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Holographic Entanglement Entropy, SUSY & Calibrations

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Abstract. Holographic calculations of entanglement entropy boil down to identifying minimal surfaces in curved spacetimes. This generically entails solving second-order equations. For higher-dimensional AdS geometries, we demonstrate that supersymmetry and calibrations reduce the problem to first-order equations. We note that minimal surfaces corresponding to disks preserve supersymmetry, whereas strips do not.

1 Introduction

Determining entanglement entropy in Quantum Field Theory is a challenging problem. For QFTs with holographic duals, Ryu-Takayanagi (RT) argued that the task is equivalent to calculating the area of codimension-two minimal surfaces in a dual geometry [1, 2]. The results for AdS_3 show perfect agreement with 2D CFTs [3–5], suggesting that this method may have merit in higher dimensions. Regularisation aside, even for time-independent QFTs, this approach is plagued by one noticeable technicality: the task of identifying minimal surfaces involves solving second-order equations.

In this letter, we consider two approaches, one inspired by physics, the second by mathematics, to recast the minimal surface problem so that the resulting equations are first-order. From the physics perspective, as demonstrated for locally AdS_3 geometries [6], e. g. BTZ black holes [7, 8], one can derive the minimal surface from supersymmetry (SUSY) [9]. An immediate corollary is that in 3D the RT minimal surfaces are supersymmetric. Here, we show in higher-dimensional AdS_{d+2} geometries that a disk minimal surface preserves supersymmetry, whereas the strip does not. Working backwards to derive the surface is expected to mirror the 3D analysis [9]. This suggests that supersymmetry may be exploited to identify minimal surfaces in supersymmetric geometries, such as LLM [10].

Our second approach involves importing calibrations [11] from mathematics. At the heart of this method is the fact that a closed differential form is guaranteed to be a minimal surface in its homology class. For locally AdS_3 geometries, it has been shown that the minimal surfaces correspond to special Lagrangian (sLag) submanifolds on a 2D spacelike hypersurface [12]. In this letter, we demonstrate the the calibration conditions for strips and disks in AdS_{d+2} geometries agree with the equations that result from extremising actions [2]. In particular, we will show that second-order equations are implied.

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2 Supersymmetric minimal surfaces

We consider two minimal surfaces embedded in unit radius AdS_{d+2} spacetimes, namely

$$\mathrm{d}s^2_{\text{disk}} = \frac{-\mathrm{d}t^2 + \mathrm{d}\eta^2 + \eta^2 \mathrm{d}s^2(S^{d-1}) + \mathrm{d}r^2}{r^2}, \tag{1}$$

and

$$\mathrm{d}s^2_{\text{strip}} = \frac{-\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \cdots + \mathrm{d}x_d^2 + \mathrm{d}r^2}{r^2}, \tag{2}$$

which we will refer to as disk and strip geometries, respectively. In each case, we assume the minimal surface is embedded along $\mathbb{R}^d$, but in the former the radial direction only depends on η, $r(\eta)$, while in the latter, it depends on x_1, $r(x_1)$. Symmetry reduces the complexity of the problem to an effective 3D problem. We consider time-independent configurations and set $t = 0$ for simplicity.

In order to discuss supersymmetric configurations, we introduce the supersymmetry condition, or Killing spinor equation (KSE), of AdS_{d+2} spacetime,

$$\nabla_\mu \epsilon = \frac{1}{2}\gamma_\mu \epsilon, \tag{3}$$

where ϵ denotes the Killing spinor, γ_μ are matrices satisfying the Clifford algebra, and $\nabla_\mu \equiv \partial_\mu + \frac{1}{4}\omega_{\mu\rho\sigma}\gamma^{\rho\sigma}$, where ω is the spin connection. All information about the geometry is encoded in the spin connection, and acting twice with ∇, one derives the Einstein equation $R_{\mu\nu} = -(d+1)g_{\mu\nu}$.

For $d = 1$, there is no distinction between strip and disk, and it was shown that RT minimal surfaces for AdS_3 spacetimes preserve *locally* half the supersymmetry [9]. Here we extend the analysis to higher dimensions. For simplicity, we focus on $d = 3$, which is most relevant to AdS/CFT [13].

2.1 Disk

We consider the geometry (1) with a circular disk on the boundary $\eta \le h$, with h constant. For this surface to be consistent with supersymmetry, we expect to be able to identity a subset of the Killing spinors that "commute" with the bulk minimal surface. More precisely, there must be Killing spinors that are solutions to the following projection condition:

$$\Gamma\epsilon = \frac{i}{\sqrt{1+\left(\frac{\mathrm{d}r}{\mathrm{d}\eta}\right)^2}}\gamma_{\theta\phi}\left(\gamma_\eta + \frac{\mathrm{d}r}{\mathrm{d}\eta}\gamma_r\right)\epsilon = \epsilon. \tag{4}$$

The projection condition is inspired by supersymmetric branes and embeddings [14], where $\Gamma^2 = 1$, so that this defines a valid projection condition.

The solution to the Killing spinor equation (3), evaluated at $t = 0$, is

$$\epsilon = \left(r^{\frac{1}{2}} + r^{-\frac{1}{2}}\eta\gamma_\eta\right)\tilde{\epsilon}_+ + r^{-\frac{1}{2}}\tilde{\epsilon}_-, \tag{5}$$

where we have defined $\tilde{\epsilon}_\pm = e^{\frac{\theta}{2}\gamma_{\eta\theta}} e^{\frac{\phi}{2}\gamma_{\theta\phi}}\epsilon_\pm$ in terms of the usual two-sphere coordinates and constant spinors satisfy $\gamma_r \epsilon_\pm = \pm\epsilon_\pm$. Evaluating this condition on the known minimal surface [2],

$$\eta^2 + r^2 = h^2, \tag{6}$$

where h is a constant, we find that the projection condition implies the relation,

$$i\gamma_{\theta\phi\eta}\,\epsilon_- = h\,\epsilon_+. \tag{7}$$

As a result, we see that $\epsilon_\pm$ are not independent, which reduces the number of supersymmetry by half. This result closely mirrors earlier 3D analysis [9] and we conclude that the disk minimal surface will be supersymmetric in other dimensions.

2.2 Strip

This time around the Killing spinor is

$$\epsilon = \left(r^{\frac{1}{2}} + r^{-\frac{1}{2}} x_i \gamma_i\right) \epsilon_+ + r^{-\frac{1}{2}} \epsilon_-, \tag{8}$$

where repeated indices $i = 1, 2, 3$ are summed and the constant spinors are subject to the same condition. The projector Γ is analogous:

$$\Gamma = \frac{i}{\sqrt{1 + \left(\frac{\mathrm{d}r}{\mathrm{d}x_1}\right)^2}} \gamma_{x_2 x_3} \left(\gamma_{x_1} + \frac{\mathrm{d}r}{\mathrm{d}x_1} \gamma_r\right), \tag{9}$$

which is simply an artifact of the large amount of symmetry. Since the surface is embedded along x_2 and x_3, for supersymmetry to be preserved, these coordinates must drop out of the projection condition. It is easy to convince oneself that this cannot happen, so supersymmetry is not preserved.

3 Calibrations

In [12], where AdS_3 geometries were largely discussed, proposals were given for candidate calibrations in higher dimensions. For geometries (1) and (2), we recall the suggested d-forms [12]:

$$\varphi_{\text{disk}} = e^{i\chi} \left(\frac{\eta}{r}\right)^{d-1} \left(\frac{\mathrm{d}\eta}{r} + i\frac{\mathrm{d}r}{r}\right) \wedge \mathrm{vol}(S^{d-1}), \tag{10}$$

and

$$\varphi_{\text{strip}} = e^{i\chi} \frac{1}{r^{d-1}} \left(\frac{\mathrm{d}x_1}{r} + i\frac{\mathrm{d}r}{r}\right) \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_d, \tag{11}$$

where χ is an arbitrary phase that is fixed through the calibration conditions:

$$0 = \mathrm{Im}(\varphi) = \mathrm{dRe}(\varphi). \tag{12}$$

Only in the special case $d = 1$ do the forms correspond to sLag calibrations and otherwise, their identity in mathematics is not immediately obvious. Admittedly, we have pulled φ_{disk}, φ_{strip} and (12) out of a hat. However, we justify the expressions by the fact that they correspond to sLag calibration conditions for $d = 1$ and also lead to reasonable results; in this section we will show that the conditions (12) recover the minimal surface equations identified in [2]. Our analysis here improves on [12] by showing agreement at the level of equations and not for specific solutions. This suggests that φ_{disk} and φ_{strip} are indeed valid calibrations. In contrast to the last section, here we consider general d.

3.1 Disk

We return to the disk geometry (1), where the minimal surface has the action [2],

$$\mathcal{L}_{\text{disk}} = \frac{\eta^{d-1}}{r^d} \sqrt{1 + \left(\frac{\mathrm{d}r}{\mathrm{d}\eta}\right)^2}. \tag{13}$$

Varying the action, we encounter a second-order equation [2]:

$$\eta r r'' + (d-1) r (r')^3 + (d-1) r r' + d\eta (r')^2 + d\eta = 0, \tag{14}$$

where $r' = \frac{dr}{d\eta}$, etc. Assuming our calibration (10) is correct, we expect that the calibration conditions (12) imply (14). To demonstrate this, let us note that we can rewrite (14) as

$$\frac{\eta r r''}{1+(r')^2} + (d-1)rr' + d\eta = 0. \tag{15}$$

Taking the derivative of the $\mathrm{Im}(\varphi) = 0$ condition, we see that

$$\partial_r \chi = -\tan\chi \partial_\eta \chi. \tag{16}$$

We can use this expression to bring $\mathrm{dRe}(\varphi) = 0$ to the form:

$$0 = -d\eta\cos\chi + r(d-1)\sin\chi + \frac{r\eta\partial_\eta\chi}{\cos\chi}. \tag{17}$$

At this point we substitute,

$$\cos\chi = \frac{1}{\sqrt{1+(r')^2}}, \quad \sin\chi = -\frac{r'}{\sqrt{1+(r')^2}}, \quad \partial_\eta\chi = -\frac{r''}{(1+(r')^2)^2}, \tag{18}$$

where all expressions follow from the first equation in (12). Substituting back into the (17), we get (14). So, we have demonstrated that any solution to the first order equations (12) satisfies the second-order equation (14), thus validating the proposed calibration φ_{disk}.

Before concluding, it is known that (6) solves (14). It is worth noting that taking the large d limit, (14) simplifies, but still retains (6) as a solution. In fact, we can rewrite (14) as

$$(\eta r r'' - r(r')^3 - rr') + d\left(r(r')^3 + rr' + \eta(r')^2 + \eta\right) = 0. \tag{19}$$

where it can be checked that both bracketed terms vanish on the known solution. The other solutions, which are not common, do not make contact with the AdS boundary at the boundary of the disk. Our observations suggest (6) is the only solution relevant for holographic entanglement entropy.

3.2 Strip

Moving along, we consider the strip geometry, where the area functional is given by the action [2],

$$\mathcal{L}_{\mathrm{strip}} = \frac{1}{r^d}\sqrt{1+\left(\frac{dr}{dx_1}\right)^2}, \tag{20}$$

which corresponds to an x_1-independent Hamiltonian. Viewing x_1 as time, we infer,

$$\frac{dr}{dx_1} = \frac{\sqrt{h^{2d} - r^{2d}}}{r^d}, \tag{21}$$

where h is a constant. Note, this first-order equation can be compared directly with (12) for φ_{strip}.

The calibration conditions that arise from (11) can be solved using the method of characteristics (see, for example [15]) and are equivalent to a systems of ODEs,

$$r\,dr = -\frac{r\sqrt{1-f^2}}{f}dx = \frac{r^2\,df}{p\,f}, \tag{22}$$

where we have defined $f = \cos\chi$. It is easy to find one first integral,

$$c_1 = \frac{f}{r^p}, \tag{23}$$

with constant c_1. Substituting this into the remaining equation and integrating, one can identify a second first integral, but we will not need the expression. It is enough to redefine $c_1 = -1/h^p$ so that we recover (21), which makes it clear that extremisation of the action (20) and the calibration conditions are equivalent.

We have thus demonstrated that φ_{disk} and φ_{strip} are valid calibrations, despite not being obviously sLag, except when $d = 1$.

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