EPJ Web of Conferences **167**, 04004 (2018) https://doi.org/10.1051/epjconf/201816704004
PPLA 2017

On a ‘time’ reparametrization in relativistic electrodynamics with travelling waves

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Abstract. We briefly report on our method [23] of simplifying the equations of motion of charged particles in an electromagnetic (EM) field that is the sum of a plane travelling wave and a static part; it is based on changes of the dependent variables and the independent one (light-like coordinate ξ instead of time t). We sketch its application to a few cases of extreme laser-induced accelerations, both in vacuum and in plane problems at the vacuum-plasma interface, where we are able to reduce the system of the (Lorentz-Maxwell and continuity) partial differential equations into a family of decoupled systems of Hamilton equations in 1 dimension. Since Fourier analysis plays no role, the method can be applied to all kind of travelling waves, ranging from almost monochromatic to socalled “impulses".

1 Introduction and set-up

The equation of motion of a particle with charge q in external electric and magnetic fields $\mathbf{E}(\mathrm{x}), \mathbf{B}(\mathrm{x})$ $[\mathrm{x} \equiv (ct, \mathbf{x})]$ in its general form is non-autonomous and highly nonlinear:

$$\begin{aligned} \dot{\mathbf{p}}(t) &= q\mathbf{E}[ct, \mathbf{x}(t)] + q\boldsymbol{\beta}(t) \wedge \mathbf{B}[ct, \mathbf{x}(t)], \\ \dot{\mathbf{x}}(t) &= \frac{c\mathbf{p}(t)}{\sqrt{m^2c^2 + \mathbf{p}^2(t)}}; \end{aligned} \tag{1}$$

here $\boldsymbol{\beta} \equiv \mathbf{v}/c$, $\mathbf{p} \equiv m\mathbf{v}/\sqrt{1-\boldsymbol{\beta}^2}$ is its relativistic momentum.

Usually, (1) is simplified assuming:

1. $\mathbf{E}, \mathbf{B}$ are constant or vary “slowly” in space/time; or
2. $\mathbf{E}, \mathbf{B}$ are “small" (so that nonlinear effects in $\mathbf{E}, \mathbf{B}$ are negligible); or
3. $\mathbf{E}, \mathbf{B}$ are monochromatic waves, or slow modulations of; or
4. the motion of the particle keeps non-relativistic.

The astonishing developments of Laser technologies (especially *Chirped Pulse Amplification* [2, 3]) today allow the construction of compact sources of extremely intense (up to 10^{23} W/cm^2) coherent EM waves, possibly concentrated in very short laser pulses ($\gtrsim$ fs). Even more intense/short (or cheaper) laser pulses by new technologies (thin film compression [4], etc.) will be soon available. In particular, these lasers can be used for making small particle-accelerators based on Laser Wake Field Acceleration (LWFA) [5] in plasmas. Extreme conditions are present also in several violent astrophysical processes (see e.g. [6] and references therein). In either case the effects are so fast, huge, highly nonlinear, ultra-relativistic that conditions 1-4 are not fulfilled. Alternative simplifying approaches are therefore desirable.

Here we summarize a new approach [23] that is especially fruitful if in the spacetime region Ω of interest (i.e., where we wish to follow the charged particles’ worldlines) $\mathbf{E}, \mathbf{B}$ can be decomposed into a static part and a plane transverse travelling wave propagating in the z direction:

$$\begin{aligned} \mathbf{E}(\mathrm{x}) &= \underbrace{\boldsymbol{\epsilon}^{\perp}(ct-z)}_{pump=travelling\,wave} + \underbrace{\mathbf{E}_s(\mathbf{x})}_{static}, \\ \mathbf{B}(\mathrm{x}) &= \underbrace{\mathbf{k} \wedge \boldsymbol{\epsilon}^{\perp}(ct-z)}_{travelling\,wave} + \underbrace{\mathbf{B}_s(\mathbf{x})}_{static}, \end{aligned} \tag{2}$$

$\mathbf{x} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, $\boldsymbol{\epsilon}^{\perp} \perp \mathbf{k}$. We decompose vectors as $\mathbf{u} = \mathbf{u}^{\perp} + u^z\mathbf{k}$. We assume *only* that $\boldsymbol{\epsilon}^{\perp}(\xi)$ is piecewise continuous and

$$\begin{aligned} &\mathbf{a)} \quad \boldsymbol{\epsilon}^{\perp} \text{ has a compact support } [0, l], \\ \text{or} \quad &\mathbf{a')} \quad \boldsymbol{\epsilon}^{\perp} \in L^1(\mathbb{R}), \end{aligned} \tag{3}$$

$$\Rightarrow \quad \boldsymbol{\alpha}^{\perp}(\xi) \equiv -\int_{-\infty}^{\xi} dy\, \boldsymbol{\epsilon}^{\perp}(y) \to 0 \quad \text{as } \xi \to -\infty; \tag{4}$$

$\boldsymbol{\alpha}^{\perp}$ is the travelling-wave part of the transverse EM potential $\mathbf{A}^{\perp}$. **a)** $\Rightarrow \boldsymbol{\alpha}^{\perp}(\xi) = 0$ if $\xi \leq 0$, $\boldsymbol{\alpha}^{\perp}(\xi) = \boldsymbol{\alpha}^{\perp}(l)$ if $\xi \geq l$. We can treat on the same footing all such $\boldsymbol{\epsilon}^{\perp}$, in particular:

1. A modulated monochromatic wave fulfilling (3):

$$\boldsymbol{\epsilon}^{\perp}(\xi) = \underbrace{\epsilon(\xi)}_{\text{modul.}} \underbrace{[\mathbf{i}a_1 \cos(k\xi+\varphi) + \mathbf{j}a_2 \sin(k\xi)]}_{\text{carrier wave } \boldsymbol{\epsilon}_o^{\perp}(\xi)} \tag{5}$$

$$\Rightarrow \quad -\boldsymbol{\alpha}^{\perp}(\xi) = \frac{\epsilon(\xi)}{k^2}\, \boldsymbol{\epsilon}_o^{\perp\prime}(\xi) + O\left(\frac{1}{k^2}\right) \simeq \frac{\epsilon(\xi)}{k^2}\, \boldsymbol{\epsilon}_o^{\perp\prime}(\xi); \tag{6}$$

$\simeq$ holds if $|\epsilon'| \ll |k\epsilon|$ (slow modulation).

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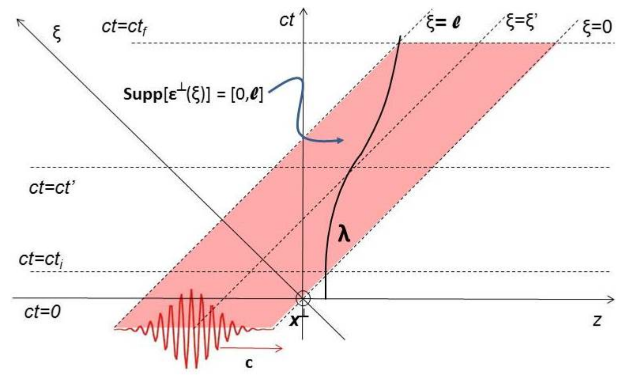


Figure 1. Every worldline λ and hyperplane ξ=C intersect once.

2. A superposition of waves of type 1.
3. An 'impulse' (few cycles, or even a fraction of).

The idea is: as no particle can reach the speed of light c, $\tilde{\xi}(t)=ct-z(t)$ is strictly growing, and we can adopt $\xi=ct-z$ as a parameter on the worldline λ (see fig. 1) and in the action functional of the particle:

$$S(\lambda) = -\int_\lambda mc^2 d\tau + qA(\mathrm{x})$$
$$= -\int_{t_0}^{t_1} dt \underbrace{\frac{mc^2+qu^\mu A_\mu}{\gamma}}_{L[\mathbf{x},\dot{\mathbf{x}},t]} = -\int_{\xi_0}^{\xi_1} \frac{d\xi}{c} \underbrace{\frac{mc^2+q\hat{u}^\mu \hat{A}_\mu}{\hat{s}}}_{\mathcal{L}[\hat{\mathbf{x}},\hat{\mathbf{x}}',\xi]};$$

$(u^\mu) = (u^0, \mathbf{u}) \equiv (\gamma, \gamma\boldsymbol{\beta})$ is the 4-velocity, i.e. the dimensionless version of the 4-momentum. $A(\mathrm{x}) = A_\mu(\mathrm{x})d\mathrm{x}^\mu = A^0(\mathrm{x})cdt - \mathbf{A}(\mathrm{x})\cdot d\mathbf{x}$ is the EM potential 1-form, $\mathbf{E} = -\partial_t\mathbf{A}/c - \nabla A^0$, $\mathbf{B} = \nabla\wedge\mathbf{A}$ (we use Gauss CGS units), and we denote $\hat{\mathbf{x}}(\xi) = \mathbf{x}(t)$, $\hat{f}(\xi,\hat{\mathbf{x}}) \equiv f(ct,\mathbf{x})$, $\hat{f}' \equiv d\hat{f}/d\xi$ for all functions $f(ct,\mathbf{x})$. Applying Hamilton's principle and the Legendre transform we find simplified Lagrange, and Hamilton equations where the argument of $\boldsymbol{\epsilon}^\perp$ is the independent variable ξ, rather than the unknown $ct-z(t)$, and the new kinetic momenta are the dependent variables.

2 General results for one particle

To parametrize λ by ξ we have to replace $d\tau/dt = 1/\gamma = \sqrt{1-\dot{\mathbf{x}}^2/c^2}$ (τ is the particle proper time) by

$$\frac{1}{\hat{s}} \equiv \frac{d(c\tau)}{d\xi} = \sqrt{1+2\hat{z}'-\hat{\mathbf{x}}^{\perp\prime 2}} > 0. \tag{7}$$

From $\mathbf{p} = md\mathbf{x}/d\tau, \gamma = dt/d\tau$ we find that the *s-factor* $\hat{s}$ is the light-like component $\hat{u}^- = \hat{\gamma} - \hat{u}^z$ of the 4-velocity $u = (u^0, \mathbf{u}) \equiv (\gamma, \gamma\boldsymbol{\beta}) = \left(\frac{p^0}{mc^2}, \frac{\mathbf{p}}{mc}\right)$ (all these are dimensionless), and $\hat{\mathbf{u}} = \hat{s}\hat{\mathbf{x}}'$. $\hat{\gamma}, \hat{u}^z, \hat{\boldsymbol{\beta}}, \hat{\mathbf{x}}'$ can be expressed as *rational functions* of $\hat{\mathbf{u}}^\perp, \hat{s}$:

$$\hat{\gamma} = \frac{1+\hat{\mathbf{u}}^{\perp 2}+\hat{s}^2}{2\hat{s}}, \quad \hat{u}^z = \hat{\gamma} - \hat{s}, \quad \hat{\boldsymbol{\beta}} = \frac{\hat{\mathbf{u}}}{\hat{\gamma}}, \tag{8}$$

$$\hat{\mathbf{x}}^{\perp\prime} = \frac{\hat{\mathbf{u}}^\perp}{\hat{s}}, \qquad \hat{z}' = \frac{1+\hat{\mathbf{u}}^{\perp 2}}{2\hat{s}^2} - \frac{1}{2} \tag{9}$$

By Hamilton's principle, any extremum λ of S is the worldline of a possible motion of the particle with initial position $\mathbf{x}_0$ at time t_0 and final position $\mathbf{x}_1$ at time t_1. Hence it fulfills Euler-Lagrange equations in both forms $\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{x}}} = \frac{\partial L}{\partial \mathbf{x}}$ and $\frac{d}{d\xi}\frac{\partial \mathcal{L}}{\partial \hat{\mathbf{x}}'} = \frac{\partial \mathcal{L}}{\partial \hat{\mathbf{x}}}$, equivalent to (1). The Legendre transform yields the Hamiltonians $H \equiv \dot{\mathbf{x}}\cdot\frac{\partial L}{\partial \dot{\mathbf{x}}} - L = \gamma mc^2 + qA^0$ and $\hat{H} \equiv \hat{\mathbf{x}}'\cdot\frac{\partial \mathcal{L}}{\partial \hat{\mathbf{x}}'} - \mathcal{L} = \hat{\gamma}mc^2 + q\hat{A}^0$. $\hat{H}$ is a *rational* function of $\hat{\mathbf{x}}, \hat{\boldsymbol{\Pi}} \equiv \frac{\partial \mathcal{L}}{\partial \hat{\mathbf{x}}'}$, or, equivalently, of $\hat{s}, \hat{\mathbf{u}}^\perp$:

$$\hat{H}(\hat{\mathbf{x}},\hat{\boldsymbol{\Pi}};\xi) = mc^2\frac{1+\hat{s}^2+\hat{\mathbf{u}}^{\perp 2}}{2\hat{s}} + q\hat{A}^0(\xi,\hat{\mathbf{x}}), \tag{10}$$

$$\text{where} \begin{cases} mc^2\hat{\mathbf{u}}^\perp = \hat{\boldsymbol{\Pi}}^\perp - q\hat{\mathbf{A}}^\perp(\xi,\hat{\mathbf{x}}), \\ mc^2\hat{s} = -\hat{\Pi}^z - q\hat{A}^-(\xi,\hat{\mathbf{x}}), \end{cases} \tag{11}$$

while $H(\mathbf{x},\mathbf{P},t) = \sqrt{m^2c^4+(c\mathbf{P}-q\mathbf{A})^2} + qA^0$ ($\mathbf{P} \equiv \frac{\partial L}{\partial \dot{\mathbf{x}}} = \mathbf{p} + \frac{q}{c}\mathbf{A}$) is not. Eq. (1) are also equivalent to the Hamilton equations $\hat{\mathbf{x}}' = \frac{\partial \hat{H}}{\partial \hat{\boldsymbol{\Pi}}}$, $\hat{\boldsymbol{\Pi}}' = -\frac{\partial \hat{H}}{\partial \hat{\mathbf{x}}}$. All the new equations (in particular these ones) can be also obtained more directly from the old ones by putting a caret on all dynamical variables and replacing d/dt by $(c\hat{s}/\hat{\gamma})d/d\xi$. Along the solutions $\hat{H}$ gives the particle energy as a function of ξ. Under the EM field (2) eqs (1) amount to (9) and

$$\begin{aligned} \hat{\mathbf{u}}^{\perp\prime} &= \frac{q}{mc^2}\left[(1+\hat{z}')\hat{\mathbf{E}}_s^\perp + (\hat{\mathbf{x}}'\wedge\hat{\mathbf{B}}_s)^\perp + \boldsymbol{\epsilon}^\perp(\xi)\right], \\ \hat{s}' &= \frac{-q}{mc^2}\left[\hat{E}_s^z - \hat{\mathbf{x}}^{\perp\prime}\cdot\hat{\mathbf{E}}_s^\perp + (\hat{\mathbf{x}}^{\perp\prime}\wedge\hat{\mathbf{B}}_s^\perp)^z\right], \end{aligned} \tag{12}$$

while the *energy gain* (normalized to mc^2) is

$$\mathcal{E} \equiv \frac{\hat{H}(\xi_1)-\hat{H}(\xi_0)}{mc^2} = \int_{\xi_0}^{\xi_1} d\xi\, q\boldsymbol{\epsilon}^\perp\cdot\frac{\hat{\mathbf{u}}^\perp}{\hat{s}} \tag{13}$$

in the interval $[\xi_0, \xi_1]$. Once solved (9-12), analytically or numerically, to obtain the solution as a function of t we just need to invert $\hat{t}(\xi) = \xi + \hat{z}(\xi)$ and set $\mathbf{x}(t) = \hat{\mathbf{x}}[\xi(t)]$. If $\mathbf{E}_s, \mathbf{B}_s$ =const then eq. (12) are solved by

$$\begin{aligned} \hat{\mathbf{u}}^\perp &= \frac{q}{mc^2}\left[\mathbf{K} - \boldsymbol{\alpha}(\xi) + (\xi+\hat{z})\mathbf{E}_s + \hat{\mathbf{x}}\wedge\mathbf{B}_s\right]^\perp, \\ \hat{s} &= \frac{-q}{mc^2}\left[K^z + \xi E_s^z - \hat{\mathbf{x}}^\perp\cdot\mathbf{E}_s^\perp + (\hat{\mathbf{x}}^\perp\wedge\mathbf{B}_s)^z\right] \end{aligned} \tag{14}$$

(K^j are integration constants) whereby (9) become three 1^{st} order ordinary differential equations (ODE) *rational* in the unknown $\hat{\mathbf{x}}(\xi)$.

Contrary to (9-12), (1) is a transcendental system, and the unknown $z(t)$ appears in the argument of the rapidly varying functions $\boldsymbol{\epsilon}^\perp, \boldsymbol{\alpha}^\perp$ in $(1)_1$, which now reads:

$$\tfrac{1}{q}\dot{\mathbf{p}}(t) = \mathbf{E}_s + \boldsymbol{\beta}\wedge\mathbf{B}_s + \boldsymbol{\epsilon}^\perp[ct-z(t)]\,(\boldsymbol{\beta}\mathbf{k}+1-\beta^z).$$

Also determining $\mathcal{E}(t)$ is more complicated.

2.1 Dynamics under $A^\mu = A^\mu(t,z)$

This applies in particular to (2) if $\mathbf{E}_s = E_s^z(z)\mathbf{k}$, $\mathbf{B}_s = \mathbf{B}_s^\perp(z)$, choosing e.g. $A^0 = -\int^z d\zeta E_s^z(\zeta)$, $\mathbf{A}^\perp = \boldsymbol{\alpha}^\perp - \mathbf{k}\wedge\int^z d\zeta \mathbf{B}^\perp(\zeta)$, $A^z \equiv 0$. As $\partial\hat{H}/\partial\hat{\mathbf{x}}^\perp = 0$, we find $\hat{\boldsymbol{\Pi}}^\perp = q\mathbf{K}^\perp$ = const, i.e. the known result $\frac{mc^2}{q}\hat{\mathbf{u}}^\perp = \mathbf{K}^\perp - \hat{\mathbf{A}}^\perp(\xi,\hat{z})$. Setting $v := \hat{\mathbf{u}}^{\perp 2}$ and replacing in $(9)_2$, $(12)_2$ we obtain

$$\hat{z}' = \frac{1+\hat{v}}{2\hat{s}^2} - \frac{1}{2}, \quad \hat{s}' = \frac{-q}{mc^2}E_s^z(\hat{z}) - \frac{1}{2\hat{s}}\frac{\partial\hat{v}}{\partial\hat{z}}. \tag{15}$$

Once solved the system (15) in the unknowns $\hat{z}(\xi)$, $\hat{s}(\xi)$, the other unknowns are obtained from

$$\hat{\mathbf{x}}(\xi) = \mathbf{x}_0 + \hat{\mathbf{Y}}(\xi), \qquad \hat{\mathbf{Y}}(\xi) \equiv \int_{\xi_0}^{\xi} dy\, \frac{\hat{\mathbf{u}}(y)}{\hat{s}(y)}. \tag{16}$$

If in addition $\mathbf{B}_s \equiv 0$, then $\mathbf{A}_s \equiv 0$ (in the Coulomb gauge), $\hat{\mathbf{u}}^\perp(\xi) = \frac{q}{mc^2}\left[\mathbf{K}^\perp - \boldsymbol{\alpha}^\perp(\xi)\right]$ and $\hat{v} = \hat{\mathbf{u}}^{\perp 2}$ are already known. The system (15) to be solved simplifies to

$$\hat{z}' = \frac{1+\hat{v}}{2\hat{s}^2} - \frac{1}{2}, \qquad \hat{s}' = \frac{-q}{mc^2} E_s^z(\hat{z}). \tag{17}$$

Some remarkables properties of the solutions are [23]:

1. Where $\boldsymbol{\epsilon}^\perp(\xi) = 0$ then $\hat{v}(\xi) = v_c$ =const, $\hat{H}$ is conserved, (17) is solved by quadrature.
2. The final transverse momentum is $mc\mathbf{u}^\perp(\xi_f)$. If ϵ of (5) varies slowly and $\mathbf{u}^\perp(0) = \mathbf{0}$, then $\mathbf{u}^\perp(\xi_f) \simeq 0$.
3. $\hat{s}(\xi)$ is insensitive to fast oscillations of $\boldsymbol{\epsilon}^\perp$, contrary to $\mathbf{u}, \gamma, \boldsymbol{\beta}$, which can be reobtained via (8).

3 Some exact solutions for $\mathbf{B}_s, \mathbf{E}_s$ =const

Let $\mathbf{b}^\perp + b\mathbf{k} \equiv q\mathbf{B}_s/mc^2$, $\mathbf{e}^\perp \equiv q\mathbf{E}_s^\perp/mc^2$ (constants), $\boldsymbol{w}(\xi) \equiv q\left[\mathbf{K} - \boldsymbol{\alpha}^\perp(\xi) + \xi\mathbf{E}_s\right]/mc^2$ (all dimensionless); (14) take the more explicit form

$$\begin{aligned} \hat{u}^x &= (e^x - b^y)\hat{z} + b\hat{y} + w^x(\xi), \\ \hat{u}^y &= (e^y + b^x)\hat{z} - b\hat{x} + w^y(\xi), \\ \hat{s} &= (e^x - b^y)\hat{x} + (e^y + b^x)\hat{y} - w^z(\xi), \end{aligned} \tag{18}$$

For any $E_s^z, B_s^z, \mathbf{E}_s^\perp$, if $\mathbf{B}_s^\perp = \mathbf{k} \wedge \mathbf{E}_s^\perp$, setting $\kappa \equiv \frac{qE_s^z}{mc^2}$ we find the following exact solutions (part of them are new):

$$\begin{aligned} (\hat{x} + i\hat{y})(\xi) &= (1-\kappa\xi)^{ib/\kappa} \int_0^\xi d\zeta\, \frac{(w^x + iw^y)(\zeta)}{(1-\kappa\zeta)^{1+ib/\kappa}}, \\ \hat{z}(\xi) &= \int_0^\xi \frac{d\zeta}{2}\left[\frac{1}{(1-\kappa\zeta)^2} + \hat{\mathbf{x}}^{\perp\prime 2}(\zeta) - 1\right], \quad \hat{s}(\xi) = 1-\kappa\xi, \\ \hat{\mathbf{u}}^\perp(\xi) &= (1-\kappa\xi)\,\hat{\mathbf{x}}^{\perp\prime}(\xi), \qquad \hat{\gamma}(\xi) = 1-\kappa\xi + \hat{u}^z(\xi) \\ \hat{u}^z(\xi) &= \frac{1}{2(1-\kappa\xi)} + (1-\kappa\xi)\,\frac{\hat{\mathbf{x}}^{\perp\prime 2}(\xi) - 1}{2}; \end{aligned} \tag{19}$$

here we have adopted the initial conditions $\mathbf{x}(0) = \mathbf{0} = \mathbf{u}(0)$. We next analyze a few special cases.

3.1 Case $\mathbf{E}_s = \mathbf{B}_s = \mathbf{0}$ (zero static fields)

Then (19) becomes [7, 8]:

$$\begin{aligned} &\hat{s} \equiv 1, \quad \hat{\mathbf{u}}^\perp = \frac{-q\boldsymbol{\alpha}^\perp}{mc^2}, \quad \hat{u}^z = \frac{\hat{\mathbf{u}}^{\perp 2}}{2}, \quad \hat{\gamma} = 1 + \hat{u}^z \\ &\hat{z}(\xi) = \int_{\xi_0}^\xi dy\, \frac{\hat{\mathbf{u}}^{\perp 2}(y)}{2}, \quad \hat{\mathbf{x}}^\perp(\xi) = \int_{\xi_0}^\xi dy\, \hat{\mathbf{u}}^\perp(y). \end{aligned} \tag{20}$$

The solutions (20) induced by two x-polarized pulses and the corresponding e^- trajectories in the zx plane are shown in fig. 2. Note that:

- The maxima of γ, $\alpha^\perp$ coincide (and approximately also of $\epsilon(\xi)$, if $\epsilon(\xi)$ is slowly varying).
- Since $u^z \geq 0$, the z-drift is positive-definite. Rescaling $\boldsymbol{\epsilon}^\perp \mapsto a\boldsymbol{\epsilon}^\perp$, $\hat{\mathbf{x}}^\perp, \hat{\mathbf{u}}^\perp$ scale like a, whereas $\hat{z}, \hat{u}^z$ scale like a^2 (hence the trajectory goes to a straight line in the limit $a \to \infty$). This is due to magnetic force $q\boldsymbol{\beta} \wedge \mathbf{B}$.
- **Corollary** The final u and energy gain read

$$\mathbf{u}_f^\perp = \hat{\mathbf{u}}^\perp(\infty), \quad u_f^z = \mathcal{E}_f = \frac{1}{2}\mathbf{u}_f^{\perp 2} = \gamma_f - 1; \tag{21}$$

Both are very small if the pulse modulation ϵ is slow [extremely small if $\epsilon \in \mathcal{S}(\mathbb{R})$ or $\epsilon \in C_c^\infty(\mathbb{R})$].

Recall the *Lawson-Woodward* Theorem [10–13] (an outgrowth of the original Woodward-Lawson Theorem [14, 15]): in spite of large energy variations during the interaction, the final energy gain $\mathcal{E}_f$ of a charged particle $\mathcal{P}$ interacting with an EM field is zero if:
i) the interaction occurs in $\mathbb{R}^3$ vacuum (no boundaries);
ii) $\mathbf{E}_s = \mathbf{B}_s = \mathbf{0}$ and $\boldsymbol{\epsilon}^\perp$ is slowly modulated;
iii) $v^z \simeq c$ along the whole acceleration path;
iv) nonlinear (in $\boldsymbol{\epsilon}^\perp$) effects $q\boldsymbol{\beta} \wedge \mathbf{B}$ are negligible;
v) the power radiated by $\mathcal{P}$ is negligible.
Our Corollary, as Ref. [9], states the same result if we relax iii), iv), but the EM field is a *plane* travelling wave.

To obtain a non-zero $\mathcal{E}_f$ one has to violate some other conditions of the theorem, as e.g. we see in next cases.

3.2 Case $\mathbf{E}_s = 0$, $\mathbf{B}_s = B_s^z\mathbf{k}$

Then (19) becomes $\hat{s} \equiv 1$ and

$$\begin{aligned} &(\hat{x} + i\hat{y})(\xi) = \int_0^\xi d\zeta\, e^{ib(\zeta-\xi)}(w^x + iw^y)(\zeta), \quad \hat{\mathbf{u}}^\perp = \hat{\mathbf{x}}^{\perp\prime}, \\ &\hat{u}^z = \hat{z}' = \frac{\hat{\mathbf{u}}^{\perp 2}}{2} = \mathcal{E} = \hat{\gamma} - 1, \quad \hat{z}(\xi) = \int_0^\xi d\zeta\, \frac{\hat{\mathbf{u}}^{\perp 2}(\zeta)}{2}. \end{aligned} \tag{22}$$

(22) reduces to the solution of [16, 17] for monochromatic $\boldsymbol{\epsilon}^\perp$. This leads to *cyclotron autoresonance* if $-b = k = \frac{2\pi}{\lambda} \gg \frac{1}{l}$: for circular polarization $w^x(\xi) + iw^y(\xi) \simeq e^{ik\xi}\mathrm{w}(\xi)$,

$$(\hat{x} + i\hat{y})(\xi) \simeq iW(\xi)e^{ik\xi}, \quad W(\xi) \equiv \int_0^\xi d\zeta\, \mathrm{w}(\zeta) > 0$$

where $\mathrm{w}(\xi) \equiv q\epsilon(\xi)/kmc^2$; clearly $W(\xi)$ grows with ξ. In particular if $\boldsymbol{\epsilon}^\perp(\xi) = \mathbf{0}$ for $\xi \geq l \equiv$, then for such ξ

$$\hat{z}'(\xi) \simeq \frac{k^2}{2}W^2(l) \simeq 2\mathcal{E}_f, \qquad \frac{|\hat{\mathbf{x}}^{\perp\prime}(\xi)|}{\hat{z}'(\xi)} \simeq \frac{2}{kW(l)} \ll 1;$$

3.3 Case $\mathbf{E}_s = E_s^z\mathbf{k}$, $\mathbf{B}_s = \mathbf{0}$

Then the solution (19) reduces to $\hat{s}(\xi) = 1 - \kappa\xi$,

$$(\hat{x} + i\hat{y})(\xi) = \int_0^\xi dy\, \frac{(w^x + iw^y)(y)}{1-\kappa y}, \quad \hat{z}(\xi) = \int_0^\xi \frac{dy}{2}\left\{\frac{1+\hat{v}(y)}{[1-\kappa y]^2} - 1\right\}; \tag{23}$$

If $\boldsymbol{\epsilon}^\perp$ is slowly modulated the energy gain (13) $\mathcal{E}_f$ is negative if $\kappa > 0$, positive if $\kappa \leq 0$ and has a unique maximum

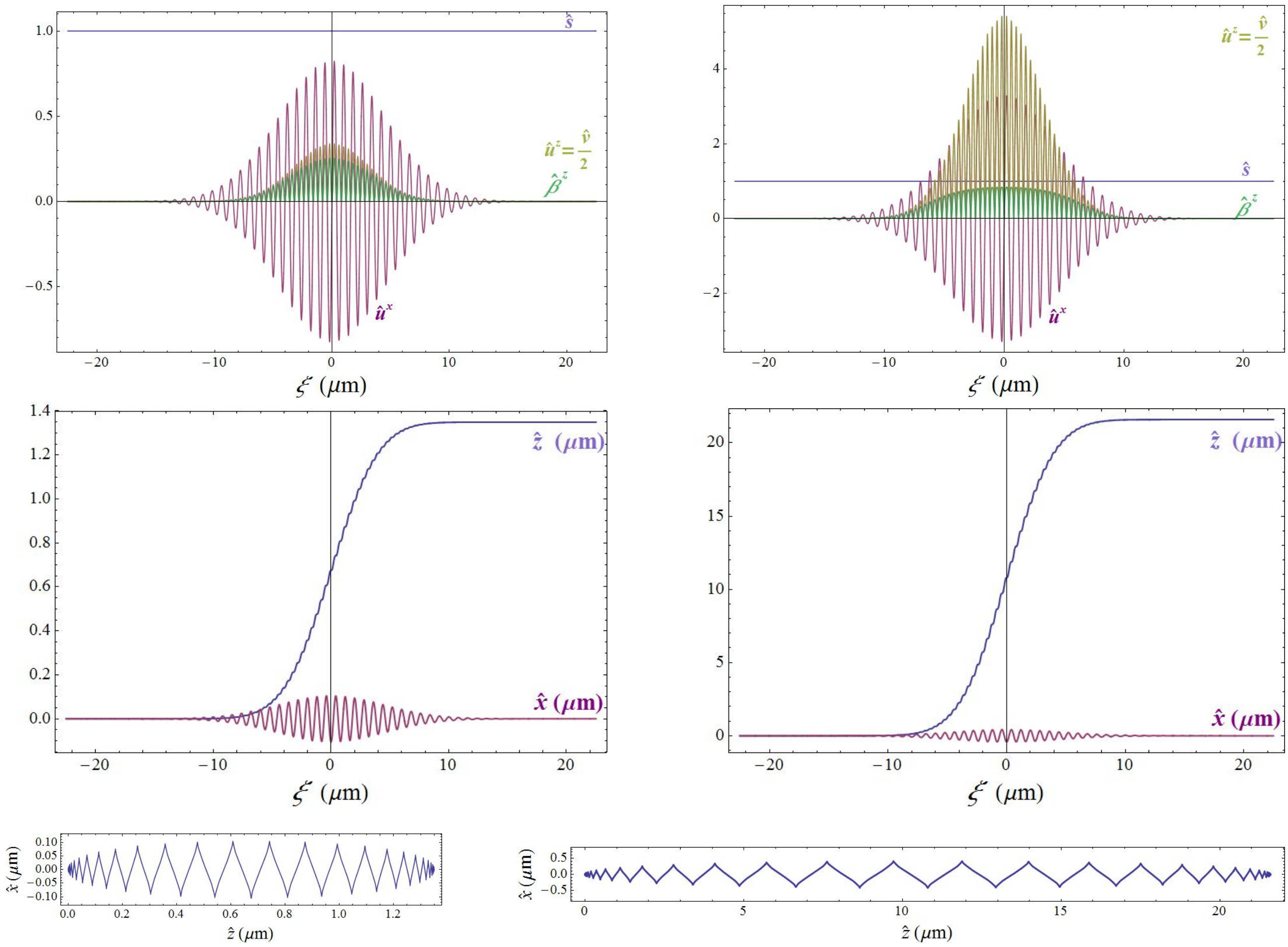
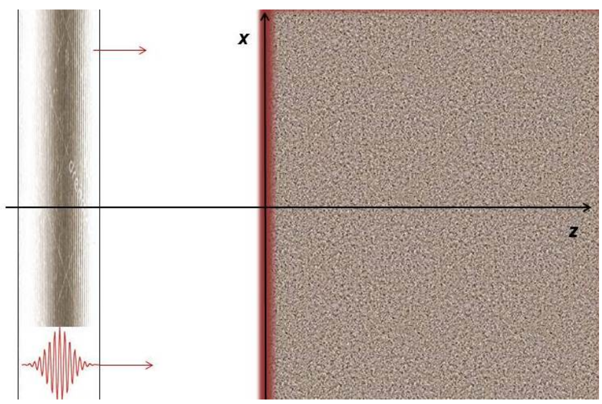


Figure 2. Solutions (20) and e^- trajectories in the zx plane induced by two x-polarized pulses with carrier wavelength $\lambda = .8\mu$m, gaussian modulation $\epsilon(\xi) = a\exp[-\xi^2/2\sigma]$, $\sigma = 20\mu\text{m}^2$, $ea\lambda/mc^2 = 4, 15$ (left, right).

point $\kappa_M < 0$ if $\epsilon(\xi)$ has a finite support with a unique maximum. Here is an acceleration device based on this solution: at $t = 0$ the particle initially lies at rest with $z_0 \lesssim 0$, just at the left of a metallic grating G contained in the $z=0$ plane and set at zero electric potential; another metallic plate P contained in a plane $z = z_p > 0$ is set at electric potential $V = V_p$. A short laser pulse $\boldsymbol{\epsilon}^\perp$ hitting the particle boosts it into the latter region through the ponderomotive force; choosing $qV_p > 0$ implies $\kappa = -qV_p/z_p mc^2 < 0$, and a backward longitudinal electric force qE_s^z. If qV_p is large enough, then $z(t)$ will reach a maximum smaller than z_p, then is accelerated backwards and exits the grating with energy $\mathcal{E}_f$ and negligible transverse momentum. A large $\mathcal{E}_f$ requires extremely large $|V_p|$, far beyond the material breakdown threshold, what prevents its realization as a static field (namely, sparks between G, P would arise and rapidly reduce $|V_p|$). A way out is to make the pulse itself generate such large $|E_s^z|$ within a plasma at the right time so as to induce the it slingshot effect, as sketchily explained at the end of next section.

4 Plane plasma problems

Assume that the plasma is initially in hydrodynamic conditions with all initial data [velocities, densities n_h, EM fields of the form (2)] not depending on $\mathbf{x}^\perp$. Then also the solutions for $\mathbf{B}, \mathbf{E}, \mathbf{u}_h, n_h, \Delta\mathbf{x}_h \equiv \mathbf{x}_h(t, \mathbf{X}) - \mathbf{X}$ (displacements) do not depend on $\mathbf{x}^\perp$. Here $\mathbf{x}_h(t,\mathbf{X})$ is the position at t of the h-th fluid material element with initial position $\mathbf{X} \equiv (X,Y,Z)$; $\mathbf{X}_h(t,\mathbf{x})$ is the inverse (at fixed t). More specifically, we consider the impact of an EM plane wave with a pump of the type (3.a) on a cold plasma at equilibrium (figure below); the initial conditions are:



$$
\begin{aligned}
&\mathbf{u}_h(0,\mathbf{x}) = \mathbf{0}, \qquad n_h(0,\mathbf{x}) = 0 \qquad \text{if } z \leq 0,\\
&j^0(0,\mathbf{x}) = \sum_h q_h n_h(0,\mathbf{x}) \equiv 0, \qquad (24)\\
&\mathbf{E}(0,\mathbf{x}) = \boldsymbol{\epsilon}^\perp(-z), \quad \mathbf{B}(0,\mathbf{x}) = \mathbf{k} \wedge \boldsymbol{\epsilon}^\perp(-z) + \mathbf{B}_s.
\end{aligned}
$$

Then Maxwell eq.s $\nabla\cdot\mathbf{E} = 4\pi j^0$, $\partial_t E^z/c + 4\pi j^z = (\nabla\wedge\mathbf{B})^z = 0$ (the current density is $\mathbf{j} = \sum_h q_h n_h \boldsymbol{\beta}_h = \sum_h q_h n_h \frac{\mathbf{u}_h}{\gamma_h}$) imply [8]

$$
E^z(t,z) = 4\pi \sum_h q_h \widetilde{N}_h[Z_h(t,z)], \qquad (25)
$$

where $\widetilde{N}_h(Z) \equiv \int_0^Z d\zeta\, n_h(0,\zeta)$: we thus reduce by one the number of unknowns, expressing E^z in terms of the (still unknown) longitudinal motion. $\mathbf{A}^\perp$ is coupled to the currents through $\Box\mathbf{A}^\perp = 4\pi\mathbf{j}^\perp$ (in the Landau gauges). Including (24) this amounts to the integral equation

$$
\mathbf{A}^\perp - \alpha^\perp - \frac{\mathbf{B}_s}{2} \wedge \mathbf{x} = 2\pi \int ds\, d\zeta\, \theta(ct - s - |z-\zeta|)\theta(s)\, \mathbf{j}^\perp\!\left(\frac{s}{c}, \zeta\right). \quad (26)
$$

The right-hand side (rhs) is zero for $t \leq 0$, because $t = 0$ is the beginning of the laser-plasma interaction. Within *short* time intervals $[0, t']$ (to be determined *a posteriori*) we can approximate $\mathbf{A}^\perp(t,z) \simeq \boldsymbol{\alpha}^\perp(ct-z) + \frac{\mathbf{B}_s}{2} \wedge \mathbf{x}$; we also neglect the motion of ions with respect to that of electrons. Then the Hamilton equations for the electron fluid with 'time' ξ and the initial conditions amount to (9) and

$$
\begin{aligned}
&mc^2 \hat{s}_e'(\xi, Z) = 4\pi e^2 \left[\widetilde{N}(\hat{z}_e) - \widetilde{N}(Z)\right] + e(\hat{\mathbf{x}}_e^{\perp\prime} \wedge \hat{\mathbf{B}}_s^\perp)^z,\\
&mc^2 \hat{\mathbf{u}}_e^{\perp\prime}(\xi, Z) = e\boldsymbol{\alpha}^{\perp\prime} - e(\hat{\mathbf{x}}_e' \wedge \hat{\mathbf{B}}_s)^\perp,
\end{aligned} \qquad (27)
$$

$$
\hat{\mathbf{x}}_e(0,\mathbf{X}) = \mathbf{X}, \quad \hat{\mathbf{u}}_e(0,\mathbf{X}) = \mathbf{0} \quad \Rightarrow \quad \hat{s}_e(0,\mathbf{X}) = 1. \qquad (28)
$$

this is a family parametrized by Z of *decoupled ODEs* which can be solved numerically. The approximation on $\mathbf{A}^\perp(t,z)$ is acceptable as long as the so determined motion makes $|\text{rhs}(26)| \ll |\alpha^\perp + \frac{\mathbf{B}_s}{2} \wedge \mathbf{x}|$; otherwise rhs(26) determines the first correction to $\mathbf{A}^\perp$; and so on.

If $\mathbf{B}_s = \mathbf{0}$, again $(27)_2$ is solved by $\hat{\mathbf{u}}_e^\perp(\xi) = e\boldsymbol{\alpha}^\perp(\xi)/mc^2$, while, setting $v = \hat{\mathbf{u}}^{\perp 2}$, $(9)_2$, $(27)_1$ take [20] the form of (17)

$$
\Delta\hat{z}_e' = \frac{1+v}{2\hat{s}^2} - \frac{1}{2}, \qquad \hat{s}_e' = \frac{4\pi e^2}{mc^2}\left\{\widetilde{N}[\hat{z}_e] - \widetilde{N}(Z)\right\}. \qquad (29)
$$

If $n_e(0,\mathbf{X}) = n_0$ =const for $Z \geq 0$, then as long as $\hat{z}_e(\xi, Z) > 0$ (29), (28) reduce to the *same* Cauchy problem *for all* Z:

$$
\Delta' = \frac{1+v}{2s^2} - \frac{1}{2}, \qquad s' = M\Delta, \qquad (30)
$$

$$
\Delta(0) = 0, \qquad s(0) = 1 \qquad (31)
$$

with $M \equiv \frac{4\pi e^2 n_0}{mc^2}$. In fig. 4 we show the solution if $\boldsymbol{\epsilon}^\perp$ is as in fig. 3 and $n_0 = 2\times10^{18}\text{cm}^{-3}$; $s(\xi)$ is indeed insensitive to the fast oscillations of $\boldsymbol{\epsilon}^\perp$ (see section 2.1). After the pulse is passed it becomes periodic: a plasma travelling-wave of spacial period $\xi_H \simeq 49\mu$m follows the pulse. The other unknowns are obtained through (16). Replacing in rhs(26) we find that $\mathbf{A}^\perp \simeq \boldsymbol{\alpha}^\perp$ is verified at least for $t \leq 5\xi_H/c$

The above results are based on a laser spot size $R = \infty$ (plane wave). When including corrections due to the finite R (based on causality and heuristic estimates), they imply: the impact of a very short and intense laser pulse on the surface of a cold low-density plasma (or gas, ionized into a plasma by the pulse itself) may induce (for carefully tuned R), beside a wakefield propagating behind the pulse [23, 24], also a backward acceleration and expulsion of surface electrons [20, 21] (*slingshot effect*), as schematically depicted in fig. 5. For reviews see also [22].

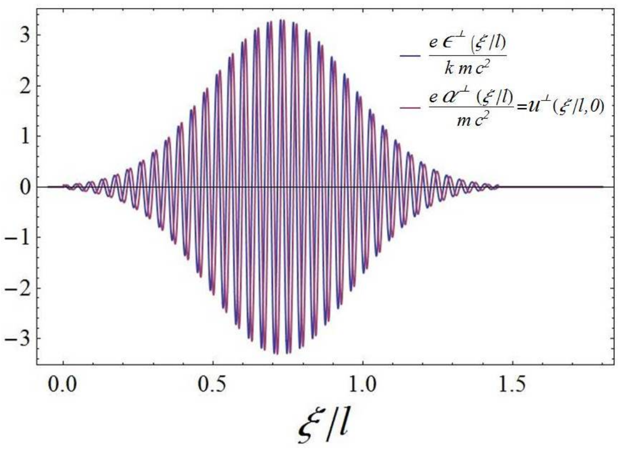


Figure 3. Normalized pump $\boldsymbol{\epsilon}^\perp$ with carrier wavelength $\lambda = 0.8\mu$m, gaussian modulation $\epsilon(\xi) = a\exp[-\xi^2/2\sigma]$, $\sigma = 20\mu\text{m}^2$, $ea\lambda/mc^2 = 15$, average pulse intensity 10^{19}W/cm^2, linear polarization. $l \simeq 27\mu$m is the length of the interval where the pump amplitude ϵ overcomes the ionization threshold for the cold gas (here helium) yielding the plasma; under such pulses the thresholds for 1^{st} and 2^{nd} ionization are overcome (i.e. Keldysh parameters become smaller than 1) almost simultaneously [18, 19].

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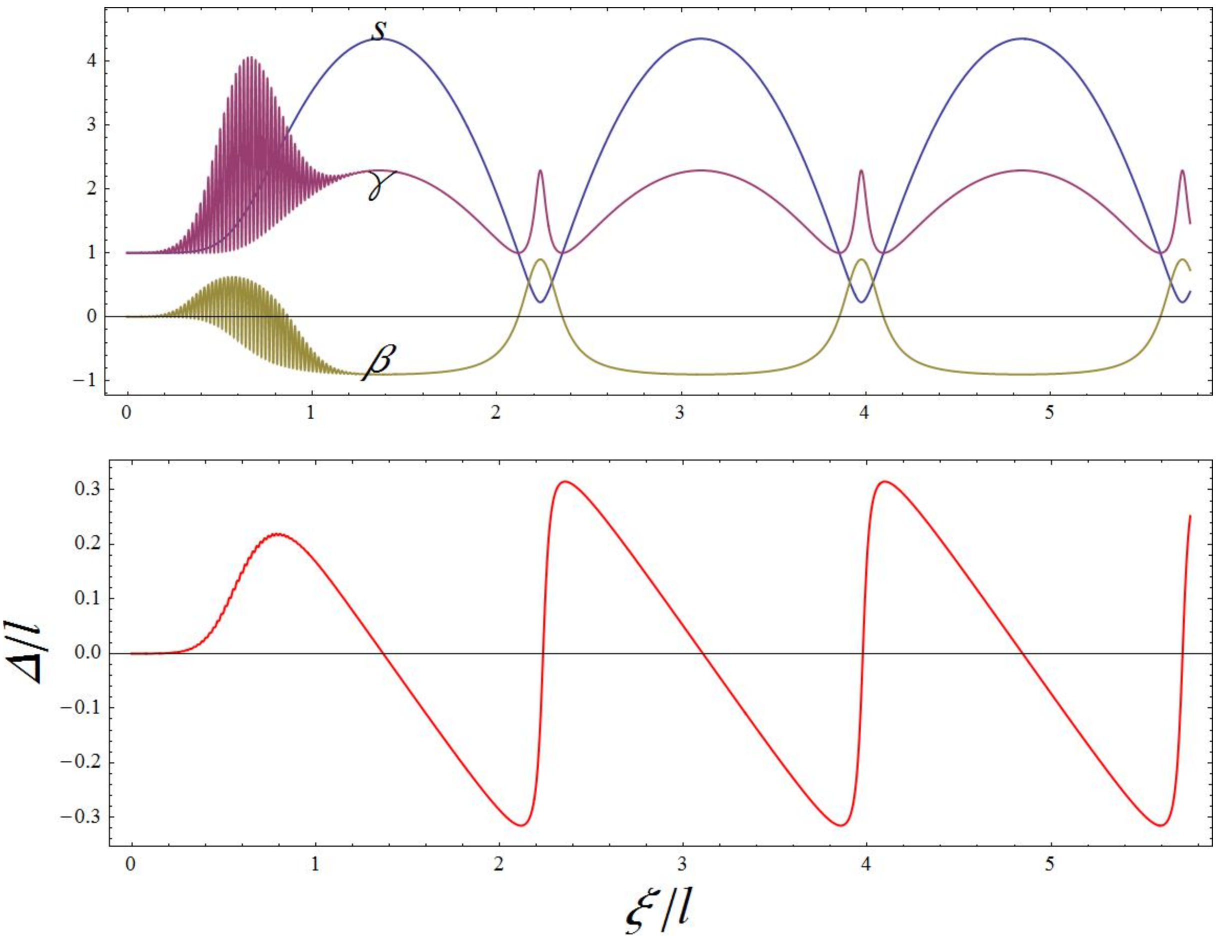

Figure 4. Solution of (29-8) corresponding to the pulse of fig. 3, initial density $\widetilde{n_{e0}}(Z) = n_0\theta(Z)$, $n_0 = 2\times10^{18}\mathrm{cm}^{-3}$.

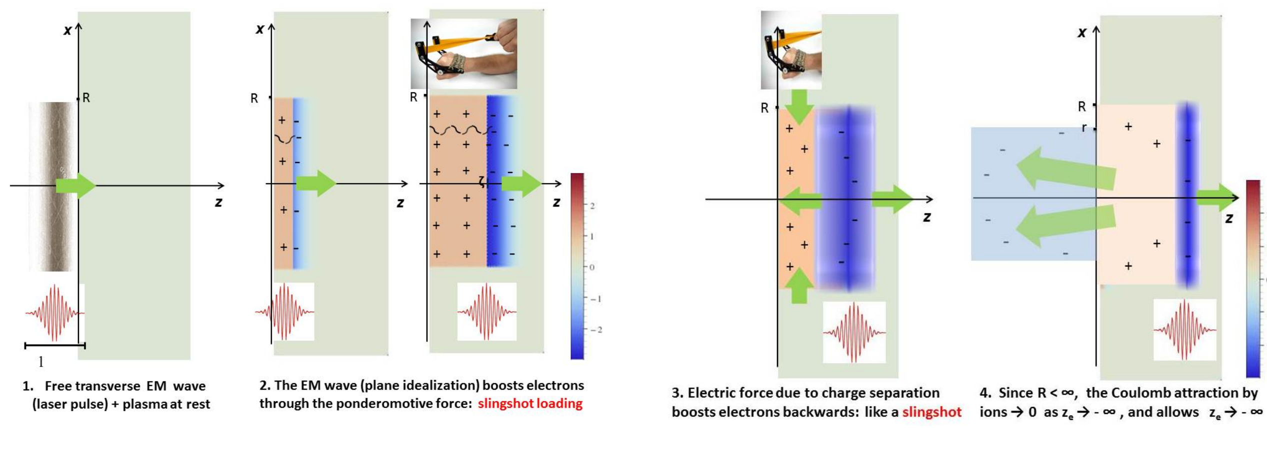

Figure 5. Schematic stages of the slingshot effect.

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