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Quasilocal angular momentum of gravitational fields in (2+2) formalism

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Abstract. Recently the Poisson algebra of a quasilocal angular momentum of gravitational fields $L(\xi)$ in (2+2) formalism of Einstein's theory was studied in detail [1]. In this paper, we will briefly review the definition of $L(\xi)$ and its remarkable properties. Especially, it will be discussed that $L(\xi)$ satisfies the Poisson algebra $\{L(\xi), L(\eta)\}_{\rm P.B.} = L([\xi, \eta]_{\rm L})$, up to a constant normalizing factor, and this algebra reduces to the standard SO(3) algebra at null infinity. It will be also argued that our angular momentum is a quasilocal generalization of A. Rizzi's geometric definition.

1 Formalism

In (2+2) formalism of Einstein's theory [2], the 4-dimensional spacetime E_4 is regarded as a fiber bundle that consists of a 2-dimensional base space M_{1+1} of the Lorentzian signature and a 2-dimensional spacelike fiber N_2 at each point on M_{1+1}. Then, the Einstein's theory can be interpreted as a gauge theory defined on M_{1+1} with *diff*(N_2) as a gauge symmetry, the diffeomorphism group of N_2 [3]. Let us introduce a coordinate system $\{u, v, y^a : a = 2, 3\}$ on E_4, where $\{u, v\}$ and $\{y^a\}$ are coordinates on M_{1+1} and N_2, respectively. The most general line element is given by [4],

$$ds^2 = 0 \cdot dv^2 - 2f du dv - 2h du^2 + \phi_{ab}(dy^a + A^a_+ du + A^a_- dv)(dy^b + A^b_+ du + A^b_- dv). \tag{1}$$

All the metric components are functions of u, v, and y^a, since we don't assume any spacetime isometry. Throughout this paper, the subscripts $+$ and $-$ denote u and v, respectively, and N_2 is assumed to be compact. The horizontal lifts $\hat{\partial}_\pm$ of the tangent vector fields $\partial_\pm$ are defined by

$$\hat{\partial}_+ = \partial_+ - A^a_+ \partial_a, \quad \hat{\partial}_- = \partial_- - A^a_- \partial_a \tag{2}$$

where $A^a_\pm$ are the gauge connections of *diff*(N_2). The *diff*(N_2)-covariant derivative of a *diff*(N_2)-tensor density $T_{ab\cdots}{}^{cd\cdots}$ with weight w is defined by

$$D_\pm T_{ab\cdots}{}^{cd\cdots} := \partial_\pm T_{ab\cdots}{}^{cd\cdots} - [A_\pm, T]_{ab\cdots}{}^{cd\cdots}, \tag{3}$$

where $[A_\pm, T]_{Lab\cdots}{}^{cd\cdots}$ denotes the Lie derivative of $T_{ab\cdots}{}^{cd\cdots}$ with respect to $A_\pm := A^a_\pm \partial_a$. It would be helpful to define the following in- and out-going null vector fields for understanding geometrical meanings of quasilocal angular momentum;

$$n = k\Big(\hat{\partial}_+ - \frac{h}{f}\hat{\partial}_-\Big), \quad l = \frac{k}{f}\hat{\partial}_-, \tag{4}$$

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with a normalization condition

$$< n, l > = -k^2, \quad k = \text{constant} > 0. \tag{5}$$

The positive-definite metric ϕ_{ab} on the two-surface N_2 defined by u, v = constant can be factored into a product of the area element e^{σ} and the conformal two-metric ρ_{ab},

$$\phi_{ab} = e^{\sigma}\rho_{ab}, \quad \det\rho_{ab} = 1. \tag{6}$$

When we identify v with a physical time in this formalism, the induced metric on the hypersurface Σ of v = constant is given by

$$ds^2|_{\Sigma} = -2hdu^2 + e^{\sigma}\rho_{ab}(dy^a + A_+^a du)(dy^b + A_+^b du). \tag{7}$$

Therefore, the phase variables on Σ consist of $q^I = (h, \sigma, A_+^a, \rho_{ab})$ and their conjugate momenta $\pi_I = (\pi_h, \pi_\sigma, \pi_a, \pi^{ab})$. Here "0", f, and A_-^a in the metric (1) are turned out to be the Lagrange multipliers. After some lengthy calculations, the first-order form of the Einstein-Hilbert action is found to be

$$S = \int dv du d^2y\{\pi_h\partial_- h + \pi_\sigma\partial_-\sigma + \pi_a\partial_- A_+^a + \pi^{ab}\partial_-\rho_{ab} - 0\cdot C_+ - f\,C_- - A_-^a C_a\}, \tag{8}$$

where the constraints are given by

$$\begin{aligned}
C_+ :=\ & \pi^{ab}D_+\rho_{ab} + \pi_\sigma D_+\sigma - hD_+\pi_h - \partial_+(h\pi_h + 2e^{\sigma}D_+\sigma) \\
& +\partial_a(h\pi_h A_+^a + 2A_+^a e^{\sigma}D_+\sigma + 2he^{-\sigma}\rho^{ab}\pi_b + 2\rho^{ab}\partial_b h), \qquad (9)\\
C_- :=\ & \mathcal{H} - \partial_+\pi_h + \partial_a(A_+^a\pi_h + e^{-\sigma}\rho^{ab}\pi_b), \qquad (10)\\
C_a :=\ & -\partial_+\pi_a + \partial_b(A_+^b\pi_a) + \pi_b\partial_a A_+^b + \pi_\sigma\partial_a\sigma - \partial_a\pi_\sigma \\
& +\pi_h\partial_a h + \pi^{bc}\partial_a\rho_{bc} - \partial_b(\pi^{bc}\rho_{ac}) - \partial_c(\pi^{bc}\rho_{ab}) + \partial_a(\pi^{bc}\rho_{bc}), \qquad (11)
\end{aligned}$$

and the functional $\mathcal{H}$ in (10) is defined as

$$\begin{aligned}
\mathcal{H} :=\ & -\frac{1}{2}e^{-\sigma}\pi_\sigma\pi_h + \frac{1}{4}he^{-\sigma}\pi_h^2 - \frac{1}{2}e^{-2\sigma}\rho^{ab}\pi_a\pi_b - e^{\sigma}R_2 + \frac{1}{2h}e^{-\sigma}\rho_{ac}\rho_{bd}\pi^{ab}\pi^{cd} \\
& +\frac{1}{8h}e^{\sigma}\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_+\rho_{bd}) + \frac{1}{2h}\pi^{ab}D_+\rho_{ab} + \frac{1}{2}\pi_h D_+\sigma. \qquad (12)
\end{aligned}$$

Notice that the conjugate momentum π^{ab} is traceless, $\rho_{ab}\pi^{ab} = 0$, since the determinant of ρ_{ab} is constant. It was shown that the constraints (9), (10), and (11) are the first-class constraints [5]. The variations over the Lagrange multipliers yield

$$C_+ = C_- = C_a = 0. \tag{13}$$

The 12 Hamilton's equations together with these constraint equations are equivalent to the vacuum Einstein' equations $R_{AB} = 0$ for $A, B = \pm, a$. In the present paper, however, we concern the constraint equation $C_a = 0$ only. Multiplying (11) with an arbitrary tangent vector field $\xi = \xi^a\partial_a$ on N_2, we have

$$\begin{aligned}
\xi^a C_a =\ & \pi^{ab}\mathcal{L}_\xi\rho_{ab} + \pi_\sigma\mathcal{L}_\xi\sigma + \pi_h\mathcal{L}_\xi h + \pi_a\mathcal{L}_\xi A_+^a + \pi_a\partial_+\xi^a \\
& -\partial_+(\xi^a\pi_a) + \partial_a(-\xi^a\pi_\sigma - 2\pi^{ab}\xi^c\rho_{bc} + A_+^a\xi^b\pi_b) = 0, \qquad (14)
\end{aligned}$$

where $\mathcal{L}_\xi$ is the Lie derivative with respect to ξ. Thus, if we integrate the constraint equation (14) over N_2, then we obtain the following integro-differential equation:

$$\frac{\partial}{\partial u}L(u, v; \xi) = \frac{1}{16\pi}\oint d^2y(\pi^{ab}\mathcal{L}_\xi\rho_{ab} + \pi_\sigma\mathcal{L}_\xi\sigma + \pi_h\mathcal{L}_\xi h + \pi_a\mathcal{L}_\xi A_+^a + \pi_a\partial_+\xi^a), \tag{15}$$

where $L(u, v; \xi)$ is *quasilocal angular momentum of gravitational fields* defined by

$$L(u, v; \xi) := \frac{1}{16\pi} \oint d^2y(\xi^a \pi_a) + L_0. \tag{16}$$

Here, L_0 is an *undetermined integration constant*, and we simply set $L_0 = 0$ throughout this paper. The equation (15) is the balance equation that relates the rate of change in L to the corresponding flux as the parameter u changes [2, 6]. It should be emphasized that the quasilocal angular momentum can be also written in terms of geometric quantities as

$$L(u, v; \xi) = \frac{1}{16\pi} \oint d\mu \frac{< \xi, [n, l]_{\rm L} >}{< n, l >} + L_0, \tag{17}$$

where $d\mu = d^2y\ e^{\sigma}$ is the invariant measure on N_2. It was proved that the sufficient and necessary condition that quasilocal angular momentum $L(\xi)$ of a two-surface along a given vector field ξ^a is insensitive to the ways of labeling the surface is

$$\tilde{\nabla}_a \xi^a = 0, \tag{18}$$

where $\tilde{\nabla}_a$ is the covariant derivative on N_2 [1]. Therefore, the definition of $L(\xi)$ makes sense only if ξ^a is a *divergence-free* vector field on N_2.

2 Poisson algebra of quasilocal angular momentum

By integrating the equation (15), we have another expression of $L(\xi)$ given by

$$L(u, v; \xi) = \frac{1}{16\pi} \int_{\Sigma} du\ d^2y\ (\pi^{ab} \mathcal{L}_{\xi} \rho_{ab} + \pi_{\sigma} \mathcal{L}_{\xi} \sigma + \pi_h \mathcal{L}_{\xi} h + \pi_a \mathcal{L}_{\xi} A^a_+ + \pi_a \partial_+ \xi^a) \quad (\tilde{\nabla}_a \xi^a = 0). \tag{19}$$

Using this form of 3-dimensional volume integral over hypersurface Σ, we can define a Poisson algebra of $L(\xi)$ on Σ by

$$\{L(\xi), L(\eta)\}_{\rm P.B.} = \int du \oint d^2y \left(\frac{\delta L(\xi)}{\delta q^I} \frac{\delta L(\eta)}{\delta p_I} - \frac{\delta L(\eta)}{\delta q^I} \frac{\delta L(\xi)}{\delta p_I} \right). \tag{20}$$

Now we state a significant theorem for the Poisson algebra of $L(\xi)$ as follows:

Theorem. Let ξ and η be two arbitrary vector fields tangent to N_2. The Poisson bracket of the 3-dimensional quasilocal angular momentum flux integrals associated with ξ and η is given by the 3-dimensional integral of the corresponding flux associated with the Lie bracket of ξ and η up to a constant factor. Precisely, it is given by

$$\{L(\xi), L(\eta)\}_{\rm P.B.} = \frac{1}{16\pi} L([\xi, \eta]_{\rm L}). \tag{21}$$

This theorem implies that there exists a homomorphism between the Poisson algebra of quasilocal angular momentum and the Lie algebra of tangent vectors on N_2. We can also define an invariant quasilocal angular momentum L^2 by

$$L^2(u, v) := \frac{1}{96\pi} \Big(\frac{\mathcal{A}}{4\pi} \Big)^2 \oint d^2y\ e^{-2\sigma} \rho^{ab} \pi_a \pi_b, \tag{22}$$

where $\mathcal{A}$ is the area of N_2, as an analog of the Casimir invariant of SO(3) algebra in the sense that

$$\{L^2, L(\xi)\}_{\rm P.B.} = 0, \tag{23}$$

for any divergence-free vector field ξ on N_2. By using this quasilocal quantity, we can avoid the problems related to the singular nature of the divergence-free tangent vector field ξ^a on a compact N_2.

3 Asymptotic limit at null infinity of asymptotically flat spacetime

At null infinity of asymptotically flat spacetimes, N_2 becomes S_2. Let $y^a = (\theta, \phi)$ be the usual angular coordinates on S_2. The generators $\bar{\xi}_{(\alpha)}$ ($\alpha = 1, 2, 3$) of the SO(3) isometry of S_2 are given by

$$\bar{\xi}_{(1)} = -\sin\phi \frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}, \quad \bar{\xi}_{(2)} = \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}, \quad \bar{\xi}_{(3)} = \frac{\partial}{\partial\phi}, \tag{24}$$

which satisfy the commutation relations $[\bar{\xi}_{(\alpha)}, \bar{\xi}_{(\beta)}]_{\rm L} = -\epsilon_{\alpha\beta}{}^{\gamma}\bar{\xi}_{(\gamma)}$. By continuity, there exist tangent vector fields $\xi_{(\alpha)}$ ($\alpha = 1, 2, 3$) on N_2 in the asymptotic zone of asymptotically flat spacetimes, which approach $\bar{\xi}_{(\alpha)}$ as N_2 approaches S_2 in the limit $v \to \infty$. Let $\bar{L}_\alpha$ be the gravitational angular momentum at null infinity defined by the limit

$$\bar{L}_\alpha := \lim_{v\to\infty} L(u, v; \xi_{(\alpha)}), \tag{25}$$

for such tangent vector fields $\xi_{(\alpha)}$. One can easily show that $\bar{\xi}_{(\alpha)}$ is divergence-free, that is, $\bar{\nabla}_a\bar{\xi}^a_{(\alpha)} = 0$ for $\alpha = 1, 2, 3$, and show that the following commutation relation holds

$$\{\bar{L}_\alpha, \bar{L}_\beta\}_{\rm P.B.} = -\frac{1}{16\pi}\epsilon_{\alpha\beta}{}^{\gamma}\bar{L}_\gamma. \tag{26}$$

as a direct consequence of the theorem proven in Section 2. This means that the algebra of gravitational angular momentum at null infinity realizes the SO(3) Lie algebra under the Poisson bracket.

The angular momentum of the Kerr spacetime can be obtained by taking the limit (25). Let us notice that the Kerr metric in the asymptotic zone can be written as

$$\begin{aligned} ds^2 = &-2dudv - (1 - \frac{2m}{v} + \cdots)du^2 + (\frac{4ma\sin^2\theta}{v} - \frac{4ma^3\sin^2\theta\cos^2\theta}{v^3} + \cdots)dud\phi \\ &+v^2(1 + \frac{a^2\cos^2\theta}{v^2} + \cdots)d\theta^2 + v^2\sin^2\theta(1 + \frac{a^2}{v^2} + \cdots)d\phi^2 \\ &+\sin^2\theta(\frac{4ma^3}{v^3} + \frac{8m^2a^3}{v^4} + \cdots)dvd\phi - (\frac{a^2\sin^2\theta}{v^2} + \cdots)dv^2, \end{aligned} \tag{27}$$

where m and a are the mass and the specific angular momentum of Kerr spacetime, respectively. It can be readily shown that

$$\bar{L}_1 = \bar{L}_2 = 0, \qquad \bar{L}_3 = ma, \qquad \lim_{v\to\infty} L^2 = (ma)^2 \tag{28}$$

which is precisely the angular momentum of the Kerr spacetime at null infinity.

Now let us show that the asymptotic limit of our quasilocal angular momentum coincides with the angular momentum defined by A. Rizzi [7]. His approach is based on an affine foliation of the asymptotic zone of a asymptotically flat spacetime with families of S_2 [8]. He introduced the in- and out-going null vector fields n and l with the normalization $< n, l >= -2$. Together with an orthonormal frame e_a ($a = 1, 2$) on S_2, n and l form a tetrad system

$$\{\, e_a, e_3, e_4 : a = 1, 2,\ e_3 = n, e_4 = l \,\}. \tag{29}$$

The twist ζ_a of n and l is defined as

$$\zeta_a = \frac{1}{2} < \nabla_3 e_4, e_a > . \tag{30}$$

Rizzi's angular momentum is defined as the following integral

$$L(\Omega_{(\alpha)}) = -\frac{1}{8\pi}\lim_{s\to\infty}\oint_{S_2}\Omega^a_{(\alpha)}\zeta_a dS_\gamma, \tag{31}$$

where $\Omega_{(\alpha)}$ are the SO(3) generators that satisfy the commutation relations $[\Omega_{(\alpha)}, \Omega_{(\beta)}]_{\rm L} = -\epsilon_{\alpha\beta}{}^{\gamma}\Omega_{(\gamma)}$, and s is an affine parameter of out-going null vector field l (Notice the change of sign from the Rizzi's original definition). The right hand side of (31) involves the limit that pulls the variables back to S_2, and dS_γ is the infinitesimal area element of S_2 with the standard metric γ. By using an identity

$$\zeta_a = \frac{1}{4}[n,\ l]_{{\rm L}a}. \tag{32}$$

one can straightforwardly show that

$$L(\Omega_{(\alpha)}) = \frac{1}{16\pi < n, l >}\lim_{s\to\infty}\oint_{S_2}\Omega^a_{(\alpha)}[n,\ l]_{{\rm L}a}dS_\gamma, \tag{33}$$

which is precisely the limit (25) of our quasilocal angular momentum, if the following identifications are made

$$dS_\gamma = d\mu, \quad \Omega^a_{(\alpha)} = \bar{\xi}^a_{(\alpha)}. \tag{34}$$

Therefore, the quasilocal angular momentum (17) is a quasilocal generalization of the angular momentum of gravitational fields proposed by A. Rizzi.

4 Summary

So far we have discussed the definition of the quasilocal angular momentum $L(\xi)$ and its noteworthy properties. In particular, the necessary and sufficient condition that ensures the gauge-independence of the quasilocal angular momentum is that ξ is divergence-free. And the Poisson algebra of the quasilocal angular momentum associated with an arbitrary tangent vector field ξ on N_2 is closed. Furthermore, the asymptotic limit of our quasilocal angular momentum is identical to the geometric angular momentum of A. Rizzi at null infinity, and its Poisson algebra reduces to the SO(3) commutation relations at null infinity.

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