Quasi-Vector Model of Propagation of Polarized Light in a Thin-Film Waveguide Lens

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**Abstract.** Maxwell equations describe the propagation with diffraction of waveguide modes through a thin-film waveguide lens. If the radius of the thin-film lens is large, then the thickness of the lens varies slowly in the \(yz\) plane. For this case we propose the model, which is based on the assumption of a small change in the electromagnetic field in a direction \(y\). Under this assumption the vector diffraction problem is reduced to a number of scalar diffraction problems. The solutions demonstrate the vector nature of the electromagnetic field, which allows us to call the proposed model a quasi-vector model.

**Figure 1.** Thin-film waveguide lens of radius \(R_l\) with variable height \(h_1(y,z)\)

**1 Introduction**

The first and the second Maxwell equations in the component representation in the Cartesian coordinate system for time-harmonic fields have the following form [1]:

\[
\begin{align*}
\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} &= -ik_0\epsilon E_x, \\
\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} &= -ik_0\epsilon E_y, \\
\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= -ik_0\epsilon E_z, \\
\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= ik_0 H_x, \\
\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= ik_0 H_y, \\
\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= ik_0 H_z.
\end{align*}
\]  

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2 Zeroth order approximation

If the waveguide lens has a radius $R_l \gg \lambda$, where $\lambda$ is the wavelength, then $|\partial h_1/\partial y| \ll 1$ and assuming in this case that the electromagnetic field also varies slowly along $y$, we introduce a small parameter defined as $\delta = \max \{|\partial E_a/\partial y|, |\partial H_a/\partial y|\}$, $\alpha = x, y, z$. In the zeroth order approximation over the small parameter $\delta$ the equations (1), (2) take the following form [2, 3]:

$$\left\{ \begin{array}{l} \frac{\partial H_x(0)}{\partial z} - \frac{\partial H_y(0)}{\partial x} = -ik_0n^2E_y(0), \quad ik_0H_x(0) = -\frac{\partial E_y(0)}{\partial z}, \\
\frac{\partial E_x(0)}{\partial z} - \frac{\partial E_y(0)}{\partial x} = ik_0H_y(0), \quad ik_0n^2E_x(0) = -\frac{\partial H_y(0)}{\partial z}, \quad -ik_0n^2E_z(0) = \frac{\partial H_y(0)}{\partial x}, \end{array} \right. \quad (3)$$

where $n(x, y, z)$ is the refractive index of the waveguide. The system of equations (1), (2) is now represented as two independent subsystems: the subsystem (3) for TE-mode and (4) for TM-mode, each subsystem can be represented as one equation of the second order for the leading component ($E_y$ for TE-mode and $H_y$ for TM-mode) and two equations relating the remaining components through the leading one [1]. We consider the problem of waveguide diffraction of one TE$_0$-mode incident on irregularity [2]. The formulation of the problem in the zeroth order approximation has the form [3]:

$$\left\{ \begin{array}{l} \left( \Delta_{zz} + k_y^2n^2(x, y, z) \right)E_y(0) = 0, \\
E_y(0)_{|z-R_l} = A(y) e^{ik_0\delta z} \varphi_0(x) + \sum_{j=1}^{N} R_{j}^{TE}(y) e^{-ik_0z_0} \varphi_j(x), \\
E_y(0)_{|z=R_l} = \sum_{j=1}^{N} T_{j}^{TE}(y) e^{ik_0\delta z} \varphi_j(x), \
E_y(0)_{x \to \pm \infty} \to 0, \end{array} \right. \quad (5)$$

where $\Delta_{zz} = \partial^2/\partial x^2 + \partial^2/\partial z^2$, $R_{j}^{TE}(y)$ and $T_{j}^{TE}(y)$ are reflection and transmission coefficients of the component $E_y$ (complex values, as in [4]), $A(y)$ determines the amplitude of the waveguide mode incident on the lens, $R_l$ is radius of the waveguide lens. The eigenvalues $\beta_j$ and eigenfunctions $\varphi_j(x)$ determine the TE-modes of the regular waveguide ($z > R_l$, $z < -R_l$). The components $H_x(0)$ and $H_z(0)$ are expressed in terms of $E_y(0)$ by means of (3). In addition, there is no TM-mode in the incident radiation, which leads us in the zeroth order approximation to $H_y(0) = E_z(0) = E_x(0) \equiv 0$, therefore the reflection and transmission coefficients of the components $E_x(0)$ and $E_z(0)$ are zeros.

We seek the approximate solution $\tilde{E}_y(0)$ of the problem (5) in the form: $\tilde{E}_y(0) = \sum_{j=1}^{N} V_j(y, z) \varphi_j(x)$ where $V_j(y, z)$ are the desired functions [2, 3, 5]. We substitute this solution into (5) and perform the Galerkin method procedure [2]. As a result we obtain the mixed boundary value problem with respect to the desired Kantorovich coefficient functions $V_j(y, z)$ [3, 5, 6]:

$$\left\{ \begin{array}{l} \tilde{\varphi}''_{zz} + Q(y, z) \tilde{\varphi} = 0, \\
\left( \tilde{\varphi}_z + ik_0\tilde{D}_{\varphi} \right)_{z=-R_l} = 2ik_0D_{\varphi} \tilde{n}_0, \\
\left( \tilde{\varphi}_z - ik_0\tilde{D}_{\varphi} \right)_{z=R_l} = 0. \end{array} \right. \quad (6)$$

For each $y = y_m$, the problem (6) is a boundary value problem for a system of ordinary differential equations, in spite of the fact that the coefficient functions $V_j(y, z)$ depend on two variables. Differentiation is carried out only on the fast variable $z$ and the dependence on the slow variable $y$ is parametric.

Solving (5), we define the leading component $E_y^{(0)}$ and the components $H_x^{(0)}$ and $H_z^{(0)}$ of the TE-mode by the formula (3). The remaining components corresponding to the TM-mode will be zero. The zeroth order approximation, therefore, does not describe the process of hybridization of modes and we now proceed to the first order approximation.
3 First order approximation

We get the Maxwell equations in the first order approximation: $\vec{E} = \vec{E}^{(0)} + \vec{E}^{(1)}$, $\vec{H} = \vec{H}^{(0)} + \vec{H}^{(1)}$ considering that $\partial E^{(1)}_\alpha / \partial y$, $\partial H^{(1)}_\alpha / \partial y$ for $\alpha = x, y, z$ have the second order of smallness. We obtain the following equations for the TE-mode:

$$\left(\Delta_{x,z} + k_0^2 n^2 (x, y, z)\right) E^{(1)}_y = 0, \quad H^{(1)}_x = -\frac{1}{ik_0} \frac{\partial E^{(1)}_y}{\partial z}, \quad H^{(1)}_z = \frac{1}{ik_0} \frac{\partial E^{(1)}_y}{\partial x}. \quad (7)$$

The formulation of the diffraction problem for the TE-mode in the first order approximation has the following form:

$$\left\{ \begin{array}{l} \left(\Delta_{x,z} + k_0^2 n^2 (x, y, z)\right) E^{(1)}_y = 0, \\
E^{(1)}_y \big|_{z<0} = \sum_{j=1}^{N} R^{(1)}_j (y) e^{-ik_0\beta_j z} \varphi_j (x), \quad E^{(1)}_y \big|_{z>0} = \sum_{j=1}^{N} T^{(1)}_j (y) e^{ik_0\beta_j z} \varphi_j (x), \quad E^{(1)}_y \big|_{x \rightarrow \pm \infty} = 0. \end{array} \right. \quad (8)$$

In the boundary conditions of the problem (8), which determine the reflected and transmitted fields, the incident field is absent, since it is completely taken into account in the zeroth order approximation, and in the first approximation gives only the first-order corrections to the reflection and transmission coefficients. Applying the incomplete Galerkin method to the diffraction problem (8), we obtain a boundary value problem for a homogeneous system of second-order differential equations with homogeneous boundary conditions of the third kind, which has only a trivial solution [2]. Thus, the correction of the first order $E^{(1)}_y$ vanishes identically and also $H^{(1)}_x = H^{(1)}_z = 0$. The first-order correction $H^{(1)}_y$ is also zero. Corrections of the first order of smallness to the remaining components of the field are equal, respectively to:

$$E^{(1)}_x = -\frac{1}{ik_0 n^2} \frac{\partial H^{(0)}_y}{\partial y}, \quad E^{(1)}_z = \frac{1}{ik_0 n^2} \frac{\partial H^{(0)}_y}{\partial y}. \quad (9)$$

It follows from (9) that the TE-mode incident on the irregularity excites the components of the TM-mode of the first order of smallness, that is, the solution obtained in the first approximation describes the hybridization of waveguide modes, which demonstrates the vector nature of the model.

4 Numerical experiment

We solve numerically the problem of waveguide diffraction of monochromatic light in the waveguide lens with the following input data: $n_e = 1.0$, $n_s = 1.47$, $n_f = 1.565$ and $n_l = 1.9$. The thickness of the waveguide layer is $2\lambda$ where $\lambda = 0.55\mu$ is the wavelength. The waveguide mode $TE_0$ with a unit amplitude ($A (y) = 1$) incidents on a waveguide lens with a radius $R_1 = 20\lambda$. We consider the transmission coefficient of the component $E^{(0)}_y$ of the second waveguide mode, which was excited by the energy redistribution between modes during the propagation of the mode $TE_0$. The order of smallness of $T^{TE}_2 (y)$, as follows from the Figure 2, corresponds to the assumption that the value of $\partial E^{(0)}_y / \partial y$ is small. The remaining components of the field have the transmission coefficients similar of the same order of smallness.

The transmission coefficients of the components $E^{(1)}_x$ and $E^{(1)}_z$ (see Fig. 3) are by an order smaller than the coefficient of transmission of $E^{(0)}_y$, which corresponds to the first approximation of the problem under consideration. The corrections of the first order of smallness to the remaining components of the field are zeros.
5 Conclusion

The paper describes a quasi-vector model of propagation of waveguide modes through a thin-film waveguide lens of large radius. Under the assumption of a small change in the electromagnetic field with respect to $y$, one can construct a perturbation theory with respect to the corresponding small parameter. The proposed model allows us to formulate in the zeroth order approximation a series of scalar diffraction problems the TE-mode, the solutions of $m$-th problem describe the TE-mode evolution in the longitudinal section $y = y_m$. In the first approximation, these solutions are used to calculate the corrections $E^{(1)}_x$, $E^{(1)}_z$ perturbing the components of the TE- and TM-modes. Thus, as a result of solving a series of scalar problems, we obtain a solution that possesses the properties of the solution of the vector problem $\vec{E} = (E^{(1)}_x, E^{(0)}_y, E^{(1)}_z)$, $\vec{H} = (H^{(0)}_x, 0, H^{(0)}_z)$.

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