S₄ solution of the transport equation for eigenvalues using Legendre polynomials

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Abstract. Numerical solution of the transport equation for monoenergetic neutrons scattered isotropically through the medium of a finite homogeneous slab is studied for the determination of the eigenvalues. After obtaining the discrete ordinates form of the transport equation, separated homogeneous and particular solutions are formed and then the eigenvalues are calculated using the Gauss-Legendre quadrature set. Then, the calculated eigenvalues for various values of the c_0 , the mean number of secondary neutrons per collision, are given in the tables.

1 Introduction

As well known, the number of neutrons and their distributions are very important in order to maintain the constant power in all nuclear power plants. In other words, the multiplication factor which describes the ratio of the produced number of fission neutrons in one generation to the number of neutrons in previous generation is desired to be equal to one. The stability of the power produced in the reactor can be thought as the isotropic scattering of the neutrons through the system.

There is no analytic solution of the transport equation which describes the interactions and the distribution of the neutrons with the materials inside the system. The methods developed for the solution of the transport equation can be interpreted as deterministic (such as polynomial expansion based techniques, integral transform methods) and stochastic (such as Monte Carlo, MCNP, source iteration). In a stochastic approach, the transport equation is usually converted into a discrete ordinate form (S_N) [1-3].

In this study, first the neutron transport equation is converted into a discrete ordinates form and then the S_N transport equation is solved numerically for the eigenvalue spectrum of the mono-energetic neutrons. The results are obtained up to the fourth order approximation (S_4) using Legendre quadrature set and they are given in the table for various values of the c , number of secondary neutrons per collision.

2 Method

The neutron transport equation for mono-energetic neutrons in a slab with isotropic scattering can be written as,

$$\mu \frac{\partial \psi(x, \mu)}{\partial x} + \sigma_T \psi(x, \mu) = \frac{\sigma_{s0}}{2} \int_{-1}^1 \psi(x, \mu') d\mu' + \frac{Q_0}{2}, \quad (1)$$

where $\psi(x, \mu)$ is the neutron angular flux at position x travelling in direction μ , cosine of the angle between the neutron velocity vector and the positive x -axis. σ_T is the total macroscopic cross-section, σ_{s0} is the differential scattering cross-section corresponding to isotropic scattering and Q_0 is the internal source [3]. When the integral in Eq. (1) is written as an integral transform with Legendre polynomials, it can be rearranged in the form of discrete ordinates S_N equations for the numerical solution,

$$\mu_m \frac{d\psi_m(x)}{dx} + \sigma_T \psi_m(x) = \frac{\sigma_{s0}}{2} \sum_{n=1}^N \psi_n(x) \omega_n + \frac{Q_0(x)}{2}, \quad (2)$$

where ω_n is the Gauss-Legendre quadrature weights or weighting factor for direction μ_n , i.e. the roots of the N th order Legendre polynomials. The roots or the zeros of the Legendre polynomials together with the weighting factors $\omega_m = 1$, one can easily analyze Eq. (2) for the solution.

Therefore, the general solution of Eq. (2) can be written as the sum of the homogeneous and the particular solutions,

$$\psi_m(x) = \psi_m^p(x) + \psi_m^h(x). \quad (3)$$

It is easy to verify a spatially constant particular solution of the form can be given as,

$$\psi_m^p(x) = \frac{Q_0}{2\sigma_T(1-c_0)}, \quad 0 \leq a \leq x, \quad 1 \leq m \leq N, \quad (4)$$

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where $c_0 = \sigma_{s0} / \sigma_T$. In order to determine the homogeneous solution $\psi_m^h(x)$ of Eq. (2), the traditional method of separation of variables can be used. Therefore, the solution of homogeneous part of Eq. (2) can be written in the form of [3],

$$\psi_m^h(x) = H_m(v) \exp(\sigma_T x / v), \quad 0 \leq x \leq a, \quad 1 \leq m \leq N. \quad (5)$$

By substituting Eq. (5) into the homogeneous part of Eq. (2), one can obtain an expression for the angular part of the neutron angular flux;

$$H_m(v) = \frac{vc_0}{2(v + \mu_m)} \sum_{n=1}^N H_n(v) \omega_n, \quad (6)$$

where the function $H_m(v)$ can be normalized as,

$$\sum_{n=1}^N H_n(v) \omega_n = 1. \quad (7)$$

Eq. (7) is multiplied by ω_m from both sides and then summed over all m to obtain an equation for the v eigenvalues:

$$\frac{1}{2} \sum_{m=1}^N \frac{vc_0}{(v + \mu_m)} \omega_m = 1, \quad v \neq -\mu_m. \quad (8)$$

Eq. (8) is referred to as the dispersion relation and the roots $v_k, 1 \leq k \leq N$, of Eq. (8) are the eigenvalues of the S_N equations and they are lying symmetrically about the origin for any c_0 satisfying $0 \leq c_0 \leq 1$.

3 Numerical results and discussion

The neutron transport equation for one-speed neutrons in a finite homogeneous slab is studied to determine the eigenvalues using S_N method with Gauss-Legendre quadrature set. The transport equation in one-dimensional geometry is first written in the form of S_N by using the integral transform technique with the even-order Gauss-Legendre quadrature set. A dispersion relation is obtained by using the reasonable homogeneous solution in the S_N form of the transport equation. The roots, i.e. the eigenvalues $v_k, 1 \leq k \leq N$ of Eq. (8) are calculated up to the fourth order approximation for various values of c_0 [4,5]. This order of approximation can be spelled to as unnecessary for the calculations in slab geometry. However, this study should be thought as the first attempts for the problem solution strategy. This method could be approved for other problems in other geometries of the transport theory.

The calculated eigenvalues for various values of the $c_0 < 1$ and $c_0 > 1$ are given in Table 1. The calculation of the eigenvalues is accepted as to be the first step by many of the researchers. Therefore, the results presented here can also be used in many studies related with transport theory.

Table 1. Eigenvalue spectrum

N	$c_0 = 0.40$	$c_0 = 0.70$	$c_0 = 0.99$	$c_0 = 1.20$	$c_0 = 2.00$
2	± 0.745355992	± 1.054092553	± 5.773502691	$\pm 1.290994449i^*$	$\pm 0.557735027i$
4	± 0.423258332	± 0.492070474	± 0.539816206	± 0.562247603	± 0.603465390
	± 1.546700021	$\pm 0.940739334i$	$\pm 0.547857547i$	$\pm 0.440083387i$	$\pm 0.280100324i$

$$*i = \sqrt{-1}$$

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