\Sigma}_i = \pi\Sigma_i, \quad (i = 1,2,3.A) \tag{10}$$

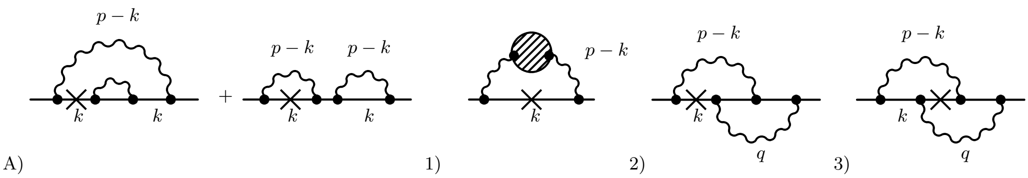

Figure 2. NLO diagrams to the dynamically generated mass $\Sigma(p)$. The shaded blob defines the two-loop polarization operator, see [1, 2] for details.

Performing the calculation of the diagrams shown in figure 2 (see [1, 2]), the gap equation (10) may be written in an explicit form as:

$$1 = \frac{(2+\xi)\beta}{L} + \frac{1}{L^2}\left[8S(\alpha,\xi) - 2(2+\xi)\hat{\Pi}\beta + \left(-\frac{5}{3} + \frac{26}{3}\xi - 3\xi^2\right)\beta^2 - 8\beta\left(\frac{2}{3}(1-\xi) - \xi^2\right)\right], \qquad (11)$$

where $\hat{\Pi} = 92/9 - \pi^2$ arises from the two-loop polarization operator in dimension $D = 3$ [20, 21] and $S(\alpha,\xi)$ contains the contributions of complicated diagrams. Considering (11) directly at the critical point $\alpha = 1/4$, *i.e.*, at $\beta = 16$, we have

$$L_c^2 - 16(2+\xi)L_c - 8\left[S(\xi) - 4(2+\xi)\hat{\Pi} - 16\left(4 - 50\xi/3 + 5\xi^2\right)\right] = 0, \qquad (12)$$

where $S(\xi) = S(\alpha = 1/4, \xi)$ and

$$8S(\xi) = 8(1-\xi)R_1 + (\xi^2-1)R_2 - (7+16\xi-3\xi^2)\frac{P_2}{16}, \quad R_1 = 163.7428, \quad R_2 = 209.175, \quad P_2 = 1260.720 \qquad (13)$$

Solving (12), we have two standard solutions:

$$L_{c,\pm} = 8(2+\xi) \pm \sqrt{d_1(\xi)}, \quad d_1(\xi) = 8\left[S(\xi) - 8\left(4 - \frac{112}{3}\xi + 9\xi^2 + \frac{2+\xi}{2}\hat{\Pi}\right)\right]. \qquad (14)$$

Combining these values with the one of $\hat{\Pi}$, yields: $N_c(\xi = 0) = 3.29$, $N_c(\xi = 2/3) = 3.09$, where "–" solutions are unphysical and there is no solution in the Feynman gauge. The range of ξ-values for which there is a solution corresponds to $\xi_- \le \xi \le \xi_+$, where $\xi_+ = 0.88$ and $\xi_- = -2.36$.

5 Resummation

Following [9], we would like to resum the LO term together with part of the NLO corrections containing terms $\sim \beta^2$. In order to do so, we will now rewrite the gap equation (11) in a form which is suitable for resummation. This amounts to extract the terms $\sim \beta$ and $\sim \beta^2$ from the complicated part of the fermion self-energy, $S(\alpha,\xi)$, yielding:

$$S(\alpha,\xi) = \frac{1}{4}(1-\xi)\beta(3\beta - 8) - \frac{1}{2}\xi(4+\xi)\beta + \tilde{S}(\alpha,\xi). \qquad (15)$$

At the critical point $\alpha = 1/4$ ($\beta = 16$), $\tilde{S}(\xi) = \tilde{S}(\alpha = 1/4, \xi)$ has the following form:

$$8\tilde{S}(\xi) = 8(1-\xi)\tilde{R}_1 + (\xi^2-1)\tilde{R}_2 - (7+16\xi-3\xi^2)\frac{\tilde{P}_2}{16}, \quad \tilde{R}_1 = 3.7428, \quad \tilde{R}_2 = 1.175, \quad \tilde{P}_2 = -19.28. \qquad (16)$$

With the help of the results (16), the gap equation (11) may be written as:

$$1 = \frac{(2+\xi)\beta}{L} + \frac{1}{L^2}\Big[8\tilde{S}(\alpha,\xi) - 2(2+\xi)\hat{\Pi}\beta + \left(\frac{2}{3}-\xi\right)(2+\xi)\beta^2 + 4\beta\left(\xi^2 - \frac{4}{3}\xi - \frac{16}{3}\right)\Big]. \tag{17}$$

At this point (11) and (17) are strictly equivalent to each other and yield the same values for $N_c(\xi)$. Equation (17) is the convenient starting point to perform a resummation of the wave function renormalization constant. To do it (see details in [2]) (17) can now be expressed as:

$$1 = \frac{8\beta}{3L} + \frac{1}{L^2}\left[8\tilde{S}(\alpha,\xi) - \frac{16}{3}\beta\left(\frac{40}{9}+\hat{\Pi}\right)\right], \tag{18}$$

which displays a strong suppression of the gauge dependence even at NLO as ξ-dependent terms do exist but they enter the gap equation only through the rest, $\tilde{S}$, which is very small numerically.

We now consider (18) at the critical point, $\alpha = 1/4$ ($\beta = 16$), which yields:

$$L_c^2 - \frac{128}{3}L_c - \left[8\tilde{S}(\xi) - \frac{256}{3}\left(\frac{40}{9}+\hat{\Pi}\right)\right] = 0\,. \tag{19}$$

Solving (19), we have two standard solutions:

$$L_{c,\pm} = \frac{64}{3} \pm \sqrt{d_2(\xi)}, \quad d_2(\xi) = \left(\frac{64}{3}\right)^2 + \left[8\tilde{S}(\xi) - \frac{256}{3}\left(\frac{40}{9}+\hat{\Pi}\right)\right]. \tag{20}$$

In order to provide a numerical estimate for N_c, we have used the values of $\tilde{R}_1$, $\tilde{R}_2$ and $\tilde{P}_2$ of (16). Combining these values together with the value of $\hat{\Pi}$, yields, for $N_c(\xi)$ ("$-$" solutions being unphysical):

$$N_c(0) = 3.08, \qquad N_c(2/3) = 3.04, \qquad N_c(1) = 3.01\,. \tag{21}$$

Actually, solutions exist for a broad range of values of ξ: $\tilde{\xi}_- \le \xi \le \tilde{\xi}_+$, where $\tilde{\xi}_+ = 4.042$ and $\tilde{\xi}_- = -8.412$; this is consistent with the weak ξ-dependence of the gap equation. Moreover, following [22], we think that the "right(est)" gauge choice is one close to $\xi = 2/3$ where the LO fermion wave function is finite. Indeed, upon resumming the theory, the value of $N_c(\xi)$ increases (decreases) for small (large) values of ξ. For $\xi = 2/3$, the value of N_c is very stable, decreasing only by 1-2% during resummation. Finally, if we neglect the rest, *i.e.*, $\tilde{S}(\xi) = 0$ in (19), the gap equation becomes ξ-independent and we have: $\overline{L}_c = 28.0981$ and therefore: $\overline{N}_c = 2.85$, a value that coincides with the one in [11].

6 Conclusion

We have presented the studies [1, 2] of DχSB in QED$_3$ by including $1/N^2$ corrections to the SD equation exactly and taking into account the full ξ-dependence of the gap equation. Following Nash, the wave function renormalization constant has been resummed at the level of the gap equation leading to a very weak gauge-variance of the critical fermion number N_c. The value obtained for the latter, (21), suggests that DχSB takes place for integer values $N \le 3$ in QED$_3$.

Notice that the large-N limit of the photon propagator in QED$_3$ has precisely the same momentum dependence as the one in the so-called reduced QED, see [22]. One difference is that the gauge fixing parameter in reduced QED is twice less than the one in QED$_3$. Such a difference can be taken into account with the help of our present results for QED$_3$ together with the multi-loop results obtained in [20, 23]. The case of reduced QED, and its relation with dynamical gap generation in graphene which is the subject of active ongoing research, see, *e.g.*, the reviews [24], was considered in our paper [25].

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