Dilatonic dyon black hole solutions in the model with two Abelian gauge fields

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Abstract. Dilatonic black hole dyon-like solutions in the gravitational $4d$ model with a scalar field, two 2-forms, two dilatonic coupling constants $\lambda_i \neq 0, i = 1, 2$, obeying $\lambda_1 \neq -\lambda_2$ and sign parameter $\varepsilon = \pm 1$ for scalar field kinetic term are considered. Here $\varepsilon = -1$ corresponds to ghost scalar field. These solutions are defined up to solutions of two master equations for two moduli functions, when $\lambda_i^2 \neq 1/2$ for $\varepsilon = -1$. A set of bounds on gravitational mass and scalar charge are presented by using a certain conjecture on parameters of solutions, when $1 + 2\lambda_i^2 \varepsilon > 0, i = 1, 2$.

1 Introduction

Here we give a brief extension of our previous work [1] devoted to dilatonic dyon black hole solutions. We consider a subclass of dilatonic black hole solutions with electric and magnetic charges $Q_1$ and $Q_2$, respectively in $4d$ model with metric $g$, scalar field $\varphi$, two 2-forms $F^{(1)}$ and $F^{(2)}$, corresponding to two dilatonic coupling constants $\lambda_1$ and $\lambda_2$, respectively. For coinciding dilatonic couplings $\lambda_1 = \lambda_2 = \lambda$ we get a trivial non-composite generalisation of dilatonic dyon black hole solutions in the model with one 2-form which was considered in ref. [1]. The dilatonic scalar field may be either an ordinary one or a phantom (or ghost) one.

Here we present lower bounds on gravitational mass $M$ and scalar charge $Q_\varphi$. As in ref. [1] this problem is solved here up to a conjecture, which states one to one (smooth) correspondence between the pair $(Q_1^2, Q_2^2)$, where $Q_1$ is electric charge and $Q_2$ is magnetic charge, and the pair of positive parameters $(P_1, P_2)$, which appear in decomposition of moduli functions at large distances. This conjecture is believed to be valid for all $\lambda_i \neq 0$ in the case of ordinary scalar field and for $0 < \lambda_i^2 < 1/2$ for the case of phantom scalar field (in both cases the inequality $\lambda_1 \neq -\lambda_2$ is assumed).

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2 Black hole dyon solutions

Let us consider a model governed by the action

\[ S = \frac{1}{16\pi G} \int d^4x \sqrt{|g|} \left( R[g] - \epsilon (\partial \varphi)^2 - \sum_{i=1}^{2} \frac{1}{2} \epsilon^{2\lambda_i \varphi} (F^{(i)})^2 \right), \]  

(2.1)

where \( g = g_{\mu\nu}(x)dx^\mu \otimes dx^\nu \) is metric, \( \varphi \) is the scalar field, \( F^{(i)} = dA^{(i)} = \frac{1}{2} F^{(i)}_{\mu\nu}dx^\mu \wedge dx^\nu \) is the 2-form with \( A^{(i)} = A^{(i)}_{\mu} dx^\mu, \epsilon = \pm 1, G \) is the gravitational constant, \((\partial \varphi)^2 = g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi, (F^{(i)})^2 = F^{(i)}_{\mu\nu}F^{(i)\mu\nu}, i = 1, 2. \) Here \( \lambda_1, \lambda_2 \neq 0 \) are coupling constants obeying \( \lambda_1 \neq -\lambda_2 \) and \( |g| = |\det(g_{\mu\nu})|. \) We put \( \lambda_i^2 \neq 1/2, i = 1, 2, \) for \( \epsilon = -1. \)

We consider a family of dyonic-like black hole solutions to the field equations corresponding to the action (2.1) which are defined on the manifold

\[ \mathcal{M} = (2\mu, +\infty) \times S^2 \times \mathbb{R}, \]  

(2.2)

and have the following form

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu = H_1^h H_2^{h_2} \left( -H_1^{-2h_1} H_2^{-2h_2} \left( 1 - \frac{2\mu}{R} \right) dt^2 \right. \]  

(2.3)

\[ + \left. \frac{dR^2}{1 - \frac{2\mu}{R}} + R^2 d\Omega_3^2 \right), \]

\[ \exp(\varphi) = H_1^{h_1,ae} H_2^{-h_2,ae}, \]

(2.4)

\[ F^{(1)} = \frac{Q_1}{R^2} H_1^{-2} H_2^{-A_{12}} dt \wedge dR, \]

(2.5)

\[ F^{(2)} = Q_2 \tau. \]  

(2.6)

Here \( Q_1 \) and \( Q_2 \) are (colored) charges - electric and magnetic, respectively, \( \mu > 0 \) is the extremality parameter, \( d\Omega_3^2 = d\theta^2 + \sin^2 \theta d\phi^2 \) is the canonical metric on the unit sphere \( S^2 \) \( (0 < \theta < \pi, 0 < \phi < 2\pi), \) \( \tau = \sin \theta d\theta \wedge d\phi \) is the standard volume form on \( S^2, \)

\[ h_i = K_i^{-1}, \quad K_i = \frac{1}{2} + \epsilon \lambda_i^2, \]

(2.7)

\[ i = 1, 2, \]

\[ A_{12} = (1 - 2\lambda_1 \lambda_2 \epsilon) h_2. \]  

(2.8)

Functions \( H_s > 0 \) obey the equations

\[ R^2 \frac{d}{dR} \left( R^2 \frac{1 - 2\mu}{H_s} \frac{dH_s}{dR} \right) = -K_s Q_s^2 \prod_{i=1,2} H_i^{-A_{i,\mu}}, \]

(2.9)

with the following boundary conditions imposed

\[ H_s \rightarrow H_{s0} > 0 \]  

(2.10)

for \( R \rightarrow 2\mu, \) and

\[ H_s \rightarrow 1 \]  

(2.11)

for \( R \rightarrow +\infty, s = 1, 2. \)
In (2.9) we denote

\[
(A_{ss'}) = \begin{pmatrix} 2 & A_{12} \\ A_{21} & 2 \end{pmatrix},
\]

where \( A_{12} \) is defined in (2.8) and

\[
A_{21} = (1 - 2\lambda_1\lambda_2\epsilon)h_1.
\]

These solutions may be obtained just by using general formulae for non-extremal (intersecting) black brane solutions from [2]. The composite analogs of the solutions with one 2-form and \( \lambda_1 = \lambda_2 \) were presented in ref. [1].

The first boundary condition (2.10) guarantees (up to a possible additional demand on analicity of \( H_s(R) \) in the vicinity of \( R = 2\mu \)) the existence of (regular) horizon at \( R = 2\mu \) for the metric (2.3). The second condition (2.11) ensures an asymptotical (for \( R \to +\infty \)) flatness of the metric.

3 Bounds on mass and scalar charge

For ADM gravitational mass we get from (2.3)

\[
GM = \mu + \frac{1}{2}(h_1P_1 + h_2P_2),
\]

where the parameters \( P_s \) appear in asymptotical relations \( H_s = 1 + P_s/R + o(1/R) \), as \( R \to +\infty \). The scalar charge just follows from (2.4)

\[
Q_\varphi = \epsilon(\lambda_1h_1P_1 - \lambda_2h_2P_2).
\]

Here we outline the following hypothesis, which is supported by certain numerical calculations [1, 3]. For \( h_1 = h_2 \) this conjecture was proposed in ref. [1].

**Conjecture.** For any \( h_1 > 0, h_2 > 0, \epsilon = \pm 1, Q_1 \neq 0, Q_2 \neq 0 \) and \( \mu > 0 \): (A) the moduli functions \( H_s(R) \), which obey (2.9), (2.10) and (2.11), are uniquely defined and hence the parameters \( P_1, P_2 \), the gravitational mass \( M \) and the scalar charge \( Q_\varphi \) are uniquely defined too; (B) the parameters \( P_1, P_2 \) are positive and the functions \( P_1 = P_1(Q_1^2, Q_2^2), P_2 = P_2(Q_1^2, Q_2^2) \) define a diffeomorphism of \( \mathbb{R}^2_+ = \{ x \mid x > 0 \} \); (C) in the limiting case we have: (i) for \( Q_2^2 \to +0 \): \( P_1 \to +0, P_2 \to -\mu + \sqrt{\mu^2 + K_1Q_1^2}, P_2 \to +0 \) and (ii) for \( Q_1^2 \to +0 \): \( P_1 \to +0, P_2 \to -\mu + \sqrt{\mu^2 + K_2Q_2^2} \).

The conjecture could be readily verified for the case \( \epsilon = 1, \lambda_1\lambda_2 = 1/2 \). Another integrable case \( \epsilon = 1, \lambda_1 = \lambda_2 = \lambda, \lambda^2 = 3/2 \) is more involved [3].

The conjecture implies the following proposition.

**Proposition.** For \( h_s > 0, Q_s \neq 0, \lambda_s \neq 0 \) (s = 1, 2) and \( \lambda_1 + \lambda_2 \neq 0 \) we have the following bounds on the gravitational mass \( M \) and the scalar charge \( Q_\varphi \) which are valid for all \( \mu > 0 \):

\[
\frac{1}{2} \sqrt{h_{\min}(Q_1^2 + Q_2^2)} < GM,
\]

\[
|Q_\varphi| < |\lambda|_{\max} \sqrt{h_{\min}(Q_1^2 + Q_2^2)},
\]

for \( \epsilon = +1 \) (0 < \( h_s < 2 \)), and

\[
\sqrt{\frac{1}{2}(Q_1^2 + Q_2^2)} < GM,
\]

\[
|Q_\varphi| < |\lambda|_{\max} \sqrt{h_{\max}(Q_1^2 + Q_2^2)},
\]
for $\varepsilon = -1$ ($h_\varepsilon > 2$). Here $h_{\text{min}} = \min(h_1, h_2)$, $h_{\text{max}} = \max(h_1, h_2)$, and $|\lambda|_{\text{max}} = \max(|\lambda|_1, |\lambda|_2)$:

$h_{\text{min}} = (\frac{1}{2} + |\lambda|_{\text{max}}^2)^{-1}$ for $\varepsilon = +1$ and $h_{\text{max}} = (\frac{1}{2} - |\lambda|_{\text{max}}^2)^{-1}$ for $\varepsilon = -1$.

In ref. [1] Proposition was proved for the case $\lambda_1 = \lambda_2$. In this case the bound (3.3) is coinciding (up to notations) with that from ref. [4], which was proved there by using certain spinor techniques.

### 4 Conclusions

Here a family of non-extremal black hole dyon-like solutions in a 4d gravitational model with a scalar field and two Abelian vector fields is considered. The scalar field is either ordinary ($\varepsilon = +1$) or phantom one ($\varepsilon = -1$). The model contain two dilatonic coupling constants $\lambda_s \neq 0, s = 1, 2$, obeying $\lambda_1 = -\lambda_2$.

The solutions are defined up to two moduli functions $H_1(R)$ and $H_2(R)$, which obey two differential equations of second order with boundary conditions imposed. For $\varepsilon = +1$ these equations are integrable for four cases, corresponding to Lie algebras $A_1 + A_1$, $A_2$, $B_2 = C_2$ and $G_2$. In the first case ($A_1 + A_1$) we have $\lambda_1, \lambda_2 = 1/2$, while in the second one ($A_2$) we get $\lambda_1 = \lambda_2 = \lambda$ and $\lambda^2 = 3/2$.

Here we have presented lower bounds on the gravitational mass and upper bounds on the scalar charge for $1 + 2\lambda_2^2 \varepsilon > 0$, which are based on the conjecture on the parameters of solutions $P_1 = P_1(Q_1^1, Q_1^2)$, $P_2 = P_2(Q_2^1, Q_2^2)$. For $\varepsilon = +1$ the lower bound on the gravitational mass is in agreement for $\lambda_1 = \lambda_2$ with that obtained earlier by Gibbons et al. by using certain spinor techniques.

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### References


